October 10, 2005

On the number of components in 2-factors of claw-free graphs

Kiyoshi Yoshimoto
Department of Mathematics, Collage of Science and Technology
Nihon University, Tokyo 101-8308, Japan
yosimoto@math.cst.nihon-u.ac.jp

Abstract

In this paper, we prove that if a claw-free graph G with minimum degree $\delta \geq 4$ has no maximal clique of two vertices, then G has a 2-factor with at most $(|G|-1)/4$ components. This upper bound is best possible. Additionally, we give a family of claw-free graphs with minimum degree $\delta \geq 4$ in which every 2-factor contains more than n/δ components.

1 Introduction

In this paper, we consider only finite graphs with no loops or no multiple edges. If no ambiguity can arise, we denote simply the order $|G|$ of G by n and the minimum degree $\delta(G)$ by δ. All notation and terminology not explained in this paper is given in [2].

A *2-factor* of a graph G is a spanning 2-regular subgraph of G, and so a Hamilton cycle is a 2-factor. It is a well known conjecture that every 4-connected claw-free graph is hamiltonian ([10]). For small connected claw-free graphs, Jackson and the author proved the following.

Theorem 1 ([6], [7]). *1. Every 3-connected claw-free graph with $\delta \geq 4$ has a 2-factor with at most $2n/15$ components.*

2. Every 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor with at most $(n+1)/4$ components.

Probably, neither of the upper bounds in Theorem 1 is best possible. For connected claw-free graphs, Faudree et al. [4] showed that a claw-free graph with $\delta \geq 4$

has a 2-factor with at most $6n/(\delta+2)-1$ components, and Gould and Jacobson [9] proved that if $\delta \geq (4n)^{\frac{2}{3}}$, then the graph has a 2-factor with at most n/δ components. In general, the second upper bound is too strong. In Section 3, we will construct examples of claw-free graphs in which every 2-factor contains more than n/δ components. Especially, for the case of $\delta = 4$, there exists a family $\{G_i\}$ of claw-free graphs such that

$$\frac{f_2(G_i)}{|G_i|} \to \frac{5}{18} \quad (|G_i| \to \infty),$$

where $f_2(G_i)$ is the minimum number of components in a 2-factor of G_i. We construct this example also in Section 3.

Both of the above examples contain bridges. Hence, it is a natural question to ask whether a bridgeless claw-free graph has a 2-factor with at most $n/4$ components or not. In this paper, we show that the following slightly weaker statement holds.

Theorem 2. *Let G be a claw-free graph with $\delta \geq 4$. If G has no maximal clique of two vertices, then G has a 2-factor with at most $(n-1)/4$ components.*

We will prove this theorem in Section 4 and describe an example in Section 3, which shows that the upper bound on the number of components in Theorem 2 is, in some sense, best possible.

The results of Egawa and Ota [3] and Choudum and Paulraj [1] implies that a claw-free graph G with $\delta \geq 4$ has a 2-factor. If G has a bridge, then the graph obtained from G by removing all bridges has a 2-factor, i.e., each block of G has a 2-factor. In general, for blocks, we can reduce the minimum degree condition.

Theorem 3. *Every 2-connected claw-free graph with $\delta \geq 3$ has a 2-factor.*

However, we cannot replace 2-connectivity by bridgeless. For example, the line graph G of the graph drawn in Figure 1 is bridgeless, $\delta(G) = 3$, and G has no 2-factor.

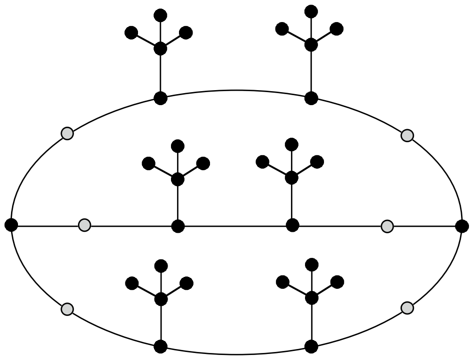

Figure 1:

2 Notation and Preliminary Results

The set of all the neighbours of a vertex x in a graph G is denoted by $N_G(x)$, or simply $N(x)$, and its cardinality by $d_G(x)$, or $d(x)$. The *edge-degree* of an edge uv is defined as $d(u) + d(v) - 2$ and the minimum edge-degree $\delta_e(G)$ is the minimum number of the edge-degrees of all edges in G. Let $e(G)$ denote the size of $E(G)$, i.e., the number of edges in G. The set of all vertices of degree k in G is denoted by $V_k(G)$ and we put $V_{\geq k}(G) = \bigcup_{i \geq k} V_i(G)$.

For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. The set of neighbours $(\bigcup_{v \in H} N_G(v)) \setminus V(H)$ is written by $N_G(H)$ or $N(H)$, and for a subgraph $F \subset G$, $N_G(H) \cap V(F)$ is denoted by $N_F(H)$. For simplicity, we denote $|V(H)|$ by $|H|$, "$u_i \in V(H)$" by "$u_i \in H$", and "$G - V(H)$" by "$G - H$".

An *even graph* is a graph in which every vertex has positive even degree. A connected even subgraph is called a *circuit*, and the $K_{1,m}$, a *star*. Let $\mathcal{S}$ be a set of edge-disjoint circuits and stars with at least three edges in a graph H. We call $\mathcal{S}$ a *system that dominates* H if every edge of H is either contained in one of the circuits or stars of $\mathcal{S}$ or is adjacent to one of the circuits. The number of elements in $\mathcal{S}$ is denoted by $\#\mathcal{S}$. We shall use the following result of Gould and Hynds.

Lemma A ([8]). *Let H be a graph. Then $L(H)$ has a 2-factor with c components if and only if there is a system that dominates H with c elements.*

3 Examples

1. We first construct a line graph in which every 2-factor contains more than n/δ components. Let $d \geq 4$ be an integer and R_d be the graph obtained from $K_2 \cup (d-1)K_{1,d}$ by adding $d-1$ edges joining a specified vertex in K_2 and the center of each $K_{1,d}$ as in Figure 2. Let us call the gray vertex in this figure a *top*.

