$\{4, 5\}$ is not coverable – a counterexample to a conjecture of Kaiser and Škrekovski

Roman Čada[1*] Shuya Chiba[2†] Kenta Ozeki[3‡] Petr Vrána[1§]
Kiyoshi Yoshimoto[4¶]

[1]*Department of Mathematics University of West Bohemia*
and Institute of Theoretical Computer Science (ITI)
Charles University Univerzitní 8 306 14 Plzeň Czech Republic.
[2]*Department of Mathematical Information Science, Tokyo University of Science,*
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan.
[3]*National Institute of Informatics,*
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan.
[4]*Department of Mathematics, Collage of Science and Technology,*
Nihon University, Tokyo 101-8308, Japan.

Abstract

For a subset A of the set of positive integers, a graph G is called A-*coverable* if G has a cycle (a subgraph in which all vertices have even degree) which intersects all edge-cuts T in G with $|T| \in A$, and A is said to be *coverable* if all graphs are A-coverable. As a possible approach to the Dominating cycle conjecture, Kaiser and Škrekovski conjectured in [Cycles intersecting edge-cuts of prescribed sizes, SIAM J. Discrete Math. **22** (2008) 861–874], that $\mathbb{N}+3$ is coverable, where $\mathbb{N}+3 = \{4, 5, 6, \cdots\}$. In this paper, we disprove Kaiser and Škrekovski's conjecture by showing that there exist infinitely many graphs which are not $\{4, 5\}$-coverable.

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*Email address: cadar@kma.zcu.cz
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†Email address: chiba@rs.kagu.tus.ac.jp
This work was supported by Grant-in-Aid for Young Scientists (B) (23740087).

‡Email address: ozeki@nii.ac.jp
Research Fellow of the Japan Society for the Promotion of Science

§Email address: vranap@kma.zcu.cz
This work was supported by the Center of Excellence – Inst. for Theor. Comp. Sci., Prague (project P202/12/G061 of GA ČR).

¶Email address: yosimoto@math.cst.nihon-u.ac.jp
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1 Introduction

In [6], Kaiser and Škrekovski studied the existence of a cycle intersecting all edge-cuts of prescribed size in a graph. In the present paper, the terminology follows the ones in [6]. A *cycle* in the present paper is a graph (not necessarily connected) in which all vertices have even degree. Possibly a cycle might have a vertex of degree zero. An *edge-cut*, in short, a *cut* in a graph G is an inclusionwise minimal set of edges whose removal increases the number of components of G.

Let $\mathbb{N}$ be the set of positive integers and $A \subseteq \mathbb{N}$. We say that a cycle in a graph G is A-*covering* if it intersects all cuts T in G with $|T| \in A$. A graph G is A-*coverable* if G has an A-covering cycle, and A is *coverable* if all graphs are A-coverable. Note that for $A' \subseteq A \subseteq \mathbb{N}$, A' is coverable if A is coverable, but generally the converse does not hold. The following is shown in [6].

Theorem 1 (Kaiser and Škrekovski [6]) $\{3, 4\}$ *is coverable.*

As mentioned in [6], the concept of "coverability" concerns several fundamental topics in graph theory, for example, the 4-Color Theorem, the 4-Flow conjecture by Tutte [12] and the Dominating cycle conjecture by Fleischner [3]. A subgraph C in G is said to be *dominating* if each edge of G is incident with a vertex in C. The Dominating cycle conjecture states the following.

Conjecture 2 (the Dominating cycle conjecture) *Every cyclically* 4*-edge-connected cubic graph has a dominating circuit.*

See the next section for the definition of the terms "cyclically 4-edge-connected" and "circuit". Note that many statements have been shown to be equivalent to Conjecture 2, see [1, 4, 9] and also a survey [2]. For example, Fleischner and Jackson [4] showed that Conjecture 2 is equivalent to the conjecture by Thomassen [11] stating that every 4-connected line graph is Hamiltonian. The conjecture by Matthews and Sumner [7] stating that every 4-connected claw-free graph is Hamiltonian is also known to be equivalent to Conjecture 2, see [9].

