October 13, 2010

Claw-Free Graphs and 2-Factors that Separate Independent Vertices

Ralph J. Faudree[1], Colton Magnant[2],
Kenta Ozeki[3][4], Kiyoshi Yoshimoto[5][6]

The authors would like to dedicate this paper to our friend and mathematical colleague Richard H. Schelp

Abstract

In this article, we prove that a line graph with minimum degree $\delta \geq 7$ has a spanning subgraph in which every component is a clique of order at least three. This implies that if G is a line graph with $\delta \geq 7$, then for any independent set S there is a 2-factor of G such that each cycle contains at most one vertex of S. This supports the conjecture that $\delta \geq 5$ is sufficient to imply the existence of such a 2-factor in the larger class of claw-free graphs.

It is also shown that if G is a claw-free graph of order n and independence number α with $\delta \geq 2n/\alpha - 2$ and $n \geq 3\alpha^3/2$, then for any maximum independent set S, G has a 2-factor with α cycles such that each cycle contains one vertex of S. This is in support of a conjecture that $\delta \geq n/\alpha \geq 5$ is sufficient to imply the existence of a 2-factor with α cycles, each containing one vertex of a maximum independent set.

1 Introduction

In this paper, we consider finite graphs. If no ambiguity can arise, we denote simply the order $|G|$ of G by n, the minimum degree $\delta(G)$ by δ and the independence number $\alpha(G)$ by α. All notation and terminology not explained in this paper is given in [2].

A *2-factor* of a graph G is a spanning 2-regular subgraph of G. It is a well known conjecture that every 4-connected claw-free graph is hamiltonian ([14]). Since a

[1]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[2]Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA
[3]National Institute of Informatics, Tokyo 101-8430, Japan
[4]Research Fellow of the Japan Society for the Promotion of Science
[5]Department of Mathematics, Nihon University, Tokyo 101-8308, Japan
[6]Research supported by JSPS. KAKENHI (14740087)

hamilton cycle is a connected 2-factor, there are many results on 2-factors of claw-free graphs. For instance, a sufficient condition for the existence of 2-factors was given by Choudum and Paulraj [4] and by Egawa and Ota [6], (i.e., it holds that every claw-free graph with $\delta \geq 4$ has a 2-factor) and Ryjáček, Saito and Schelp [17] proved that a claw-free graph G has a 2-factor with at most k cycles if and only if $cl(G)$ has a 2-factor with at most k cycles, where $cl(G)$ is the Ryjáček closure [15] of G. In this paper, we study the existence of a maximum independent set and a 2-factor of a claw-free graph G which together dominate G in some sense.

First, we begin with the following question.

Question A. *What is the lower bound of minimum degrees such that for any independent set S, there exists a 2-factor in which each cycle contains at most one vertex of S?*

For this question, we will show the following result in Section 2.

Theorem 1. *A line graph with $\delta \geq 7$ has a spanning subgraph in which every component is a clique of order at least three.*

This implies the following immediately.

Theorem 2. *If G is a line graph with $\delta \geq 7$, then for any independent set S, G has a 2-factor such that each cycle contains at most one vertex in S.*

Ryjáček [16] pointed out that the minimum degree condition in Theorem 1 is best possible by showing that the line graph of $K_7 - E(C_7)$ has no desired 2-factors, where C_7 is a hamilton cycle of the complete graph K_7 of order seven. For Theorem 2, we can construct a line graph with $\delta = 4$ of a multigraph and choose a maximum independent set S such that the graph has no 2-factor in which every cycle contains at most one vertex of S. The example will be explained at the end of this section. Hence we propose the following conjecture.

Conjecture 1. *If G is a claw-free graph with $\delta \geq 5$, then for any independent set S, there exists a 2-factor such that each cycle contains at most one vertex in S.*

Next we consider the existence of a 2-factor and a maximum independent set S such that every cycle of the 2-factor contains exactly one vertex of S, i.e., the following is our question.

Question B. *Which degree conditions guarantee the existence of a maximum independent set S and a 2-factor with α cycles such that each cycle contains one vertex of S?*

For this question, we have to look for a 2-factor with α cycles. For the number of cycles, we know the following result.

Theorem 3 (Broersma, Paulusma and Yoshimoto [3]). *A claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\max\left\{\frac{n-3}{\delta-1}, 1\right\}$ cycles.*

In this result, if $\alpha \geq \frac{n-3}{\delta-1}$, then we can replace the upper bound by α. Hence the following corollary holds immediately.

Corollary 4. *A claw-free graph with $\delta \geq \frac{n-3}{\alpha} + 1$ has a 2-factor with at most α cycles.*

On the other hand, the fourth author of this paper constructed an infinite family of line graphs in which every 2-factor contains more than n/δ cycles in [18]. By considering these, we can obtain the following fact, which will be shown in Section 4.

Fact 5. *For any positive integer d with $\frac{n}{\alpha} - \frac{1}{2d} < d < \frac{n}{\alpha}$, there exists an infinite family of claw-free graphs with minimum degree d such that every 2-factor contains more than α cycles.*

Furthermore, Ryjáček [16] constructed claw-free graphs with $3 \leq \delta \leq 4$ and $\delta > n/\alpha$ in which any 2-factor contains fewer than α cycles. Therefore, we propose the following conjecture.

Conjecture 2. *A claw-free graph with $\delta \geq \frac{n}{\alpha} \geq 5$ has a 2-factor with α cycles.*

Possibly a stronger statement might hold.