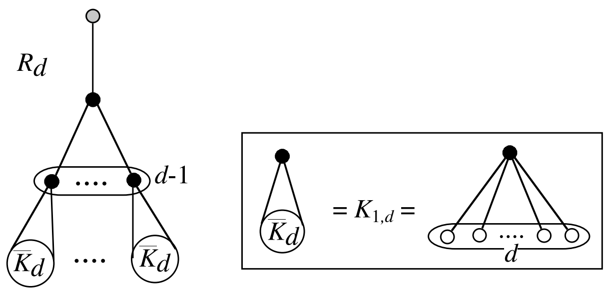

Figure 2: R_d

We define a tree $H^*_{m,d}$ from the path $P_m = u_1u_2\cdots u_m$ and a number of R_d as follows. For each inner vertex of P_m, we add $(d-2)R_d$ and $d-2$ edges joining the inner vertex and the top of each R_d as in Figure 3, and for each end of P_m, we add $(d-1)R_d$ and

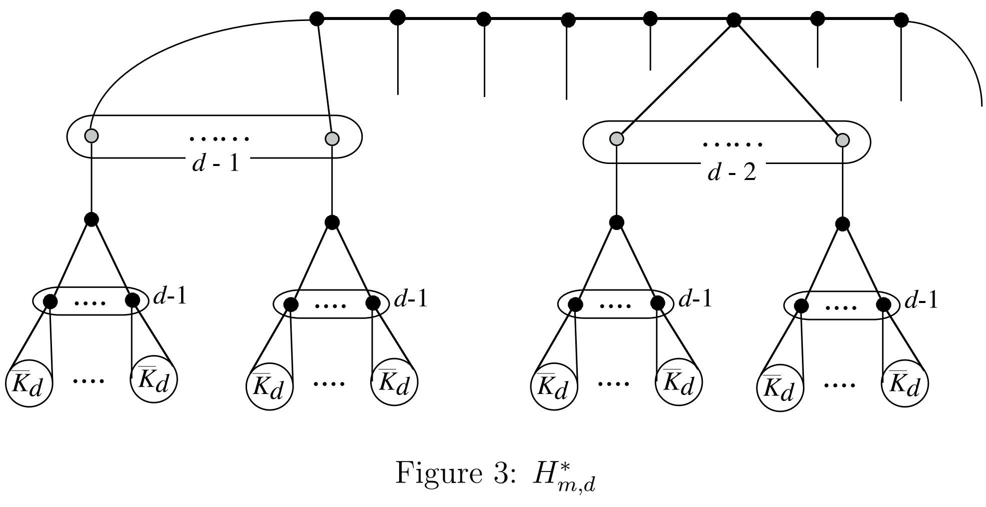

Figure 3: $H^*_{m,d}$

$d-1$ edges. It is easy to check that $\delta_e(H^*_{m,d}) \geq d$, and so $\delta(L(H^*_{m,d})) \geq d \geq 4$. Hence $L(H^*_{m,d})$ has a 2-factor, and by Lemma A, there exists a system $\mathcal{S}$ that dominates $H^*_{m,d}$. We show that the cardinality $\#\mathcal{S}$ must be greater than e/d, where e is the size of $H^*_{m,2d}$.

Let S be the set of the centers of all the stars in $\mathcal{S}$, and we show that $S = V_{\geq 3}(H^*_{m,d})$. By the definition of a system, $S \subseteq V_{\geq 3}(H^*_{m,d})$. Let us label the neighbours of P_m as follows

$$N(P_m) = \{y_{ij} \mid 1 \leq j \leq d-1 \text{ if } i = 1 \text{ or } m; \text{ and } 1 \leq j \leq d-2 \text{ if } 2 \leq i \leq m-1\}.$$

For each y_{ij}, let x_{ij} be the neighbour of y_{ij} which is not u_i. Since $d(y_{ij}) = 2$, the edges $u_i y_{ij}, y_{ij} x_{ij}$ must be covered by the stars in $\mathcal{S}$ whose center are u_i, x_{ij}, respectively. This implies $\{u_i\} \cup \{x_{ij}\} \subset S$. Similarly, every pendant edge is also covered by a star in $\mathcal{S}$ whose center is in $N(V_1(H^*_{m,d}))$. Therefore, $V_{\geq 3}(H^*_{m,d}) \subseteq S$, which are colored black in Figure 3. Thus, $\#\mathcal{S} = |V_{\geq 3}(H^*_{m,d})|$.

Since the order of R_d is $(d+1)(d-1)+2 = d^2+1$,

$$|H^*_{m,2d}| = m + (d^2+1)(d-2)m + 2(d^2+1) = (d^3 - 2d^2 + d - 1)m + 2(d^2+1).$$

Hence,

$$e = (d^3 - 2d^2 + d - 1)m + 2d^2 + 1 \quad \text{and} \quad m = \frac{e - (2d^2+1)}{d^3 - 2d^2 + d - 1}. \tag{1}$$

Since each R_d contains d vertices of degree at least three,

$$\begin{aligned}
|V_{\geq 3}(H^*_{m,d})| &= m + d(d-2)m + 2d = (d^2 - 2d + 1)m + 2d \\
&= (d^2 - 2d + 1)\frac{e - (2d^2+1)}{d^3 - 2d^2 + d - 1} + 2d \qquad (\text{ by } (1)) \\
&= \frac{(d^2 - 2d + 1)e - (2d^2+1)(d^2 - 2d + 1) + 2d(d^3 - 2d^2 + d - 1)}{d^3 - 2d^2 + d - 1} \\
&= \frac{(d^2 - 2d + 1)e - (d^2+1)}{d^3 - 2d^2 + d - 1} > \frac{e}{d} \\
\Longleftrightarrow\ & e > d(d^2+1) \\
\Longleftrightarrow\ & (d^3 - 2d^2 + d - 1)m + 2d^2 + 1 > d(d^2+1) \qquad (\text{ by } (1)) \\
\Longleftrightarrow\ & m > \frac{d^3 - 2d^2 + d - 1}{d^3 - 2d^2 + d - 1} = 1.
\end{aligned}$$

Hence if $m \geq 2$, then $|V_{\geq 3}(H^*_{m,d})| > e/d$. Therefore, by Lemma A, any 2-factor of $L(H^*_{m,d})$ has more than n/d components.