As a possible approach to Conjecture 2, in [6], the following conjecture was posed. Let $\mathbb{N} + 3 = \{4, 5, 6, \cdots\}$.

Conjecture 3 (Kaiser and Škrekovski [6]) *The set* $\mathbb{N} + 3$ *is coverable.*

As mentioned in [6], if Conjecture 3 is true, then together with many equivalent conjectures, Conjecture 2 is also true. A result of Thomassen [10] implies that Conjecture 3 is true for the class of planar graphs. However, there is a counterexample to Conjecture 3 and the main purpose of this paper is to construct it. Indeed, we show the following.

Theorem 4 *There exist infinitely many cubic graphs which are not* $\{4, 5\}$*-coverable.*

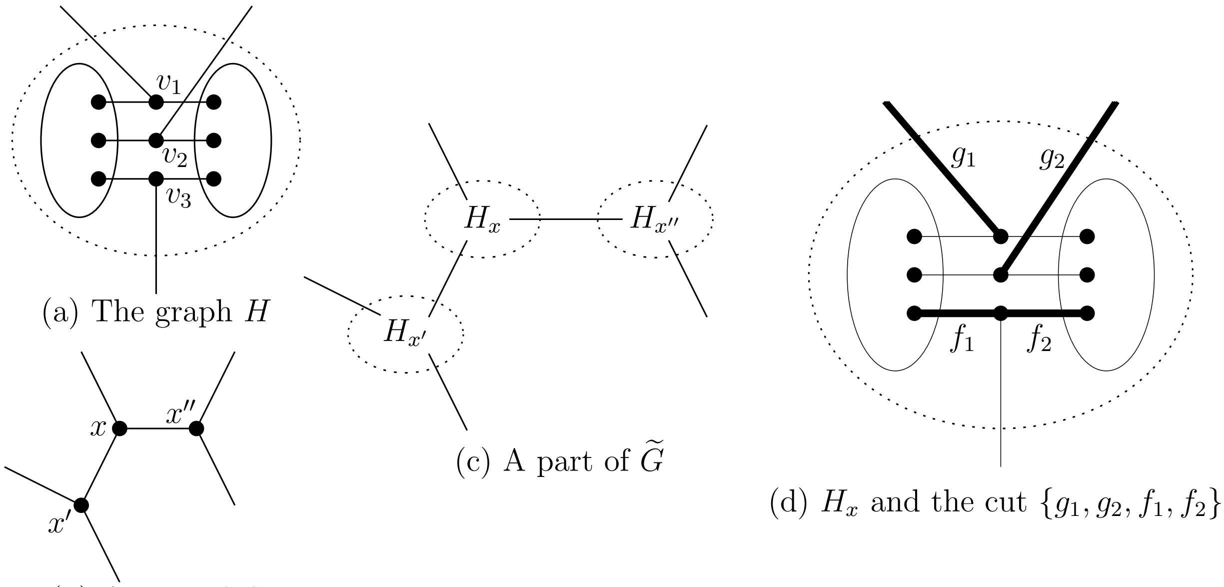

Figure 1: The graphs used in the proof of Theorem 4.

2 Proof of Theorem 4

Here we define some terminology needed for the proof of Theorem 4. A graph G is called *cyclically k-edge-connected* if it contains no cut of size at most $k-1$ such that after removing all edges in it from G, both of the two resulting components contain a cycle. A *2-factor* of a graph G is a spanning subgraph of G in which every vertex has degree exactly two. A connected cycle is said to be a *circuit*.

We use the following lemma, which was shown by Jackson and Yoshimoto [5]. Note that they only mentioned in Section 4 in [5] that the graphs they constructed are cyclically 4-edge-connected graphs in which every 2-factor has a circuit of order at most five. Although they did not explicitly mention other properties in Lemma 5, it is clear from their construction that we can find infinitely many graphs with the desired properties. Notice also that Lukot'ka, Máčajová, Mazák and Škoviera [8] independently showed Lemma 5.

Lemma 5 *There exist infinitely many cyclically 4-edge-connected cubic graphs in which every 2-factor has a circuit of order exactly five.*

Proof of Theorem 4.