Conjecture 3. *If G is a claw-free graph with $\delta \geq \dfrac{n}{\alpha} \geq 5$, then there exist a maximum independent set S and a 2-factor with α cycles such that each cycle contains a vertex of S.*

In Section 3, we show the following result.

Theorem 6. *If G is a claw-free graph with $\delta \geq \dfrac{2n}{\alpha} - 2$ and $n \geq \dfrac{3\alpha^3}{2}$, then for any maximum independent set S, G has a 2-factor with α cycles such that each cycle contains one vertex in S.*

Notice that it is well known that the minimum degree of a claw-free graph is at most $2n/\alpha - 2$ (for instance, see Fact 8 in Section 3), and so the minimum degree condition of the above theorem is maximal. However, the conclusion is stronger than Conjecture 3 because we show the existence of a desired 2-factor for any maximum independent set. Accordingly, the following question is proposed.

Question C. *What is the lower bound of minimum degrees such that for any maximum independent set S, there exists a 2-factor with α cycles in which each cycle contains one vertex of S?*

For this third question, we can construct the following examples. Let R_i be the complete graph of order r_i where

$$r_i \geq \begin{cases} \lceil (p-1)/2 \rceil & \text{if } i \text{ is odd} \\ \lfloor (p-1)/2 \rfloor & \text{if } i \text{ is even} \end{cases} \tag{1}$$

for $1 \leq i \leq \alpha$ and for some integer p. Let R be the graph obtained from $\bigcup_{i=1}^{\alpha} R_i$ by joining all pairs of R_i and R_{i+1} for all $1 \leq i \leq \alpha$ (mod α), (i.e., the resultant graph R is like a torus). Let $R_0 \simeq K_\alpha$ and $t_1, t_2, \ldots, t_\alpha$ be the vertices, and $S = \{s_1, s_2, \ldots, s_\alpha\}$. The example R^* is constructed from $R_0 \cup S \cup R$ by joining s_i and all vertices in $\{t_i\} \cup V(R_i) \cup V(R_{i+1})$ for all i (mod α). See Figure 1. Notice that R^* is a line graph of a multigraph.

If the equality holds for all i in (1), then the resultant graph is denoted by $R^*(\alpha, p)$. Obviously any cycle passing through a vertex in R_0 either contains no vertex in S or at least two vertices in S. Furthermore:

$$\delta(R^*) = \min\{r_1 + r_2 + 1, r_2 + r_3 + 1, \ldots, r_\alpha + r_1 + 1, \alpha\} \geq \min\{p, \alpha\}$$

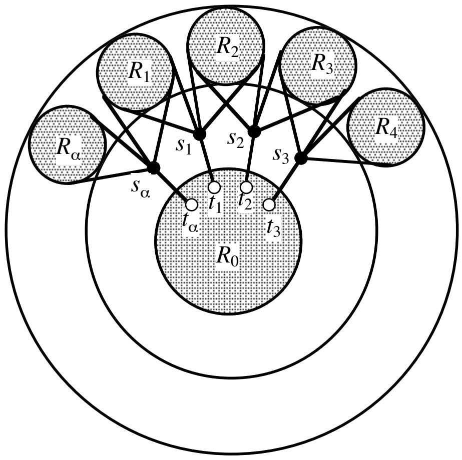

Figure 1:

and the order is

$$|R^*| = \sum_{i=1}^{\alpha} r_i + 2\alpha \geq \frac{(p-1)\alpha}{2} + 2\alpha$$

if α is even. Especially, if $p \geq \alpha$, then

$$\delta(R^*) = \alpha \text{ and } |R^*| \geq \frac{\alpha^2}{2} + \frac{3\alpha}{2}.$$

Therefore, we need the condition that $\delta \geq \alpha + 1$ for our third question. If $\delta \geq \alpha + 1$, then for any vertex $u \in V(G) \setminus S$, there is a cycle C such that $u \in C$ and $|C \cap S| = 1$. Indeed, if there is an edge joining a vertex in $N_G(u) \cap S$ and a vertex in $N_G(u) - S$, then these two vertices and u induce a triangle. Suppose there is no edge between $S \cap N(u)$ and $N(u) \setminus S$. Since $|N(u) \setminus S| \geq \delta - |S \cap N(u)| \geq \alpha + 1 - |S \cap N(u)| = |S \setminus N(u)| + 1$ and $S \setminus N(u)$ has to dominate $N(u) \setminus S$, there are two vertices v, v' in $N(u) \setminus S$ which are adjacent to some vertex $w \in S \setminus N(u)$. The cycle $uvwv'u$ is a desired cycle. Does this fact suggest the existence of a desired 2-factor?

Conjecture 4. *A claw-free graph with $\delta \geq \alpha + 1$ has a 2-factor with α cycles.*

Conjecture 5. *If G is a claw-free graph with $\delta \geq \alpha + 1$, then for any maximum independent set S, there exists a 2-factor with α cycles such that each cycle contains a vertex in S.*

Notice that if $p \geq 5$ and if we choose the independent set $S' = (S \setminus \{s_1\}) \cup \{t_1\}$, then there is a 2-factor with α cycles in $R^*(\alpha, p)$ such that each cycle contains a vertex in S'. Especially $R^*(\alpha, 4)$ has no 2-factor in which every cycle contains at most one vertex in S. Therefore $R^*(\alpha, 4)$ is an extremal graph for Conjecture 1.