On the other hand, since $|V_{\geq 3}(H^*_{m,d})| < e/(d-1)$, the following problem still remains.

Problem 4. *Does every claw-free graph with $\delta \geq 4$ have a 2-factor with less than $n/(\delta - 1)$ components?*

2. The second example is complicated. First we define a tree B_T^m inductively from $B_T^0 = K_1$ as follows; B_T^m is obtained from B_T^{m-1} by adding, for each end vertex of B_T^{m-1}, two new vertices and two edges joining the end and the new vertices. The graph B_T^2 is drawn in Figure 4(i). Let $\widetilde{B_T^m}$ be the graph obtained from B_T^m by

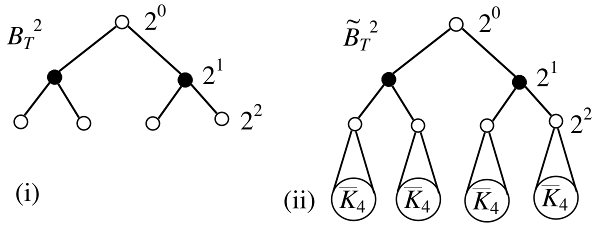

Figure 4:

replacing each end vertex of B_T^m by $K_{1,4}$ as in Figure 4(ii). Then

$$|B_T^m| = \sum_{0 \leq i \leq m} 2^i = 2^{m+1} - 1 \text{ and}$$
$$|\widetilde{B_T^m}| = |B_T^m| + 4(2^m) = 2^{m+1} - 1 + 2(2^{m+1}) = 3(2^{m+1}) - 1.$$

Let u_0 be the vertex of degree two in B_T^m and

$$U_0^m = \{u \in V(B_T^m) \mid d(u, u_0) \equiv 0 \pmod 2\ \} \text{ and}$$
$$U_1^m = V(B_T^m) \setminus U_0^m.$$

Let $m = 2k$ and then

$$|U_0^{2k}| = \sum_{0 \leq i \leq k} 2^{2i} = \frac{2^{2k+2} - 1}{3} \text{ and}$$
$$|U_1^{2k}| = |B_T^{2k}| - |U_0^{2k}| = 2^{2k+1} - 1 - \frac{2^{2k+2} - 1}{3} = \frac{2^{2k+1} - 2}{3}.$$

Let

$$\widetilde{U_i^{2k}} = U_i^{2k} \cup V_1(B_T^{2k}),$$

for $i \in \{0,1\}$, and then

$$|\widetilde{U_0^{2k}}| = |U_0^{2k}| = \frac{2^{2k+2}-1}{3} \text{ and } |\widetilde{U_1^{2k}}| = |U_1^{2k}| + 2^{2k} = \frac{5(2^{2k})-2}{3}.$$

For simplicity, let $x = 2^{2k}$ and then

$$|\widetilde{B_T^m}| = 6x-1,\ |\widetilde{U_0^{2k}}| = \frac{4x-1}{3},\ \text{and } |\widetilde{U_1^{2k}}| = \frac{5x-2}{3}. \tag{2}$$

Notice that $\widetilde{B_T^{2k}}$ has only one system, i.e., the set of all the stars of which centers are the vertices of $\widetilde{U_1^{2k}}$. Note that in order to make these stars edge-disjoint, the star with center in U_1^{2k} can be taken as the vertex with all its neighbours, while the stars with center in $V_1(B_T^{2k})$ must avoid the edge to its neighbour u which is at distance $d(u,u_0) = 2k-1$ from u_0. The cardinality of the system is $(5x-2)/3$ and the ratio of $|\widetilde{U_1^{2k}}|$ and $|\widetilde{B_T^{2k}}|$ is

$$\frac{|\widetilde{U_1^{2k}}|}{|\widetilde{B_T^{2k}}|} = \frac{5x-2}{18x-3} \to \frac{5}{18} \quad (2k \to \infty),$$

but the minimum edge-degree is three. Hence, next we construct a tree of which minimum edge-degree is four using $\widetilde{B_T^{2k}}$.

Let $B_{m,2k}$ be the graph obtained from P_m and $mK_{1,5}$ and $(m+2)\widetilde{B_T^{2k}}$ by adding $(2m+2)$ edges as in Figure 5. It is easy to check that $\delta_e(B_{m,2k}) = 4$. Hence, there is a system that dominates $B_{m,2k}$ by Lemma A. Let $\mathcal{S}$ be a system that dominates $B_{m,2k}$ such that the cardinality is minimum, and let S be the set of the centers of all the stars in $\mathcal{S}$.

Since $V_2(B_{m,2k}) \cap S = \emptyset$, the center of each $K_{1,5}$ and $V(P_m)$ are included in S. Thus $S \cap V(\widetilde{B_T^{2k}})$ is $\widetilde{U_0^{2k}}$ or $\widetilde{U_1^{2k}}$ obviously. However, the degrees of vertices in P_m are four and those are adjacent consecutively. Therefore, except one $\widetilde{B_T^{2k}}$, for every $\widetilde{B_T^{2k}}$,

$$S \cap V(\widetilde{B_T^{2k}}) = \widetilde{U_1^{2k}}.$$

In Figure 5, S is the set of all black vertices. Hence by (2),

$$\begin{aligned} \#\mathcal{S} = |S| &= m + m + (m+1)|\widetilde{U_1^{2k}}| + |\widetilde{U_0^{2k}}| = 2m + (m+1)\frac{5x-2}{3} + \frac{4x-1}{3} \\ &= \frac{5x+4}{3}m + (3x-1). \end{aligned}$$

