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The upper bound of the number of cycles in a 2-factor of a line graph

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Abstract

Let G be a simple graph with order n and minimum degree at least two. In this paper, we prove that if every odd branch-bond in G has an edge-branch, then its line graph has a 2-factor with at most $\frac{3n-2}{8}$ components. For a simple graph with minimum degree at least three also, the same conclusion holds.

1 Introduction

We consider only simple graphs G and the order is denoted by n and the minimum degree by δ throughout this article. The length of a path is defined by the number of edges on the path, and the $K_{1,m}$ is called a *star*. *A circuit* is a connected graph with at least three vertices in which every vertex has even degree.

There are various results about the number of the components in a 2-factor which is a 2-regular spanning subgraph, see [1],[2],[7],[10],[12]. In this article, we study the upper bound of the number of cycles in 2-factors in a line graph. By results of Egawa and Ota [6] and Choudum and Paulraj [4], the line graph of a graph with $\delta \geq 3$ has a 2-factor. In general, if there is a family $\mathcal{S}$ of edge-disjoint circuits and stars with at least three edges in a graph G such that:

$$\textit{every edge in } E(G) \setminus \bigcup_{S \in \mathcal{S}} E(S) \textit{ is incident to a circuit in } \mathcal{S},$$

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then obviously the line graph $L(G)$ has a 2-factor in which every component is induced by an element in $\mathcal{S}$ or a circuit in $\mathcal{S}$ together with some edges in $E(G) \setminus \cup_{S \in \mathcal{S}} E(S)$. Gould and Hynds [9] showed the above condition is a necessary and sufficient one for the existence of a 2-factor with $|\mathcal{S}|$ components in $L(G)$. Let us call the family $\mathcal{S}$ a *k-system that dominates G* (or simply *k-system*), where $k = |\mathcal{S}|$.

A *branch* in a graph G is a nontrivial path such that all of the internal vertices have degree two and neither of the ends have degree two. Especially, a branch of length one is called an *edge-branch*. A set $\mathcal{B}$ of branches is called a *branch cut* if the graph obtained from $G \setminus \bigcup_{B \in \mathcal{B}} E(B)$ by deleting all the internal vertices in the branches has more components than G. A *branch-bond* is a minimal branch cut. Some results about hamiltonicity of $L(G)$ and branches or branch-bonds have been known, see [3],[13],[14],[15].

A branch-bond is called *odd* if it consists of an odd number of branches. If the maximum number $l(G)$ of the lengths of shortest branches in all odd branch-bonds in G is at least three, then obviously G has no k-system for any k. In the case of $l(G) = 2$, also there exist many graphs without a k-system. For example, the line graph of the 2-connected graph G in Figure 1 has no 2-factor, while $l(G) \leq 2$ since

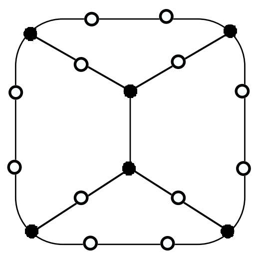

Figure 1:

the subgraph obtained by removing the internal vertices in all branches of length three is connected. However, if $l(G) = 1$, i.e., all odd branch-bonds have an edge-branch, and $\delta \geq 2$, then its line graph contains a 2-factor. We show the following fact in this paper.

Theorem 1. *Let G be a simple graph of order $n \geq 4$ and minimum degree $\delta \geq 2$. If every odd branch-bond in G has an edge-branch, then its line graph has a 2-factor with at most $\frac{3n-2}{8}$ cycles.*

If a graph has minimum degree at least three, then all branches are edges, and so the same conclusion holds.