2 Proof of Theorem 1

The *edge degree* of an edge uv in G is defined by the number of edges joining uv and $G - \{u, v\}$. For a multigraph, we call a subgraph S a *star* if S consists of a vertex (called a *center*) and edges incident with the center. So a star in this paper is not necessarily a tree. It is enough to show the following lemma because the subgraph in $L(H)$ induced by the vertices corresponding to edges in a star is a clique.

Lemma 7. *A multigraph H with minimum edge degree at least seven has a set $\mathcal{S}$ of edge-disjoint stars with at least three edges such that $E(H) = \bigcup_{S \in \mathcal{S}} E(S)$.*

Proof. Suppose that H is a multigraph with the minimum edge degree at least 7. We look for a set $\mathcal{S}$ of edge-disjoint stars with at least three edges such that $E(H) = \bigcup_{S \in \mathcal{S}} E(S)$. In the following, a desired set $\mathcal{S}$ is called a *star-cover* of H. For $i \geq 0$, let $V_i(H)$ and $V_{\geq i}(H)$ be the set of vertices whose degree in H are exactly i and at least i, respectively. By the minimum edge degree condition, we have $N_H(u) \subset V_{\geq 9-i}(H)$ for any $u \in V_i(H)$ and $1 \leq i \leq 6$. In particular the following claim holds.

Claim 1. $\bigcup_{i=1}^{4} V_i(H)$ *is independent.*

Let $u \in V_i(H)$ with $i \geq 2$ and $N_H(u) \cap V_1(H) = \emptyset$, and let $N_H(u) = \{v_1, v_2, \ldots, v_i\}$. Now we consider the following operation; Replace u with i vertices $u_1, u_2, \ldots, u_i$ and replace i edges $uv_1, \ldots, uv_i$ with $u_1v_1, \ldots, u_iv_i$, respectively. We call the graph obtained by this operation *a division of H at u*. (See Figure 2). Note that the division of H at u does not change the number of edges and the degree of vertices other than u. Since $N_H(u) \cap V_1(H) = \emptyset$, the division of H at u does not have a component consisting of only one edge.

Let $H^0 = H, H^1, \ldots, H^l$ be a graph sequence such that for any $0 \leq j \leq l-1$, H^{j+1} is the division of H^j at u for some $u \in V_i(H^j)$ $(2 \leq i \leq 4)$. By Claim 1,

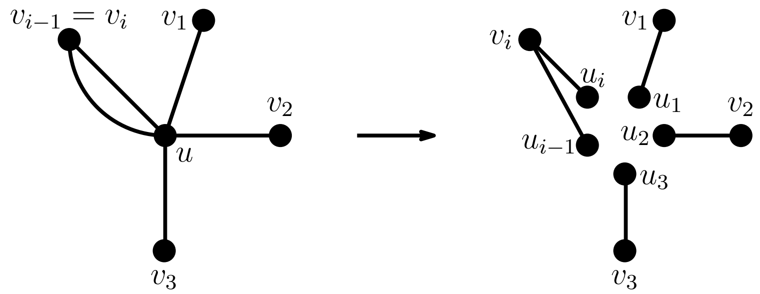

Figure 2: A division of H at u.

$N_{H^j}(v) \cap V_1(H^j) = \emptyset$ for any $v \in \bigcup_{i=2}^{4} V_i(H^j)$ and for any j, and hence we can perform the operation until the vertices with degree 2, 3 or 4 disappear. Notice that the operation strictly decreases the number of vertices of degree 2 or 3 or 4.

Again we take a graph sequence $H^l, H^{l+1}, \dots, H^p$ so that for any $l \leq j \leq p-1$, H^{j+1} is the division of H^j at u for some $u \in V_{\geq 5}(H^j)$ with $N_{H^j}(u) \cap V_1(H^j) = \emptyset$. We perform this operation consecutively as many times as possible and let $H_1 := H^p$. By the choice of H_1, we have the following claim.

Claim 2. *$V_i(H_1) = \emptyset$ for any $2 \leq i \leq 4$, and $N_{H_1}(u) \cap V_1(H_1) \neq \emptyset$ for any $u \in V_{\geq 5}(H_1)$. Moreover, $V_1(H_1)$ is an independent set.*

We will find a mapping $\varphi : E(H_1) \longrightarrow V(H_1)$ so that

(i) $\varphi(e) = x$ or $\varphi(e) = y$ for any $e = xy \in E(H_1)$,

(ii) $|\varphi^{-1}(u)| = 0$ for any $u \in V_1(H_1)$,

(iii) $|\varphi^{-1}(u)| \geq 3$ for any $u \in V_{\geq 5}(H_1)$.

If we can find such a mapping φ, $\mathcal{F} := \{C_u : u \in V_{\geq 5}(H_1)\}$ is a star-cover of H_1, where C_u is a star consisting of a vertex u (as a center) and the edges in $\varphi^{-1}(u)$. Moreover, a star-cover of H_1 corresponds to a star-cover of H, because the edge set of H is the same as that of H_1. Thus, it suffices to show the existence of a mapping φ satisfying the conditions (i)–(iii).

Suppose $e = xy \in E(H_1)$ with $x \in V_1(H_1)$. By Claim 2, $y \in V_{\geq 5}(H_1)$. Let $\varphi(e) = y$. This implies φ satisfies the condition (ii).