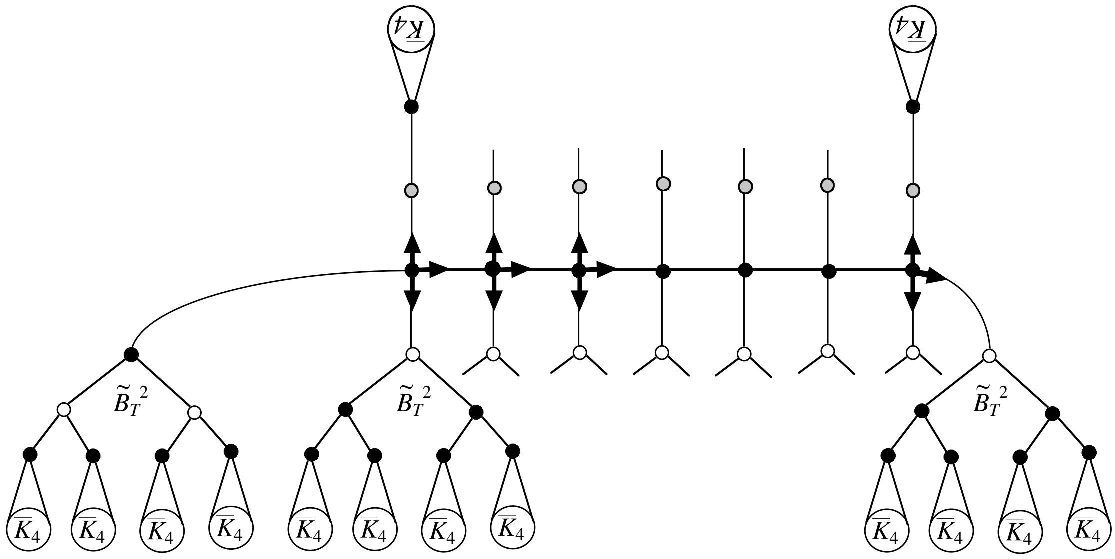

Figure 5: $B_{m,2k}$

Since $|K_{1,5}| = 6$ and $|\widetilde{B_T^{2k}}| = 6x - 1$,

$$|B_{m,2k}| = m + 6m + (6x-1)m + 2(6x-1) = (6x+6)m + 2(6x-1)$$

and so

$$e = e(B_{m,2k}) = (6x+6)m + 12x - 3.$$

Thus the ratio of $|S|$ and e, i.e., the ratio of the the minimum number of cycles in a 2-factor of $L(B_{m,2k})$ and $|L(B_{m,2k})|$, is

$$\frac{|S|}{e} = \frac{\frac{5x+4}{3}m + (3x-1)}{(6x+6)m + 12x - 3)} = \frac{5xm + 4m + 9x - 3}{18xm + 18m + 36x - 9} \to \frac{5}{18} \quad (2k, m \to \infty).$$

Now, the following problem remains.

Problem 5. *Does every claw-free graph with $\delta \geq 4$ have a 2-factor with at most $5n/18$ components?*

3. Finally we construct line graphs which show that the upper bound in Theorem 2 is best possible. Let $P_{2m} = u_1u_2 \cdots u_{2m}$ be the path and let $H_{2m,4}$ be the graph obtained from $P_{2m} \cup (2m+2)K_{1,4}$ by adding $2m+2$ edges as in Figure 6.

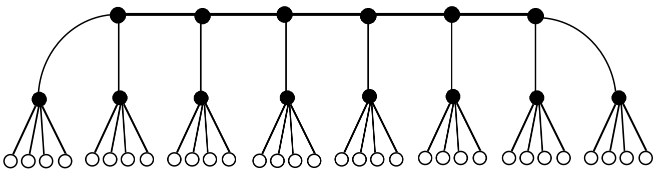

Figure 6: $H_{2m,4}$

Clearly $\delta_e(H_{2m,4}) = 4$, and so its line graph $L(H_{2m,4})$ has minimum degree four. Moreover, $L(H_{2m,4})$ has no maximal clique of two vertices because there is no vertex of degree two in $H_{2m,4}$. Let $\mathcal{S}$ be a system that dominates $H_{2m,4}$ and S be the set of the centers of all stars in $\mathcal{S}$.

Since every edge u_iu_{i+1} in P_{2m} is covered by a star in $\mathcal{S}$ with center u_i or u_{i+1}, S have to contain at least half vertices in P_{2m}. On the other hand, since $V(P_{2m}) \subset V_3(H_{2m,4})$, no consecutive two vertices are contained in S. Therefore, $|S \cap V(P_{2m})| = m$. Since $S \cap V_1(H_{2m,4}) = \emptyset$, S contains all vertices in $V_5(H_{2m,4})$; otherwise, there is a pendant edge which is not covered by a star in $\mathcal{S}$. Thus

$$\#\mathcal{S} = |S| = m + (2m+2) = 3m + 2.$$

Since the order of $H_{2m,4}$ is

$$2m + 5(2m+2) = 12m + 10,$$

then, $e = e(H_{2m,4}) = 12m + 9$. Therefore

$$\#\mathcal{S} = 3m + 2 = 3\frac{e-9}{12} + 2 = \frac{e-1}{4},$$

and any 2-factor of $L(H_{2m,4})$ has at least $(|L(H_{2m,4})|-1)/4$ components by Lemma A.

Easily we can generalize this example as follows. Let $H_{2m,d}$ be the graph obtained from $H_{2m,4}$ by replacing each $K_{1,4}$ adjacent to internal vertices of P_{2m} by $(d-2)/2K_{1,d}$ and by replacing each $2K_{1,4}$ adjacent to the ends by $(d/2)K_{1,d}$ as in Figure 7. Then as in the case of $H_{2m,4}$, it is easy to see that the minimum edge-degree is d and $L(H_{2m,d})$ has no maximal clique of two vertices.