Let H' be a 3-edge-connected graph having a cut $\{e_1, e_2, e_3\}$ of size exactly three. Subdivide all of the three edges e_1, e_2 and e_3, and let v_1, v_2 and v_3 be the vertices obtained by the subdivision of e_1, e_2 and e_3, respectively. Adding an edge to each of v_1, v_2 and v_3, we obtain the graph H. (We do not specify the other end vertex of the edges yet.) See Figure 1 (a).

By Lemma 5, there exist infinitely many cyclically 4-edge-connected cubic graphs G in which every 2-factor has a circuit of order exactly five. We replace each vertex x of G with a copy of H, say H_x, and regard the three edges incident to x in G as the three (half) edges in H_x incident to v_1, v_2 and v_3, respectively. Let $\widetilde{G}$ be the obtained graph. See Figures 1 (b) and (c). In order to complete the proof of Theorem 4, it suffices to show that $\widetilde{G}$ is not $\{4,5\}$-coverable.

Suppose, for a contradiction, that $\widetilde{G}$ is $\{4,5\}$-coverable. Then $\widetilde{G}$ has a $\{4,5\}$-covering cycle $\widetilde{C}$, that is, $\widetilde{C}$ intersects all cuts T in $\widetilde{G}$ with $|T| \in \{4,5\}$. Let C be the cycle obtained from $\widetilde{C}$ by contracting each subgraph H_x of $\widetilde{G}$ to the vertex x. Now we regard C as a spanning subgraph of G. Since G is cubic, every vertex of G has degree zero or two in C.

Suppose that there exists no vertex of degree zero in C. Then all vertices of G have degree two in C, and hence C is a 2-factor of G. By the choice of G, C has a circuit D of order exactly five. Since G is cyclically 4-edge-connected, D has no chords, and hence there are exactly five edges leaving from D (recall that G is cubic). Let T be the set consisting of such five edges. Then T is a cut in G which does not intersect C. Furthermore, T is also a cut in $\widetilde{G}$ of size exactly five which does not intersect $\widetilde{C}$, contradicting the assumption that $\widetilde{C}$ is a $\{4,5\}$-covering cycle of $\widetilde{G}$.

Hence there exists a vertex, say x, of degree zero in C. Now we restrict $\widetilde{C}$ to H_x. Since x has degree zero in C, $C_x = \widetilde{C} \cap H_x$ is also a cycle of H_x. On the other hand, since $\{e_1, e_2, e_3\}$ is a cut of H' of odd size, at least one of the three vertices v_1, v_2 and v_3 has degree zero in C_x. By symmetry, we may assume that v_3 has degree zero in C_x. Then letting g_1 (resp. g_2) be the edge in $\widetilde{G}$ connecting v_1 (resp. v_2) and the outside of H_x, and f_1 and f_2 be the two edges in H_x obtained by the subdivision of e_3 in H', $\{g_1, g_2, f_1, f_2\}$ is a cut of $\widetilde{G}$ of size four which does not intersect $\widetilde{C}$, a contradiction. See Figure 1 (d). This completes the proof of Theorem 4. $\square$

3 Concluding remarks

In this paper, we have disproved Conjecture 3, showing that there exist infinitely many cubic graphs which are not $\{4,5\}$-coverable, therefore not $(\mathbb{N}+3)$-coverable. Since our construction depends on cyclic 3-cuts in a crucial way, we here pose two new conjectures.

Conjecture 6 *Every cyclically 4-edge-connected cubic graph is $(\mathbb{N}+3)$-coverable.*

Conjecture 7 *Every cyclically 4-edge-connected cubic graph is $\{4,5\}$-coverable.*

As mentioned in [6], for a cyclically 4-edge-connected cubic graph G, a circuit is dominating in G if and only if it is $(\mathbb{N}+3)$-covering. This implies that Conjecture 6 is equivalent to Conjecture 2. On the other hand, Conjecture 6 easily implies Conjecture 7, and hence attacking Conjecture 7 before Conjectures 2 and 6 might be helpful when we try to resolve them.

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