The upper bound in Theorem 1 is best possible as follows. Let P_{2m} be a path of length $2m-1$. We add $2m+2$ edges to $P_{2m} \cup (2m+2)K_3$ as in Figure 2. Then

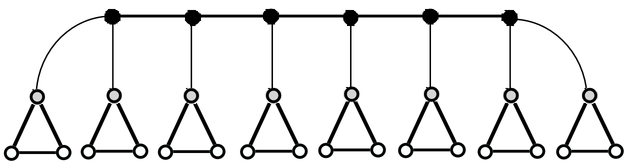

Figure 2:

the resultant graph $H_{2m,3}$ has order $8m+6$, and so $(3|V(H_{2m,3})|-2)/8 = 3m+2$. Because the edges on P_{2m} are covered only by exactly m stars and each cycle K_3 is covered only by itself, $H_{2m,3}$ does not have a k-system for $k < m + (2m+2)$. Moreover, the graph obtained by removing all the triangles which are adjacent to the ends of P_{2m} has no k-system for any k. Hence we can not relax the minimum degree condition also.

In general, the following conjecture seems to hold.

Conjecture 2. *If G is a simple graph with order n and minimum degree δ (≥ 3), then its line graph has a 2-factor with at most $\frac{(2\delta-3)n}{2(\delta^2-\delta-1)}$ ($< \frac{n}{\delta}$) cycles.*

If this conjecture is true, then the upper bound of the number of cycles is almost best possible by the graph obtained from $H_{2m,3}$ by replacing each K_3 adjacent to internal vertices of P_{2m} by $(\delta-2)K_{\delta+1}$ and by replacing each $2K_3$ adjacent to the ends by $(\delta-1)K_{\delta+1}$. See Figure 3.

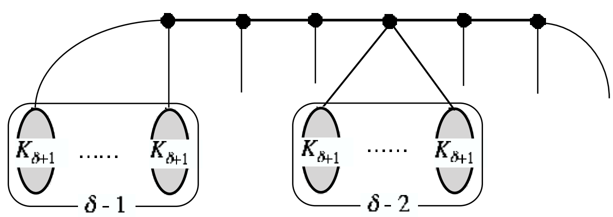

Figure 3:

Notice that in [10], it was shown that: if a claw-free graph with order n' and minimum degree δ' has an integer k such that $n'/\delta' \le k \le \sqrt[3]{n'/16}$, then the graph has a 2-factor with at most k cycles. However, this fact implies neither of Theorem 1 nor Conjecture 2 because if a graph G has an edge whose ends have degree δ, then its line graph has no integer k satisfying the condition of the statement. Actually, $n' = |E(G)| \ge \delta|V(G)|/2$ and $\delta' = 2(\delta - 1)$ implies $\delta > |V(G)|^2/2$.

Finally we give some additional definitions and notation. The set of all the neighbours of a vertex $x \in V(G)$ is denoted by $N_G(x)$ or simply $N(x)$, and its cardinality by $d_G(x)$ or $d(x)$. For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. For simplicity, we denote $|V(H)|$ by $|H|$ and "$u_i \in V(H)$" by "$u_i \in H$". The set of neighbours $\bigcup_{v \in H} N_G(v) \setminus V(H)$ is written by $N_G(H)$ or $N(H)$, and for a subgraph $F \subset G$, $N_G(H) \cap V(F)$ is denoted by $N_F(H)$. For vertex-disjoint subgraphs H, H', we denote the set of all the edges joining H and H' by $E[H, H']$. For subgraphs $H \subset F$, let $\mathrm{Int}_F H = \{u \in V(H) \mid d_F(u) \neq 1\}$.

We use [5] for notation and terminology not explained here.

2 Proof of Theorem 1

The following lemma implies the existence of a 2-factor in $L(G)$.

Lemma 3. *A graph G has a set of vertex-disjoint circuits containing all vertices of degree two if every odd branch-bond in G has an edge-branch.*

Proof. Let $C_1, C_2, \ldots, C_l$ be vertex-disjoint circuits in G such that $C = \bigcup_i C_i$ contains vertices of degree two as many as possible. Let $F = G - V(C)$, and suppose F contains a vertex x of degree two. Notice that every vertex in G of degree two is contained in a branch or a cycle in which all but one vertex have degree two. It follows from the choice of $C_1, C_2, \ldots, C_l$ that x is contained in a branch, say P. Since $\mathrm{Int}_G(P) \subset V(F)$, $E(P) \subset E(G) \setminus E(C)$. Let T be a maximal tree such that $P \subset T$ and

$$\textit{if there is an edge in } T \cap C\textit{, then neither of the ends have degree two.} \qquad (1)$$

If we remove all the internal vertices of P from T, then two trees T_1 and T_2 are remained. Let $\mathcal{B}$ be a branch-bond joining T_1 and $G - V(T_1) \cup \mathrm{Int}_G(P)$ in which P is one of branches.