Since the order is

$$2m + (d+1)\frac{d-2}{2}2m + 2(d+1) = d(d-1)m + 2(d+1),$$

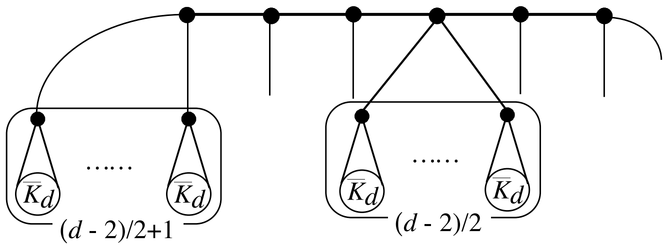

Figure 7: $H_{2m,d}$

then, $e = e(L(H_{2m,d})) = d(d-1)m + 2d + 1$. As in the case of $H_{2m,4}$, it is easy to check that the number of stars of any system that dominates $H_{2m,d}$ is at least

$$m + \frac{d-2}{2}2m + 2 = (d-1)m + 2 = (d-1)\frac{e-(2d+1)}{d(d-1)} + 2 = \frac{e-1}{d}.$$

Problem 6. *Does every bridgeless claw-free graph with $\delta \geq 4$ have a 2-factor with at most $(n-1)/\delta$ components?*

4 Proofs of Theorems 2 and 3

Let x be a vertex of a claw-free graph G. If the subgraph induced by $N(x)$ is connected, we add edges joining all pairs of nonadjacent vertices in $N(x)$. This operation is called *local completion* of G at x. The *closure* $cl(G)$ of G is a graph obtained by recursively repeating the local completion operation, as long as this is possible. Ryjáček [11] showed that the closure of G is uniquely determined and G is hamiltonian if and only if $cl(G)$ is hamiltonian. The latter result was extended to 2-factors as follows.

Theorem B (Ryjáček, Saito and Shelp [12])**.** *Let G be a claw-free graph. If $cl(G)$ has a 2-factor with k components, then G has a 2-factor with at most k components.*

Since G is a spanning subgraph of $cl(G)$, Theorem B implies that

$$f_2(G) = f_2(cl(G)),$$

where $f_2(G)$ is the minimum number of components in a 2-factor of G. Ryjáček also proved:

Theorem C ([11]). *If G is a claw-free graph, then there is a triangle-free graph H such that*

$$L(H) = cl(G).$$

If a claw-free graph G has no maximal clique of two vertices, then obviously $cl(G)$ also has no such cliques. Moreover, $L(H)$ has no maximal clique of two vertices if and only if H has no vertex of degree two. Thus for Theorem 2, it is sufficient to prove the following lemma, by Theorems B and C.

Lemma 7. *Let H be a triangle-free graph with $\delta_e(H) \geq 4$. If $V_2(H) = \emptyset$, then H has a system of cardinality at most $(e(H)-1)/4$ that dominates H.*

A graph H is *essentially k-edge-connected* if for any edge set E_0 of at most $k-1$ edges, $H - E_0$ contains at most one component with edges. Since $L(H)$ is k-edge-connected if and only if H is an essentially k-edge-connected, for Theorem 3, it is sufficient to prove the following lemma, by Theorems B and C.

Lemma 8. *If H is an essentially 2-edge-connected graph with $\delta_e(H) \geq 3$, then there exists a system $\mathcal{S}$ that dominates H such that the even subgraph in $\mathcal{S}$ passes through all vertices in $V_{\geq 3}(H - V_1(H))$.*

4.1 Proof of Lemma 7

We first show the following lemma.

Lemma 9. *Let H be a tree with $\delta_e(H) \geq 4$. If $V_2(H) = \emptyset$, then H has a system of cardinality at most $(e(H)-1)/4$ that dominates H.*

Proof. We proceed by contradiction. Suppose the lemma is false and choose a counterexample H with $e(H)$ as small as possible. Let $F = H - V_1(H)$ and $Pr(H) = N(V_1(H))$.

Claim 1. *$d_H(x) = 5$ for all $x \in Pr(H)$.*

Proof. Since $\delta_e(H) \geq 4$, $d_H(x) \geq 5$ for $x \in Pr(H)$. Label the vertices of $N_H(x)$ as follows:

$$\begin{aligned} N_H(x) \cap V_1(H) &= \{u_i \mid i \leq |N_H(x) \cap V_1(H)|\}, \\ N_F(x) &= \{y_j \mid j \leq |N_F(x)|\}, \end{aligned} \tag{3}$$

and for each $y_j \in N_F(x)$, let F_j be the component of $H - x$ containing y_j. Assume that $d_H(x) \geq 6$. Suppose $|N_H(x) \cap V_1(H)| \geq 2$ and let $H' = H - u_1$. Since $d_{H'}(x) \geq 5$, $\delta_e(H') \geq 4$. As $e(H') < e(H)$, there exists a system $\mathcal{S}'$ that dominates H', of cardinality at most $(e(H') - 1)/4 = (e(H) - 2)/4$. Let A be the star in $\mathcal{S}'$ containing the edge xu_2. Clearly, the center of A is x, and so $A' = A \cup xu_1$ is a star. Hence $(\mathcal{S}' \setminus \{A\}) \cup \{A'\}$ is a system that dominates H and its cardinality is at most $(e(H) - 2)/4$. This contradicts the choice of H.