Let $H_2 := H_1 \setminus V_1(H_1)$ and let $o(H_2)$ be the set of vertices whose degree in H_2 are odd. Since the number of vertices of odd degree is even in each component of H_2, there exists a collection of paths $P_1, \dots, P_q$ such that each vertex in $o(H_2)$ appears in the set of end vertices of them exactly once. Note that $q = |o(H_2)|/2$. By considering the symmetric difference of them, we may assume that $P_1, \dots, P_q$ are pairwise edge disjoint. Let $P_i := x_0^i x_1^i x_2^i \dots$ and let $e_j^i = x_{j-1}^i x_j^i \in E(P_i)$. Then we define $\varphi(e_j^i) = x_j^i$.

Let $H_3 = H_2 \setminus \bigcup_{i=1}^q E(P_i)$. By the definition of $P_1, \dots, P_q$, we have $o(H_3) = \emptyset$, and hence the edges of H_3 can be covered by cycles. For each cycle, written by $y_0 y_1 y_2 \dots y_{r-1} y_r (= y_0)$, we define $\varphi(e_j) = y_j$, where $e_j = y_{j-1} y_j$ for $1 \le j \le r$.

We can easily check that this definition of φ satisfies the condition (i). Let $u \in V_{\ge 5}(H_1)$ and let $h = |N_{H_1}(u) \cap V_1(H_1)|$. By Claim 2, $h \ge 1$. Then

$$\begin{aligned} |\varphi^{-1}(u)| &\ge h + \frac{d_{H_2}(u) - 1}{2} \\ &= h + \frac{d_{H_1}(u) - h - 1}{2} \\ &= \frac{d_{H_1}(u) + h - 1}{2} \\ &\ge \frac{5}{2}, \end{aligned}$$

because $d_{H_1}(u) \ge 5$. Since $|\varphi^{-1}(u)|$ is an integer, we obtain the condition (iii). □

3 Proof of Theorem 6

3.1 Lemmas for the proof

Before giving the proof of Theorem 6, we first prove some lemmas which will be useful in the proof. For a vertex subset A of a graph G, the quantity $\min\{d_G(v) \mid v \in A\}$ is denoted by $\delta(A)$.

Fact 8. *If G is a claw-free graph, then for any maximum independent set S of G, $\delta(S) \le \frac{2n}{\alpha} - 2$.*

Proof. Note that $|N_G(u) \cap S| \le 2$ for any $u \in V(G) \setminus S$, because otherwise we can find a claw with center u. Thus, $e(V(G) \setminus S,\ S) \le 2|V(G) \setminus S| = 2(n - \alpha)$. On the other hand, $e(S,\ V(G)\setminus S) \geq \alpha \cdot \delta(S)$, and hence $\alpha \cdot \delta(S) \leq 2|V(G)\setminus S| = 2(n-\alpha)$, or $\delta(S) \leq \frac{2n}{\alpha} - 2$. $\square$

Lemma 9. *Let G be a claw-free graph with $\delta \geq \dfrac{2n}{\alpha} - 2$ and S be a maximum independent set of G. Then for any $v \in V(G) \setminus S$, $|N_G(v) \cap S| = 2$.*

Proof. Suppose that there exists a vertex $v \in V(G) \setminus S$ such that $|N_G(v) \cap S| \neq 2$. Note that $|N_G(u) \cap S| \leq 2$ for any $u \in V(G) \setminus S$. So, $|N_G(v) \cap S| \leq 1$ and hence, $e(V(G) \setminus S, S) \leq \sum_{u \in V(G)\setminus S} |N_G(u) \cap S| \leq 2|V(G) \setminus S| - 1 = 2(n-\alpha) - 1$. On the other hand, since S is an independent set and $\delta(G) \geq \frac{2n}{\alpha} - 2$, we obtain $e(S, V(G) \setminus S) \geq \alpha\big(\frac{2n}{\alpha} - 2\big) = 2n - 2\alpha$, a contradiction. $\square$

Lemma 10. *Let G be a claw-free graph with $\delta \geq 6$ and let S be an independent set of order r in G. Then there exists r vertex-disjoint triangles $C_1, C_2, \ldots, C_r$ such that $|S \cap C_i| = 1$ for any $1 \leq i \leq r$.*

Note that this implies each vertex of S is in a triangle.

Proof. Let $S = \{s_1, s_2, \ldots, s_r\}$. We will find r sets $T_1, T_2, \ldots, T_r$ such that (i) $|T_i| = 3$, (ii) $T_i \subset N_G(s_i)$ and (iii) $T_i \cap T_j = \emptyset$ for any $1 \leq i \neq j \leq r$. Suppose that a set T_i satisfies (i) and (ii). Since G is claw-free, there exists at least one edge connecting two vertices in T_i, and hence we find a triangle containing s_i in $\{s_i\} \cup T_i$. Furthermore if r sets $T_1, T_2, \ldots, T_r$ satisfy (iii), such triangles are pairwise disjoint. Hence it suffices to show that G has r sets satisfying (i)–(iii).

We construct a bipartite graph H as follows; one partite set of H is the union of three copies of S, say $\widetilde{S}$, and the other is $\bigcup_{i=1}^{r} N_G(s_i)$. For $\widetilde{s} \in \widetilde{S}$ and $x \in \bigcup_{i=1}^{r} N_G(s_i)$, we let $\widetilde{s}x \in E(H)$ if and only if $sx \in E(G)$, where s is the vertex in S corresponding to $\widetilde{s}$.