We choose a branch B in $\mathcal{B}$ as follows. If $\mathcal{B}\backslash P$ has a branch which is edge-disjoint to C, then let B be the branch. In the case that $\mathcal{B} \setminus P$ has no such a branch, $\mathcal{B}$ is an odd branch-bond, and so $\mathcal{B}$ has an edge-branch. We choose the edge-branch as B. Notice that if $E(B) \cap E(C) \neq \emptyset$, then B is an edge-branch and neither of the ends have degree two by the definition of a branch. In either case, as the maximality of T, B is joining T_1 and T_2, and so $T \cup B$ contains a cycle D. Then

$$C' = (C \cup D) \setminus E(C \cap D) - \mathrm{Int}_{C \cap D}(C \cap D)$$

is a set of circuits. Because $P \subset C'$ and $\mathrm{Int}_{C \cap D}(C \cap D)$ does not contain a vertex of degree two by (1), the set C' of the circuits contains more vertices of degree two than C, a contradiction. □

Proof of Theorem 1

By Lemma 3, we can choose vertex-disjoint circuits $C_1, C_2, \dots, C_\alpha$ in G such that:

1. $C = \bigcup_{i \leq \alpha} C_i$ contains all the vertices of degree two;
2. Subject to 1, $|V(C)|$ is maximal;
3. Subject to the above, α is as small as possible.

Then $F = G - V(C)$ is a forest. Let $F_1, F_2, \dots, F_\beta$ be the components of F. As F is a bipartite graph, there are partite sets X and Y of $V(F)$. Suppose $|X| \leq |Y|$, and for each $x \in X$, let $S(x)$ be the star $\{xu_i \mid u_i \in N_G(x)\}$. Since $d_G(v) \geq 3$ for every $v \in V(F)$, $S(x)$ has at least three ends for all $x \in X$. As F is a forest, every edge in G is contained in C or $\bigcup_{x \in X} S(x)$ or incident to C. Therefore

$$\mathcal{S} = \{C_1, C_2, \dots, C_\alpha\} \cup \{S(x) \mid x \in X\}$$

is an $(\alpha + |X|)$-system that dominates G. We prove the number $\alpha + |X|$ is at most $(3n - 2)/8$.

First suppose that $|F| \leq (n - 6)/4$, then

$$\alpha + |X| \leq \alpha + \frac{|F|}{2} \leq \frac{n - |F|}{3} + \frac{|F|}{2} = \frac{2n + |F|}{6} \leq \frac{3n - 2}{8}. \tag{2}$$

Next suppose that $F = \emptyset$. Then (2) holds for $n \geq 6$. In case of $n = 4$ or 5, since we cannot take two vertex disjoint circuits in G, $\alpha = 1$. Therefore $\alpha + |X| < (3n - 2)/8$ holds.

Hence we may assume that $F \neq \emptyset$ and

$$|F| > \frac{n-6}{4}. \tag{3}$$

Claim 1. $|E[e, F_k]| \leq 1$ *for any edge* $e \in E(C)$ *and any* $k \leq \beta$.

Proof. Suppose there is an edge $e \in E(C_i)$ such that $|E[e, F_k]| \geq 2$. Let $uv, u'v' \in E[e, F_k]$ be different edges, where $u, u' \in V(e)$, and $P_{v,v'}$ be the path in F_k joining v and v'. If $u = u'$, then $v \neq v'$ as G is simple. Hence $C \cup \{uv, uv'\} \cup P_{v,v'}$ is the set of circuits containing $V(C)$ and $V(P_{v,v'})$. This contradicts the requirement 2 of C. See Figure 4i. Similarly if $u \neq u'$, then $C \cup \{uv, u'v'\} \cup P_{v,v'} \setminus \{uu'\}$ is the set of