Hence, $|N_H(x) \cap V_1(H)| = 1$. See Figure 8(i). Let $H'_1 = F_1 \cup F_2 \cup \{y_1 y_2\}$. Let

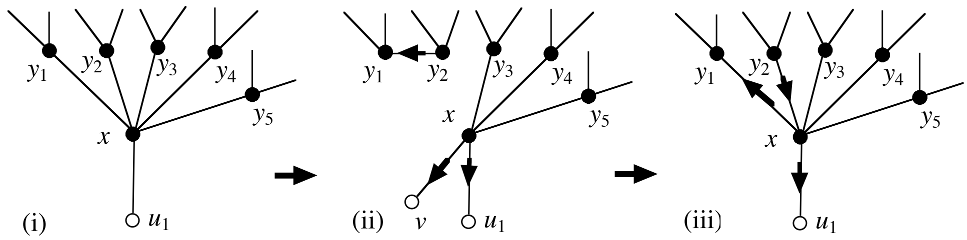

Figure 8:

v be a new vertex and $H'_2 = (H - (F_1 \cup F_2)) \cup \{v, xv\}$. See Figure 8(ii). Because $\delta_e(H'_i) \geq 4$, there exists a system $\mathcal{S}_i$ that dominates H'_i, of cardinality at most $(e(H'_i) - 1)/4$ for each $i \in \{1, 2\}$. Let A_1 be the star in $\mathcal{S}_1$ containing the edge $y_1 y_2$ and A_2 be the star in $\mathcal{S}_2$ containing xv. By symmetry, we may assume that the center of A_1 is y_2. Let $A'_1 = (A_1 - y_1) \cup y_2 x$ and $A'_2 = (A_1 - v) \cup x y_1$. Then, $(\mathcal{S}_1 \cup \mathcal{S}_2 \setminus \{A_1, A_2\}) \cup \{A'_1, A'_2\}$ is a system that dominates H and its cardinality is

$$\begin{aligned} \#\mathcal{S}_1 + \#\mathcal{S}_2 &\leq \frac{e(H'_1) - 1}{4} + \frac{e(H'_2) - 1}{4} \\ &= \frac{e(F_1) + e(F_2) + 1 - 1}{4} + \frac{e(H) - e(F_1) - e(F_2) - 2 + 1 - 1}{4} \\ &= \frac{e(H) - 2}{4}. \end{aligned}$$

This contradicts again the choice of H. □

Claim 2. $Pr(H) = V_1(F)$.

Proof. Since $V_1(F) \subseteq Pr(H)$, it is sufficient to prove that $Pr(H) \subseteq V_1(F)$. Suppose that there is $x \in Pr(H) \setminus V_1(F)$ and let us label its neighbours $\{u_i\}, \{y_j\}$ as in (3), and define $\{F_j\}$ as before. We divide our argument into three cases.

1. $|N_H(x) \cap V_1(H)| = 3$.
By Claim 1, $d_F(x) = 2$ and

$$\sum_{1 \le j \le d_F(x)} e(F_j) = e(H) - 5. \tag{4}$$

See Figure 9(i). Since the tree $H' = F_1 \cup F_2 \cup \{y_1 y_2\}$ has minimum edge-degree

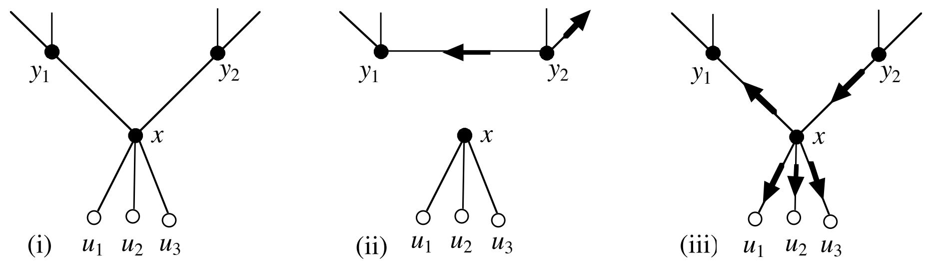

Figure 9:

at least four and $|e(H')| < |e(H)|$. As $e(H') < e(H)$, there exists a system $\mathcal{S}'$ that dominates H', of cardinality at most

$$\frac{e(F_1) + e(F_2) + 1 - 1}{4} = \frac{e(F_1) + e(F_2)}{4} = \frac{e(H) - 5}{4}.$$

See Figure 9(ii). By symmetry, we may assume that the center of the star $A \in \mathcal{S}$ containing the edge $y_1 y_2$ is y_2. Let A' be the star $(A - y_1) \cup y_2 x$ and B be the star $xy_1 \cup xu_1 \cup xu_2 \cup xu_3$. See Figure 9(iii). Then $(\mathcal{S}' \setminus \{A\}) \cup \{A', B\}$ is a system that dominates H and its cardinality is

$$\#\mathcal{S}' + 1 \le \frac{e(H) - 5}{4} + 1 = \frac{e(H) - 1}{4}.$$

This contradicts our choice of H.

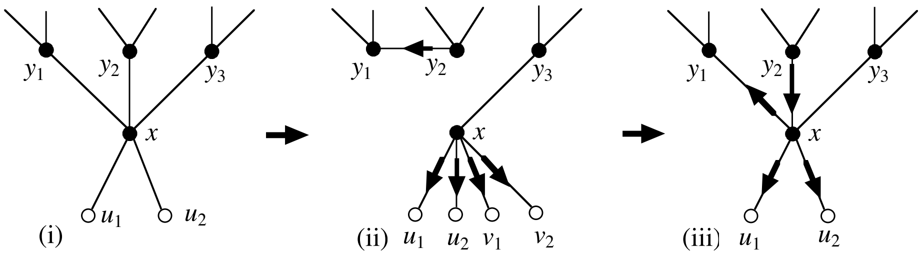

Figure 10:

2. $|N_H(x) \cap V_1(H)| = 2$.