We will find a matching in H covering $\widetilde{S}$. Let $\widetilde{X} \subset \widetilde{S}$. Note that $d_H(\widetilde{s}) \geq 6$ for any $\widetilde{s} \in \widetilde{S}$, because $\delta(G) \geq 6$. This implies that

$$e\big(\widetilde{X}, N_H(\widetilde{X})\big) \geq 6|\widetilde{X}|.$$

On the other hand, we have $d_H(x) \leq 6$ for any $x \in \bigcup_{i=1}^{r} N_G(s_i)$, because otherwise $|N_G(x) \cap S| \geq 3$, and hence we can find a claw with center x in G. This implies that

$$e\big(N_H(\widetilde{X}), \widetilde{X}\big) \leq 6|N_H(\widetilde{X})|.$$

It follows from these two inequalities that $|N_H(\widetilde{X})| \geq |\widetilde{X}|$. By Hall's Theorem, H has a matching M covering $\widetilde{S}$.

For $1 \leq i \leq r$, let $T_i := \{x \in N_M(\widetilde{s_i}) : \widetilde{s_i}$ is a vertex corresponding to $s_i\}$. By the definition of H, T_i satisfies (i): $|T_i| = 3$ and (ii): $T_i \subset N_G(s_i)$ for any $1 \leq i \leq r$. Moreover $T_1, T_2, \ldots, T_r$ satisfy (iii) because M is a matching in H. This completes the proof of Lemma 10. $\square$

For the sake of the next lemma, we define some more notation. An *end block* of a graph G is a block that has at most one cut vertex of G. Let C be a cycle of a graph G. We give an orientation to C and denote the oriented cycle by $\overrightarrow{C}$. The directed cycle with reverse orientation is denoted by $\overleftarrow{C}$. For $x \in V(C)$, let x^+ be a successor vertex of x along $\overrightarrow{C}$.

The following lemma is shown in [1, Lemma 2] and [5, Lemma 5].

Lemma 11. *Let B be an end block of a graph G. For any $u, v \in B$ $(u \neq v)$, there exists a path in B connecting u and v of order at least $\delta(B) + 1$.*

3.2 Proof of Theorem 6

If G is a complete graph, there is nothing to prove. Thus, we may assume that $\alpha \geq 2$. Let $\mathcal{C}$ be a set of disjoint cycles such that each cycle in $\mathcal{C}$ has exactly one vertex in S. By Lemma 10 and by the fact $\delta \geq \frac{2n}{\alpha} - 2 \geq 3\alpha^2 - 2 \geq 10$, we can take such a set $\mathcal{C}$. Take such a set of cycles $\mathcal{C}$ so that $\sum_{C \in \mathcal{C}} |C|$ is as large as possible. Let $H := G \backslash \bigcup_{C \in \mathcal{C}} V(C)$. Suppose that there exists a vertex v in H such that $d_C(v) \geq \alpha$ for some cycle $C \in \mathcal{C}$. Let $R := \{x^+ : x \in N_C(v)\}$. Since $|R \cup \{v\}| \geq \alpha + 1$, $R \cup \{v\}$ is not an independent set. Let $D := vx^+\overrightarrow{C}xv$ if $vx^+ \in E(G)$ for some $x \in N_C(v)$; otherwise let $D := vx_2\overleftarrow{C}x_1^+x_2^+\overrightarrow{C}x_1v$, where $x_1^+x_2^+ \in E(G)$ with $x_1, x_2 \in N_C(v)$. This contradicts the maximality of $\mathcal{C}$. So, $d_C(v) \leq \alpha - 1$ for any vertex $v \in V(H)$ and for any cycle $C \in \mathcal{C}$. Thus, for any $v \in V(H)$, $d_H(v) \geq \delta(G) - \alpha(\alpha-1) \geq \frac{2n}{\alpha} - 2 - \alpha(\alpha-1)$. Note that $|H| \geq 2$ because $n \geq \frac{3\alpha^3}{2}$ and $\alpha \geq 2$.

Let B be an end block of H and let $v_1v_2 \in E(B)$. By Lemma 9, there exist $s, s' \in S$ such that $s, s' \in N_G(v_1)$. If $sv_2 \notin E(G)$ and $s'v_2 \notin E(G)$, then $\{v_1, s, s', v_2\}$ induces a claw, a contradiction. Thus, we may assume that $s \in N_G(v_2)$. By Lemma 11, there exists a path P in B connecting v_1 and v_2 of order at least $\delta(H) + 1 \geq \frac{2n}{\alpha} - 1 - \alpha(\alpha - 1)$. Rename $s_1 := s$ and let C_1 be a cycle in $\mathcal{C}$ containing s_1. Let u_1, u_2 be neighbors of s_1 in C_1. If $u_1u_2 \notin E(G)$, then $v_2u_1 \in E(G)$ or $v_2u_2 \in E(G)$, because otherwise we can find an induced claw. We may assume that $v_2u_1 \in E(G)$. Then when we consider a cycle $s_1v_1Pv_2u_1\overrightarrow{C}u_2s_1$, this contradicts the maximality of $\mathcal{C}$. So $u_1u_2 \in E(G)$, and hence $C_1 \setminus \{s_1\}$ has a hamilton cycle.