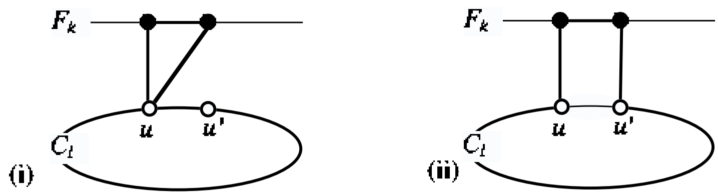

Figure 4:

circuits containing $V(C)$ and $V(P_{v,v'})$. See Figure 4ii. □

Let $C_i = u_1u_2\ldots u_pu_1$. Using Claim 1, we define $D_i \subset C_i$ such that $V(D_i) = V(C_i)$ and $E[Z, F_k] \leq 1$ for any component Z of D_i and any $k \leq \beta$, as follows.

1. If p is even, say $2m$, then let

$$D_i = \{u_{2j-1}u_{2j} \mid 1 \leq j \leq m\}.$$

In Figure 5i, the spanning subgraph determined by heavy edges is D_i.

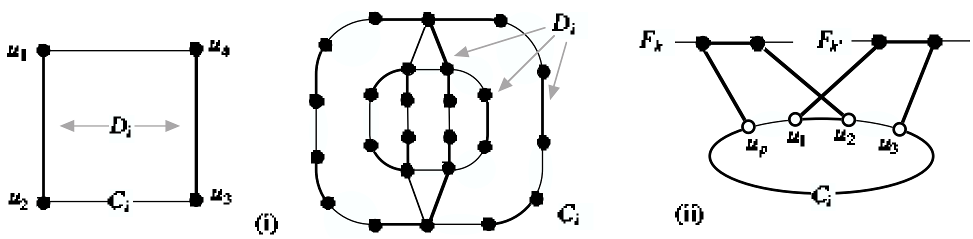

Figure 5:

2. Suppose p is odd, say $2m+1$. Assume C_i is an odd cycle. If $E[C_i, F] = \emptyset$, then let

$$D_i = \{u_p u_1 u_2\} \cup \{u_{2j-1}u_{2j} \mid 2 \leq j \leq m\}. \qquad (4)$$

Suppose $E[C_i, F] \neq \emptyset$. By symmetry, we may assume $N_F(u_1) \neq \emptyset$. If u_p and u_2 are not adjacent to the same tree, then we define D_i by (4).

Assume both of u_p and u_2 have neighbours on the same tree F_k. Now we prove that u_1 and u_3 are not adjacent to the same tree in F. If both of u_1 and u_3 also are adjacent to the same tree $F_{k'}$, then $k \neq k'$ and $u_p \notin N(F_{k'})$ by Claim 1. As $u_3 \in N(F_{k'})$, $u_3 \neq u_p$, and so $G[C_i \cup F_k \cup F_{k'}]$ contains a circuit longer than C_i. See Figure 5ii. Therefore u_1 and u_3 are not adjacent to the same tree. Thus we define

$$D_i = \{u_1 u_2 u_3\} \cup \{u_{2j}u_{2j+1} \mid 2 \leq j \leq m\}.$$

Note that $d_G(u_2) \geq 3$.

Assume C_i is not an odd cycle. Then there is a vertex of which the degree is at least four in C_i. By symmetry, we can suppose u_1 is such a vertex. If both of u_p and u_2 are adjacent to some tree F_k, then $C \cup \{u_p v, u_2 v'\} \cup P_{v,v'} \setminus \{u_p u_1, u_1 u_2\}$ is a set of circuits containing $V(C) \cup V(P_{v,v'})$, where $v \in N_{F_k}(u_p)$, $v' \in N_{F_k}(u_2)$ and $P_{v,v'}$ is the path joining v and v'. This contradicts the requirement 2. Therefore u_p and u_2 are not adjacent to the same tree in F. Let us define D_i by (4).

By the definition of D_i, immediately the following fact holds.