By Claim 1, $d_F(x) = 3$ and (4) holds. See Figure 10(i). Let $H_1' = F_1 \cup F_2 \cup \{y_1y_2\}$. Let v_1, v_2 be new vertices and

$$H_2' = (H - (F_1 \cup F_2)) \cup \{v_1, v_2, xv_1, xv_2\}.$$

See Figure 10(ii). As $\delta_e(H_i') \geq 4$ and $e(H_i') < e(H)$, there exists a system $\mathcal{S}_i'$ that dominates H_i' for each $i \in \{1, 2\}$, such that

$$\begin{aligned} \#\mathcal{S}_1' &\leq \frac{e(F_1) + e(F_2) + 1 - 1}{4} = \frac{e(F_1) + e(F_2)}{4} \\ \#\mathcal{S}_2' &\leq \frac{e(F_3) + 5 - 1}{4} = \frac{e(F_3) + 4}{4}. \end{aligned}$$

By symmetry, we may assume that the center of the star $A_1 \in \mathcal{S}_1$ containing the edge y_1y_2 is y_2, and let $A_2 \in \mathcal{S}_2$ be the star containing the edge xu_1. Let $A_1' = (A_1 - y_1) \cup y_2x$ and $A_2' = (A_2 - \{v_1, v_2\}) \cup xy_1$. See Figure 10(iii). Then $(\mathcal{S}_1 \cup \mathcal{S}_2 \setminus \{A_1, A_2\}) \cup \{A_1', A_2'\}$ is a system that dominates H and, by (4), its cardinality is

$$\#\mathcal{S}_1 + \#\mathcal{S}_2 \leq \frac{e(F_1) + e(F_2) + e(F_3) + 4}{4} = \frac{e(H) - 1}{4},$$

a contradiction.

3. $|N_H(x) \cap V_1(H)| = 1$.

By Claim 1, $d_F(x) = 4$ and (4) holds. See Figure 11(i). Let $H_1' = F_1 \cup F_2 \cup \{y_1y_2\}$ and $H_2' = F_3 \cup F_4 \cup \{y_3y_4\}$, and then as in the previous case, there exists a system $\mathcal{S}_i$ that dominates H_i', of cardinality at most

$$\frac{e(F_{2i-1}) + e(F_{2i}) + 1 - 1}{4} = \frac{e(F_{2i-1}) + e(F_{2i})}{4}$$

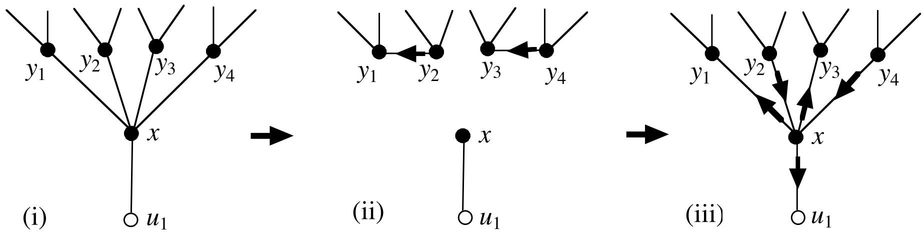

Figure 11:

for each $i \in \{1, 2\}$. By symmetry, we may assume that the center of the star A_i containing the edge $y_{2i-1}y_{2i}$ in $\mathcal{S}_i$ is y_{2i}, $(i = 1, 2)$. Let $A'_i = (A_i - y_{2i-1}) \cup y_{2i}x$ $(i = 1, 2)$ and B be the star $xy_1 \cup xy_3 \cup xu_1$. Then $(\mathcal{S}_1 \cup \mathcal{S}_2 \setminus \{A_1, A_2\}) \cup \{A'_1, A'_2, B\}$ is a system that dominates H and its cardinality is

$$\#\mathcal{S}_1 + \#\mathcal{S}_2 + 1 \leq \frac{e(F_1) + e(F_2) + e(F_3) + e(F_4)}{4} + 1 = \frac{e(H) - 1}{4},$$

a contradiction. □

Now, we construct a required system that dominates H. Let (Z, Z') be a bipartition of $V(F) \setminus V_1(F)$ with $|Z| \leq |Z'|$. Let

$$X_1 = \{x \in V_1(F) \mid N_F(x) \cap Z = \emptyset\} \text{ and } X_2 = V_1(F) \setminus X_1.$$

Let

$St(z)$ be the star with the center z and the ends $N_H(z)$ for $z \in Z$

$T_1(x)$ be the star with the center x and the ends $N_H(x)$ for $x \in X_1$

$T_2(x)$ be the star with the center x and the ends $N_H(x) \cap V_1(H)$ for $x \in X_2$,

and let

$$\mathcal{S} = \{St(z) \mid z \in Z\} \cup \{T_1(x) \mid x \in X_1\} \cup \{T_2(x) \mid x \in X_2\}.$$

Since $d_H(x) \geq 3$ for $x \in V(F)$, every star in $\mathcal{S}$ has at least three edges. Obviously $E(H) = \bigcup_{S \in \mathcal{S}} E(S)$ and all the stars in $\mathcal{S}$ are mutually edge-disjoint, and so $\mathcal{S}$ is a system that dominates H and its cardinality is

$$\#\mathcal{S} = |Z| + |X_1| + |X_2| \leq \frac{|F - V_1(F)|}{2} + |V_1(F)|. \tag{5}$$

Claim 2 and $V_2(H) = \emptyset$ imply $V_2(F) = \emptyset$. Therefore

$$|V_1(F)| = \sum_{i\geq 3}(i-2)|V_i(F)| + 2 \geq |F - V_1(F)| + 2,$$

and so

$$e(F) = |F - V_1(F)| + |V_1(F)| - 1 \geq 2|F - V_1(F)| + 1.$$

Since $e(H) - e(F) = 4|V_1(F)|$, the upper bound of (5) is

$$\frac{|F - V_1(F)|}{2} + |V_1(F)| \leq \frac{e(F)-1}{4} + \frac{e(H)-e(F)}{4} = \frac{e(H)-1}{4},$$

a contradiction. □

Proof of Lemma 7. Without loosing generality, we may assume that H is connected. Let X be a maximum even subgraph of H. If $V(H) = V(X)$, then X is a system that dominates H. If $E(X) = E(H)$, then the number $\#X$ of the components in X is $1 < (e(H)-1)/4$. If $E(X) \subsetneq E(H)$, then $\#X \leq e(X)/4 \leq (e(H)-1)/4$ because H is triangle-free. Thus X constitutes a desired system that dominates H.