Let $C_1, C_2, \ldots, C_j$ be j cycles in $\mathcal{C}$ and let s_i be the vertex in S contained in C_i. We call $(C_1, C_2, \ldots, C_j)$ *a cycle system of order* j, if for any $1 \leq i \leq j$, there exist j cycles $D_1^i, D_2^i, \ldots, D_j^i$ such that

(S1) $\bigcup_{r=1}^{j} V(D_r^i) = \left(\bigcup_{r=1}^{j} V(C_r) \setminus V(C_i)\right) \cup V(P) \cup \{s_i\}$,

(S2) $s_r \in V(D_r^i)$ for any $1 \leq r \leq j$,

(S3) $C_i \setminus \{s_i\}$ has a hamilton cycle.

Note that (C_1) is a cycle system of order 1.

Claim 3. *Let $(C_1, C_2, \ldots, C_j)$ be a cycle system of order j. Then for any $1 \leq i \leq j$, $|C_i| \geq \frac{2n}{\alpha} - \alpha(\alpha - 1)$.*

Proof. By the definition of a cycle system, for any $1 \leq i \leq j$, there exists j cycles $D_1^i, D_2^i, \ldots, D_j^i$ satisfying (S1)–(S3). Let $\mathcal{D} := \left(\mathcal{C} \setminus \{C_1, \ldots, C_j\}\right) \cup \{D_1^i, \ldots, D_j^i\}$. By (S1), we obtain $\sum_{D \in \mathcal{D}} |D| = \sum_{C \in \mathcal{C}} |C| - |C_i| + |P| + 1 \geq \sum_{C \in \mathcal{C}} |C| - |C_i| + \frac{2n}{\alpha} - \alpha(\alpha - 1)$, and hence $|C_i| \geq \frac{2n}{\alpha} - \alpha(\alpha - 1)$, by the maximality of $\mathcal{C}$. □

Claim 4. *For any $1 \leq j \leq \alpha$, there exists a cycle system of order j.*

Proof. We will prove Claim 4 using induction on j. Since (C_1) is a cycle system of order 1, we may assume that $j \geq 2$. Suppose that there exists a cycle system $(C_1, \ldots, C_{j-1})$ of order $j-1$.

First we will show that there exist a vertex $s \in S \setminus \{s_1, \ldots, s_{j-1}\}$ and a cycle C_l with $1 \leq l \leq j-1$ such that $d_{C_l}(s) \geq \alpha$. Suppose that for any $s \in S \setminus \{s_1, \ldots, s_{j-1}\}$ and for any C_l with $1 \leq l \leq j-1$, we have $d_{C_l}(s) \leq \alpha - 1$. Then

$$\begin{aligned} & e\Big(S \setminus \{s_1, \ldots, s_{j-1}\}, \bigcup_{l=1}^{j-1} \big(V(C_l) \setminus \{s_l\}\big)\Big) &\leq& (\alpha - j + 1)(j-1)(\alpha - 1), \\ \text{and} \quad & e\Big(\{s_1, \ldots, s_{j-1}\}, \bigcup_{l=1}^{j-1} \big(V(C_l) \setminus \{s_l\}\big)\Big) &\leq& \sum_{r=1}^{j-1} d_G(s_r) \\ & &=& (j-1)\Big(\frac{2n}{\alpha} - 2\Big), \end{aligned}$$

because $d_G(s_r) = \frac{2n}{\alpha} - 2$ for every $s_r \in S$ by Fact 8. Thus,

$$e\Big(S, \bigcup_{l=1}^{j-1} \big(V(C_l) \setminus \{s_l\}\big)\Big) \leq (\alpha - j + 1)(j-1)(\alpha - 1) + (j-1)\Big(\frac{2n}{\alpha} - 2\Big).$$

On the other hand, it follows from Lemma 9 and Claim 3 that

$$\begin{aligned} e\Big(\bigcup_{l=1}^{j-1} \big(V(C_l) \setminus \{s_l\}\big), S\Big) &= 2\sum_{l=1}^{j-1} \big(|C_l| - 1\big) \\ &\geq 2(j-1)\frac{2n}{\alpha} - 2(j-1)\alpha(\alpha - 1) - 2(j-1). \end{aligned}$$

These two inequalities and the fact that $j \geq 2$ imply that

$$\begin{aligned} & (\alpha - j + 1)(j-1)(\alpha - 1) + (j-1)\Big(\frac{2n}{\alpha} - 2\Big) \\ & \quad \geq 2(j-1)\frac{2n}{\alpha} - 2(j-1)\alpha(\alpha - 1) - 2(j-1) \\ \text{or} \quad & n \leq \frac{3\alpha^3 - 2\alpha^2 - j\alpha(\alpha - 1) - \alpha}{2} \leq \frac{3\alpha^3 - 4\alpha^2 + \alpha}{2} < \frac{3\alpha^3}{2}, \end{aligned}$$

contradicting the assumption "$n \geq \frac{3\alpha^3}{2}$". So, there exist a vertex $s \in S \setminus \{s_1, \ldots, s_{j-1}\}$ and a cycle C_l with $1 \leq l \leq j-1$ such that $d_{C_l}(s) \geq \alpha$. Take such a vertex s and rename $s_j := s$ and let C_j be the cycle in $\mathcal{C}$ that contains s_j. Next, we shall prove that $(C_1, C_2, \ldots, C_j)$ is a cycle system of order j.