Fact 4. *If $E[C_i, F] \neq \emptyset$, then for any $u_l \in \mathrm{Int}_{D_i}(D_i)$, $d_G(u_l) \geq 3$. Especially if C_i is not an odd cycle, then $d_{C_i}(u_l) = 2s$ for some $s \geq 2$.*

Let r_i be the number of components in D_i and $\{Z_i^1, Z_i^2, \dots, Z_i^{r_i}\}$ the set of all the components in D_i for $i \leq \alpha$. By the definition of D_i, $V(D_i) = V(C_i)$ and

$$r_i \leq \frac{|C_i|}{2} \qquad (5)$$

because each component Z_i^j contains at least two vertices.

Claim 2. $|E[Z_i^j, F_k]| \leq 1$ *for any component Z_i^j in D_i and $k \leq \beta$.*

Proof. Suppose $|E[Z_i^j, F_k]| \geq 2$, and let $u_a, u_b \in N_{Z_i^j}(F_k)$ and Q_{u_a,u_b} a path in Z_i^j joining u_a and u_b. By Claim 1, Q_{u_a,u_b} is not an edge, and so C_i is not a cycle by the definition of D_i. Therefore, for any $u_l \in \mathrm{Int}_{Q_{u_a,u_b}}(Q_{u_a,u_b})(\subset \mathrm{Int}_{D_i}(D_i))$, $d_{C_i}(u_l) = 2m$ for some $m \geq 2$ by Fact 4. Hence, for $v_a \in N_{F_k}(u_a)$ and $v_b \in N_{F_k}(u_b)$ and the path P_{v_a,v_b} in F_k joining v_a and v_b, the subgraph

$$C' = C \cup \{u_a v_a, u_b v_b\} \cup P_{v_a,v_b} \setminus E(Q_{u_a,u_b})$$

is a set of circuits containing $V(C) \cup V(P_{v_a,v_b})$ because for any $u_l \in \mathrm{Int}_{Q_{u_a,u_b}}(Q_{u_a,u_b})$, $d_{C'}(u_l) = d_C(u_l) - 2$ is a positive even number and for any $u_l \in V(C) \setminus \mathrm{Int}_{Q_{u_a,u_b}}(Q_{u_a,u_b})$, $d_{C'}(u_l) = d_C(u_l)$. This contradicts the requirement 2 of C. □

Let $D = \bigcup_{i \leq \alpha} D_i$ and H the graph obtained from $F \cup E[F, C] \cup D$ by contracting all edges in $E(F) \cup E(D)$.

Claim 3. *H is a forest.*

Proof. Let z_i^j and f_k be vertices in H corresponding to Z_i^j and F_k, respectively, and

$$V_Z = \{z_i^j \mid i \leq \alpha \text{ and } j \leq r_i\} \text{ and } V_F = \{f_k \mid k \leq \beta\}.$$

By the definition of H, H is a bipartite graph with partite sets V_Z and V_F and there is an edge $z_i^j f_k \in E(H)$ if and only if $E[Z_i^j, F_k] \neq \emptyset$. By Claim 2, there is no multiple edges in H.

Suppose there is a cycle. By symmetry, we may assume the cycle is

$$f_1 z_{\varphi(1)}^{\psi(1)} f_2 z_{\varphi(2)}^{\psi(2)} \cdots f_r z_{\varphi(r)}^{\psi(r)} f_1.$$

Let

$$e_i^1 = v_i^1 u_{\varphi(i)}^1 \in E[F_i, Z_{\varphi(i)}^{\psi(i)}] \text{ and } e_i^2 = u_{\varphi(i)}^2 v_{i+1}^2 \in E[Z_{\varphi(i)}^{\psi(i)}, F_{i+1}]$$

corresponding to $f_i z_{\varphi(i)}^{\psi(i)}$ and $z_{\varphi(i)}^{\psi(i)} f_{i+1}$, respectively, where $i \leq r$ and $f_{r+1} = f_1$. Let

$$\begin{cases} P_i \text{ be the path joining } v_i^2 \text{ and } v_i^1 & \text{in } F_i \\ Q_{\varphi(i)} \text{ be a path joining } u_{\varphi(i)}^1 \text{ and } u_{\varphi(i)}^2 & \text{in } Z_{\varphi(i)}^{\psi(i)}, \end{cases}$$

where $i \leq r$ and $v_0^2 = v_r^2$. Let

$$\widetilde{C} = \{\bigcup_{i \leq r} (C_{\varphi(i)} \cup \{e_i^1, e_i^2\} \cup P_i)\} \setminus \{\bigcup_{i \leq r} E(Q_{\varphi(i)})\}.$$