Suppose $H - V(X)$ is not empty. Let $\{Y_i\}$ be the set of all the components in $H - V(X)$ and S_i be the set of all the edges joining Y_i and X. Let Y_i^* be the graph obtained from $Y_i \cup S_i \cup kK_{1,4}$ by identifying each vertex in $V_1(Y_i \cup S_i) \cap V(S_i)$ and each center of $K_{1,4}$, where $k = |S_i|$, as in Figure 12. Then $\delta_e(Y_i^*) \geq 4$ and

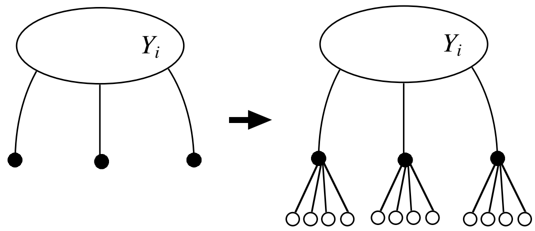

Figure 12:

$V_2(Y_i^*) = \emptyset$. Hence, by Lemma 9, there exists a system $\mathcal{S}_i^*$ of cardinality at most $(e(Y_i) + 5|S_i| - 1)/4$ that dominates Y_i^*. Let $\mathcal{T}_i$ be the set of all the stars in $\mathcal{S}_i^*$ with centers in $V_1(Y_i \cup S_i) \cap V(S_i)$. Then $\#\mathcal{T}_i = |S_i|$. Since the set of the stars $\mathcal{S}_i = \mathcal{S}_i^* \setminus \mathcal{T}_i$ contains all edges in Y_i and every edge in $\bigcup_i S_i$ is incident to X,

$$\mathcal{S} = \{ \text{ all circuits in } X \ \} \cup \bigcup_i \mathcal{S}_i$$

is a system that dominates H. As H is triangle-free, $\#X \leq e(X)/4$ and so

$$\begin{aligned} \#\mathcal{S} &= \frac{e(X)}{4} + \sum_i (\#\mathcal{S}_i^* - \#\mathcal{T}_i) = \frac{e(X)}{4} + \sum_i (\frac{e(Y_i) + 5|S_i| - 1}{4} - |S_i|) \\ &= \frac{e(X)}{4} + \sum_i \frac{e(Y_i) + |S_i| - 1}{4} = \frac{e(X) + \sum_i (e(Y_i) + |S_i|) - i}{4} \leq \frac{e(H) - i}{4}. \end{aligned}$$

Hence, $\mathcal{S}$ is a desired system that dominates H. □

4.2 Proof of Lemma 8

We use the following lemma.

Lemma D (Fleischner [5]). *Every bridgeless multigraph with $\delta \geq 3$ has a spanning even subgraph.*

If $V_1(H) = \emptyset$, then H has no bridge, and so the graph H' obtained from H by suppressing all vertices of degree two, i.e., remove a vertex of degree two and join the neighbours by an edge, is a bridgeless multigraph with $\delta(H') \geq 3$. Hence, by Lemma D, H' has a spanning even subgraph X'. Because $V_2(H)$ is a stable set in H, the even subgraph X in H corresponding to X' is a system that dominates H such that $V_{\geq 3}(H) \subset V(X)$.

Suppose $V_1(H) \neq \emptyset$, and let $F = H - V_1(H)$ and $Pr(H) = N(V_1(H))$. Let F' be the graph obtained from F by suppressing all vertices in $V_2(F)$. Then by Lemma D, F' has a spanning even subgraph X'. Let X be the even subgraph in H corresponding to X' and let Q be the forest obtained from $F - E(X)$ by removing all isolated vertices. Notice that each component in Q is a path as $V_{\geq 3}(F) \subset V(X)$. Because $V_2(H)$ is a stable set, easily we can assign direction to every edge in Q such that the initial vertex is a vertex in $Pr(H)$ and for each vertex $x \in Pr(H)$, there is a directed edge with initial vertex x.

For $x \in Pr(H) \cap V_2(F)$, let $St(x)$ be the star with center x and all pendant edges incident to x and all directed edges with initial vertex x. Since there are at least two pendant edges incident to $x \in Pr(H) \cap V_2(F)$, $St(x)$ has at least three edges. Thus, $\{St(x) \mid x \in Pr(H)\}$ and X constitutes a system that dominate H such that the even subgraph X passes through all vertices in $V_{\geq 3}(F)$. □

References

[1] S.A. Choudum and M.S. Paulraj, *Regular factors in $K_{1,3}$-free graphs*, J. Graph Theory **15** (1991) 259-265.

[2] R. Diestel, *Graph Theory, Second edition*, Graduate Texts in Mathematics **173**, Springer (2000)

[3] Y. Egawa and K. Ota, *Regular factors in $K_{1,n}$-free graphs*, J. Graph Theory **15** (1991) 337-344.

[4] R.J. Faudree, O. Favaron, E. Flandrin, H. Li, Z. Liu, *On 2-factors in claw-free graphs*, Discrete Math. **206** (1999) 131-137.

[5] H. Fleischner, *Spanning Eulerian subgraphs, the splitting lemma, and Petersen's theorem*, Disc. Math. **101** (1992) 33-37.

[6] B. Jackson and K. Yoshimoto, *Even subgraphs of bridgeless graphs and 2-factors of line graphs*, submitted

[7] B. Jackson and K. Yoshimoto, *Spanning Even Subgraphs of 3-edge-connected Graphs*, submitted

[8] R. Gould and E. Hynds, *A note on cycles in 2-factors of line graphs*, Bull. of ICA. 26 (1999), 46-48.

[9] R.J. Gould and M.S. Jacobson, *Two-factors with few cycles in claw-free graphs*, Discrete Math. **231** (2001), 191-197.

[10] M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, J. Graph Theory **8** (1984), 139-146.

[11] Z. Ryjáček, *On a closure concept in claw-free graphs*, J Combin Theory Ser B **70** (1997) 217-224.

[12] Z. Ryjáček, A. Saito and R.H. Schelp, *Closure, 2-factor, and cycle coverings in claw-free graphs*, J. Graph Theory **32** (1999) 109-117.