Fix an integer i with $1 \leq i \leq j-1$. Since $(C_1, C_2, \ldots, C_{j-1})$ is a cycle system of order $j-1$, there exist $j-1$ cycles $D^i_1, D^i_2, \ldots, D^i_{j-1}$ satisfying (S1)–(S3). Let $D^i_j := C_j$. Then j cycles $D^i_1, D^i_2, \ldots, D^i_j$ satisfy (S1): $\bigcup_{r=1}^{j} V(D^i_r) = \big(\bigcup_{r=1}^{j-1} V(C_r) \setminus V(C_i)\big) \cup V(P) \cup \{s_i\} \cup V(C_j) = \big(\bigcup_{r=1}^{j} V(C_r) \setminus V(C_i)\big) \cup V(P) \cup \{s_i\}$, (S2): $s_r \in V(D^i_r)$ for any $1 \leq r \leq j$, and (S3): $C_i \setminus \{s_i\}$ has a hamilton cycle. So for any $1 \leq i \leq j-1$, there exist j cycles $D^i_1, D^i_2, \ldots, D^i_j$ satisfying (S1)–(S3).

Therefore it suffices to show that for $i = j$, there exists j cycles $D^j_1, D^j_2, \ldots, D^j_j$ satisfying (S1)–(S3). Again since $(C_1, C_2, \ldots, C_{j-1})$ is a cycle system of order $j-1$, there exist $j-1$ cycles $D^l_1, D^l_2, \ldots, D^l_{j-1}$ satisfying (S1)–(S3). Recall that l be the index satisfying $d_{C_l}(s_j) \geq \alpha$.

Let C'_l be a hamilton cycle of $C_l \setminus \{s_l\}$. Since $s_j s_l \notin E(G)$, $d_{C'_l}(s_j) = d_{C_l}(s_j) \geq \alpha$. Let $R := \{x^+ : x \in N_{C_l}(s_j)\}$. Since $|R \cup \{s_j\}| \geq \alpha + 1$, there exists an edge between two vertices of $R \cup \{s_j\}$. Let $D^j_r := D^l_r$ for any $1 \leq r \leq j-1$. Let $D^j_j := s_j x^+ \overrightarrow{C_l} x s_j$ if $s_j x_1^+ \in E(G)$ for some $x_1 \in N_{C_l}(s_j)$; otherwise let $D^j_j := s_j x_2 \overleftarrow{C_l} x_1^+ x_2^+ \overrightarrow{C_l} x_1 s_j$, where $x_1^+ x_2^+ \in E(G)$ with $x_1, x_2 \in N_{C_l}(s_j)$.

Then $D^j_1, \ldots, D^j_j$ satisfy (S1) and (S2), because

$$\begin{aligned}
\bigcup_{r=1}^{j} V(D^j_r) &= \Big(\bigcup_{r=1}^{j-1} V(C_r) \setminus V(C_l)\Big) \cup V(P) \cup \{s_l\} \cup V(D^j_j) \\
&= \Big(\bigcup_{r=1}^{j-1} V(C_r) \setminus V(C_l)\Big) \cup V(P) \cup \{s_l\} \cup V(C'_l) \cup \{s_j\} \\
&= \Big(\bigcup_{r=1}^{j} V(C_r) \setminus V(C_j)\Big) \cup V(P) \cup \{s_j\}.
\end{aligned}$$

Let u_1, u_2 be neighbors of s_j in C_j. Suppose that $u_1 u_2 \notin E(G)$. Since G is a claw-free graph, $x_1 u_1 \in E(G)$ or $x_1 u_2 \in E(G)$, by the symmetry, we may assume that $x_1 u_1 \in E(G)$. Then $D'_j := s_j \overrightarrow{D^j_j} x_1 u_1 \overrightarrow{C_j} s_j$ is a cycle containing s_j. Then $\mathcal{D} := \mathcal{C} \setminus \{C_1, C_2, \ldots, C_j\} \cup \{D^j_1, \ldots, D^j_{j-1}, D'_j\}$ is a set of disjoint cycles such that each cycle in $\mathcal{D}$ has exactly one vertex in S and $\sum_{D \in \mathcal{D}} |D| = \sum_{C \in \mathcal{C}} |C| + |P|$, contradicting the maximality of $\mathcal{C}$. So $u_1 u_2 \in E(G)$, and hence (S3) $C_j \setminus \{s_j\}$ has a hamilton cycle $u_1 \overrightarrow{C_j} u_2 u_1$. Therefore for $i = j$, there exists j cycles satisfying (S1)–(S3). Hence there exists a cycle system $(C_1, C_2, \ldots, C_j)$ of order j. $\square$

By Claim 4, there exists a cycle system $(C_1, C_2, \ldots, C_\alpha)$ of order α. It follows from Claim 3 that $|C_i| \geq \frac{2n}{\alpha} - \alpha(\alpha-1)$ for any $1 \leq i \leq \alpha$. Thus,

$$\begin{aligned} n &> \sum_{i=1}^{\alpha} |C_i| \\ &\geq \alpha\Big(\frac{2n}{\alpha} - \alpha(\alpha-1)\Big) \\ &= 2n - \alpha^2(\alpha-1), \\ \text{or} \qquad n &< \alpha^3 - \alpha^2 \ < \ \alpha^3, \end{aligned}$$

contradicting $n \geq \frac{3\alpha^3}{2}$. This completes the proof of Theorem 6. □

4 Proof of Fact 5

Let $d \geq 4$ be an integer and R_d be the graph obtained from $K_2 \cup (d-1)K_{1,d}$ by adding $d-1$ edges joining a specified vertex in K_2 and the center of each $K_{1,d}$. We define a tree $H^*_{m,d}$ from the path P_m of length $m-1$ and a number of R_d as follows. For each inner vertex of P_m, we add $(d-2)R_d$ and $d-2$ edges joining the inner vertex and the top of each R_d as in Figure 3, and for each end of P_m, we add