As $V(\widetilde{C}) \subset V(\bigcup_{i \leq r} (C_{\varphi(i)} \cup F_i))$,

$$\widetilde{C} \text{ is vertex-disjoint to } C_l \text{ for all } l \neq \varphi(1), \varphi(2), \ldots, \varphi(r).$$

Moreover, it holds that

$$\begin{cases} d_{\widetilde{C}}(v) = 2 & \text{for } v \in V(\bigcup_{i\le r} P_i) \\ d_{\widetilde{C}}(u_l) = d_C(u_l) & \text{for } u_l \in V(\bigcup_{i\le r} C_{\varphi(i)}) \setminus \{\bigcup_{i\le r} \mathrm{Int}_{Q_{\varphi(i)}}(Q_{\varphi(i)})\}. \\ d_{\widetilde{C}}(u_l) = d_C(u_l) - 2 & \text{for } u_l \in \bigcup_{i\le r} \mathrm{Int}_{Q_{\varphi(i)}}(Q_{\varphi(i)}). \end{cases}$$

If there exists $u_l \in \mathrm{Int}_{Q_{\varphi(i)}}(Q_{\varphi(i)})$ such that $d_C(u_l) - 2 = 0$, then, by Fact 4 and the definition of $D_{\varphi(i)}$, the circuit $C_{\varphi(i)}$ is an odd cycle and $Q_{\varphi(i)}$ is the component in $D_{\varphi(i)}$ of length two. As $D_{\varphi(i)}$ has only one such a component,

$$M = \{u_l \in \bigcup_{i\le r} \mathrm{Int}_{Q_{\varphi(i)}}(Q_{\varphi(i)}) \mid d_{\widetilde{C}}(u_l) = d_C(u_l) - 2 = 0\}$$

contains at most r vertices. Because $d_G(u_l) \ge 3$ for all $u_l \in M$ by Fact 4,

$$C' = (\widetilde{C} \setminus M) \cup \bigcup_{i \ne \varphi(1),\varphi(2),\dots,\varphi(r)} C_i$$

is a set of circuits satisfying the requirement 1 of C. Since $\sum_{i\le r} |P_i| \ge r$ and $|M| \le r$,

$$|\widetilde{C} \setminus M| = \sum_{i\le r} (|C_{\varphi(i)}| + |P_i|) - |M| \ge \sum_{i\le r} |C_{\varphi(i)}|,$$

and so

$$|C'| \ge |C|.$$

If $|M| < r$ or $|P_i| \ge 2$ for some $i \le r$, then $|\widetilde{C} \setminus M| > \sum_{i\le r} |C_{\varphi(i)}|$, and so $|C'| > |C|$. This contradicts the requirement 2 of C.

If $|M| = r$ and $|P_i| = 1$ for all $i \le r$, then $|C'| = |C|$ and $C_{\varphi(i)}$ is an odd cycle and $Q_{\varphi(i)}$ is the component in $D_{\varphi(i)}$ of length two for any $i \le r$. As $D_{\varphi(i)}$ has only one such a component,

$$C_{\varphi(i)} \ne C_{\varphi(j)} \text{ if } i \ne j.$$

Hence, the number of the components in $\bigcup_{i\le r} C_{\varphi(i)}$ is r and $\widetilde{C} \setminus M$ is a cycle. See Figure 6. Therefore, the number of the components in C' is $\alpha - r + 1 < \alpha$. This contradicts the requirement 3 of C. □