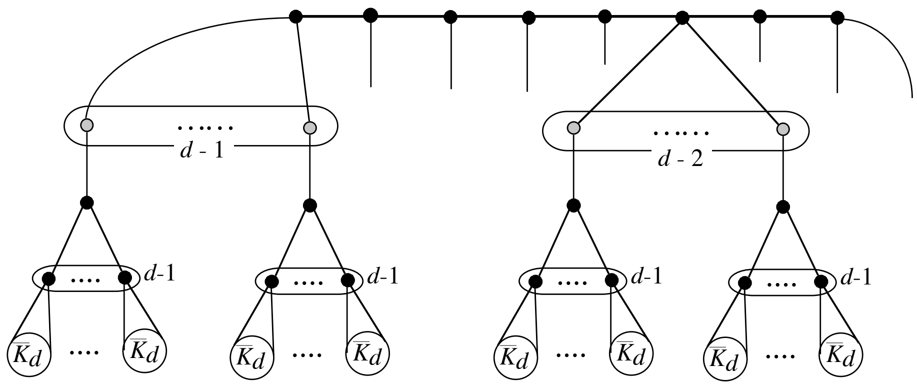

Figure 3: $H^*_{m,d}$

$(d-1)R_d$ and $d-1$ edges. The order n and the minimum number f_2 of cycles of 2-factors of $L(H^*_{m,d})$ are:

$$n = (d^3 - 2d^2 + d - 1)m + 2d^2 + 1 \text{ and } f_2 = (d^2 - 2d + 1)m + 2d$$

See [18]. It is easy to check the independence number α of $L(H^*_{m,d})$ is:

$$\alpha = f_2 - \lceil \frac{m}{2} \rceil \geq \frac{(2d^2 - 4d + 1)m + 4d - 1}{2}.$$

Therefore

$$0 < \frac{(d-2)m + 2}{(2d^2 - 4d + 1)m + 4d} \leq \frac{n}{\alpha} - d \leq \frac{(d-2)m + 2}{(2d^2 - 4d + 1)m + 4d - 1} < \frac{1}{2d}.$$

Since the minimum degree of $L(H^*_{m,d})$ is d, we obtain

$$\frac{n}{\alpha} - \frac{1}{2d} < d < \frac{n}{\alpha}.$$

Acknowledgements

We would like to express our sincere gratitude to Professor Zdenek Ryjáček for his helpful comments.

References

[1] J.A. Bondy and B. Jackson, *Long paths between specified vertices of a block*, Annals of Discrete Mathematics **27** (1985), 195-200.

[2] Bondy, J.A.; Murty, U.S.R.: Graph Theory with Applications. Macmillan, London and Elsevier, New York, 1976.

[3] H.J. Broersma, D. Paulusma and K. Yoshimoto, *Sharp upper bounds on the minimum number of components of 2-factors in claw-free graphs*, Graphs and Combin. **25** (2009) 427–460.

[4] S.A. Choudum and M.S. Paulraj, *Regular factors in $K_{1,3}$-free graphs*, Journal of Graph Theory **15** (1991) 259–265.

[5] Y. Egawa, R. Glas and S. Locke, *Cycles and paths through specified vertices in k-connected graphs*, Journal of Combinatorial Theory, Series B **52** (1991) 20–29.

[6] Y. Egawa and K. Ota, *Regular factors in $K_{1,n}$-free graphs*, Journal of Graph Theory **15** (1991) 337–344.

[7] R.J. Faudree, O. Favaron, E. Flandrin, H. Li, and Z. Liu, *On 2-factors in claw-free graphs*, Discrete Mathematics **206** (1999) 131–137.

[8] R. Faudree, E. Flandrin, and Z. Ryjáček *Claw-free graphs—a survey*, Discrete Mathematics **164** (1997) 87–147.

[9] B. Jackson and K. Yoshimoto, *Even subgraphs of bridgeless graphs and 2-factors of line graphs*, Discrete Mathematics **307** (2007) 2775–2785.

[10] B. Jackson and K. Yoshimoto, *Spanning Even Subgraphs of 3-edge-connected Graphs*, J. Graph Theory **62** (2009) 37–47.

[11] R.J. Gould and E. Hynds, *A note on cycles in 2-factors of line graphs*, Bulletin of the ICA **26** (1999) 46–48.

[12] R.J. Gould and M.S. Jacobson, *Two-factors with few cycles in claw-free graphs*, Discrete Mathematics **231** (2001) 191–197.

[13] F. Harary and C. St. J.A. Nash-Williams *On eulerian and hamiltonian graphs and line graphs*, Canadian Mathematical Bulletin **8** (1965) 701-710.

[14] M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, Journal of Graph Theory **8** (1984) 139–146.

[15] Z. Ryjáček, *On a closure concept in claw-free graphs*, Journal of Combinatorial Theory, Series B **70** (1997) 217–224.

[16] Z. Ryjáček, *private communication.*

[17] Z. Ryjáček, A. Saito, and R.H. Schelp, *Closure, 2-factors, and cycle coverings in claw-free graphs*, Journal of Graph Theory **32** (1999) 109–117.

[18] K. Yoshimoto, *On the number of components in 2-factors of claw-free graphs*, Discrete Mathematics **307** (2007) 2808–2819.