Next, we calculate $|E[F, C]|$. Let $k \le \beta$ and let $p_l(k) = |\{v \in V(F_k) \mid d_{F_k}(v) = l\}|$. Since F_k is a tree,

$$p_1(k) = \sum_{i\ge 3} (i-2) p_i(k) + 2.$$

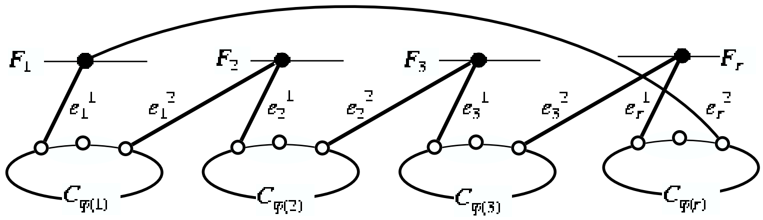

Figure 6:

Because $d_G(v) \geq 3$ for any $v \in V(F_k)$ by the requirement 1 of C,

$$\begin{aligned} |E[F_k, C]| &\geq 2p_1(k) + p_2(k) \\ &= p_1(k) + p_2(k) + \sum_{i\geq 3}(i-2)p_i(k) + 2 \\ &\geq \sum_{i\geq 1} p_i(k) + 2 \\ &= |F_k| + 2. \end{aligned}$$

Hence

$$|E[F, C]| = \sum_{i\leq\beta} |E[F_i, C]| \geq \sum_{i\leq\beta}(|F_i| + 2) = |F| + 2\beta. \tag{6}$$

Because H is a forest with partite sets V_Z and V_F, there is a set R of at most $|V_F| - 1 = \beta - 1$ edges such that $H \setminus R$ is a set of vertex-disjoint stars whose central vertices are contained in V_F. Let $\overline{R}$ be the set of all the edges in $E[F, C]$ corresponding to edges in R and $L = E[F, C] \setminus \overline{R}$. Then

$$|L| \geq |F| + \beta + 1 \geq |F| + 2 \text{ and} \tag{7}$$

$$|E[Z_i^j, F] \cap L| \leq 1 \text{ for all } Z_i^j. \tag{8}$$

Let

$$\gamma_j = |\{C_i \mid |E[C_i, F] \cap L| = j\}|.$$

Then

$$\sum_{j\geq 0} \gamma_j = \alpha \quad \text{and} \quad \sum_{j\geq 0} j\gamma_j = \sum_{j\geq 1} j\gamma_j = |L|. \tag{9}$$

If there are j edges incident to C_i in L, then $r_i \geq j$ by (8), and so

$$|C_i| \geq 2r_i \geq 2j$$

by (5). Because any circuit has at least three vertices, (9) implies

$$\begin{aligned} n - |F| &= |C| \\ &\geq 3\gamma_0 + 3\gamma_1 + 2\sum_{j\geq 2} j\gamma_j \\ &= 3\gamma_0 + \gamma_1 + 2\sum_{j\geq 1} j\gamma_j \\ &= 3\gamma_0 + \gamma_1 + 2|L|. \end{aligned} \tag{10}$$

And also by (9),

$$\begin{aligned} |L| &= \sum_{j\geq 1} j\gamma_j \\ &= \sum_{j\geq 2} j\gamma_j + \gamma_1 \\ &\geq 2\sum_{j\geq 2} \gamma_j + \gamma_1 \\ &= 2\sum_{j\geq 0} \gamma_j - 2\gamma_0 - \gamma_1 \\ &= 2\alpha - 2\gamma_0 - \gamma_1. \end{aligned} \tag{11}$$

Taking sum of (3), (7), (10) and (11), we obtain

$$\begin{aligned} & |F| + |L| + n - |F| + |L| > \frac{n-6}{4} + |F| + 2 + 3\gamma_0 + \gamma_1 + 2|L| + 2\alpha - 2\gamma_0 - \gamma_1 \\ \Longrightarrow \quad & n > \frac{n-6}{4} + |F| + 2 + \gamma_0 + 2\alpha. \end{aligned}$$

Therefore,

$$\begin{aligned} 2\alpha + |F| &< n - \frac{n-6}{4} - 2 - \gamma_0 \\ &\leq \frac{3n-2}{4} - \gamma_0 \\ . \quad &\leq \frac{3n-2}{4}, \end{aligned}$$

which implies

$$\alpha + |X| \leq \alpha + \frac{|F|}{2} < \frac{3n-2}{8}.$$

Now the proof is completed.

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