Note on locating pairs of vertices on Hamiltonian cycles

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Abstract. Given a fixed positive integer $k \geq 2$, let G be a simple graph of order $n \geq 6k$. It is proved that if the minimum degree of G is at least $n/2+1$, then for every pair of vertices x and y, there exists a Hamiltonian cycle such that the distance between x and y along that cycle is precisely k.

Key words. Hamiltonian cycle, panconnected graph, Enomoto's conjecture, dominating cycle
MSCI 05C45

1. INTRODUCTION

We deal only with finite simple graphs and our notation generally follows the notation of Chartrand and Lesniak in [1]. Given an ordered set of vertices $S = \{x_1, x_2, \cdots, x_k\}$ in a graph, there are a series of results giving minimum degree conditions that imply the existence of a Hamiltonian cycle such that the vertices in S are located in order on the cycle with restrictions on the distance between consecutive vertices of S. Examples include results by Kaneko and Yoshimoto [6], Sárközy and Selkow [9], and Faudree, Gould, Jacobson, and Magnant [4].

Here we will consider only a pair of vertices, and will require the distance between the vertices on the Hamiltonian cycle to be precise. For a Hamiltonian cycle C, and distinct vertices x and y, let $d_C(x,y)$ denote the length of x and y along C. The minimum degree of G is denoted by $\delta(G)$.

It was conjectured by Enomoto [3] that if G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2+1$, then for any x, y, there is a Hamiltonian cycle C of G such that $d_C(x,y) = \lfloor n/2 \rfloor$. The following natural generalization of Enomoto's conjecture was stated and investigated by Faudree and Hao Li [5].

Conjecture 1 ([5]) *If G is a graph of order n with $\delta(G) \geq n/2+1$, then for any integer $2 \leq k \leq n/2$ and any vertices x and y, there is a Hamiltonian cycle C of G such that $d_C(x,y) = k$.*

In [5] the cases $k = 2$ and 3 have been answered in the affirmative, and Conjecture 1 was supported by solving the case when k was fixed and n was sufficiently large. Along

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the same line in the present note we will show that if G is a graph of order $n \geq 6k$ and $\delta(G) \geq n/2+1$, then for any vertices x and y, G has a Hamiltonian cycle C such that $d_C(x, y) = k$.

We will use a classical result of Nash-Williams [8] on dominating cycles, and a result on panconnected graphs due to Williamson [11]. A cycle C is called a *dominating cycle* in G if $G - V(C)$ is an independent set.

Theorem 1 ([8]) *Let G be a 2-connected graph on n vertices with $\delta(G) \geq (n+2)/3$. Then every longest cycle of G is a dominating cycle.* □

Theorem 2 ([11]) *If G is a graph of order n with $\delta(G) \geq n/2+1$, then for any $2 \leq k \leq n-1$ and for any vertices x and y, G has an x, y-path of length k.* □

The minimum degree condition in Theorem 2 for panconnectivity is sharp, thus it is obviously sharp in Conjecture 1 as well. Our main result supports further the conjecture, but leaves open the range $n/6 < k \leq n/2$. In the next section we prove the following theorem.

Theorem 3 *Let $k \geq 2$ be a fixed positive integer. If G is a graph of order $n \geq 6k$ and $\delta(G) \geq n/2+1$, then for any vertices x and y, G has a Hamiltonian cycle C such that $d_C(x, y) = k$.*

2. PROOF

Let $\kappa(G)$ be the vertex connectivity of G, that is the minimum number of vertices in a cut set, and let $\alpha(G)$ be the independence number of G, that is the maximum number of vertices in an independent set. The lemma below will be useful in the proof of Theorem 3.

Lemma 1 *If G is a graph of order n with $\delta(G) \geq n/2+1$, then $\kappa(G) \geq \alpha(G)$.*

Proof: Let $\kappa(G) = s$, and let S be a minimum cut set of G, so that $|S| = s$. Let H_1 and H_2 be connected components of $G - S$, with h_1 and h_2 vertices, respectively. Let H_i^* be the subgraph spanned by $H_i \cup S$, for $i = 1, 2$. Any independent set in H_i^* with a vertex in H_i will have at most $h_i + s - (n/2+1)$ vertices. Hence, any independent set in G containing a vertex in H_1 or H_2 will have at most $h_1 + h_2 + 2s - 2(n/2+1) = s-2$ vertices. Since S cannot contain an independent set with more than s vertices, $\alpha(G) \leq s = \kappa(G)$ follows. □

In the proof of Theorem 3 additional standard terminology will be used as follows. For the vertex set $V(G)$ and for the edge set $E(G)$ of a graph G we will eventually use just G whenever the context is clear. The set of all adjacencies of a vertex $v \in G$ in $S \subset G$ is denoted by $N_S(v)$, and we set $d_S(v) = |N_S(v)|$.

A cycle (or path) with an ordered set of vertices $\{x_1, x_2, \cdots, x_k\}$ will be denoted by $(x_1, x_2, \cdots, x_k)$. If x_i is a vertex of a cycle (path) then x_i^+ will denote the successor x_{i+1}, and if S is a subset of its vertices, then S^+ will denote the set of all successors of the vertices of S. The set S^- of predecessors is defined similarly.

In the proof we fix an integer k and a pair of vertices $x, y \in G$. An x, y-path of length k will be called a *good x, y-path* of G. We assume that G has $n \geq 6k$ vertices and that every vertex has at least $n/2+1$ neighbors. Then, by Theorem 2, G contains good x, y-paths.

A cycle containing a good x, y-path will be called a *good cycle* of G. Assume that the required good Hamiltonian cycle does not exist for x, y; we will show that this leads to a contradiction. We may assume that $k \geq 4$, since in [5], Conjecture 1 was solved for $k = 2$ and 3. Furthermore, the minimum degree condition $\delta(G) \geq n/2 + 1$ implies easily that the connectivity $\kappa(G) \geq 4$.

Claim 1: *There is a good cycle for x, y that has length at least $n - k + 1$.*

Step 1. First we shall find a good path P such that $G' = G - V(P)$ is 2-connected. If $\kappa(G) \geq k + 3$, then this is obviously true for any good x, y-path P, guaranteed by Theorem 2.

Next we consider the case when $\kappa(G) < k + 3$. Then G has a minimum cutset S of order s, $4 \leq s \leq k + 2$. The condition $s \leq k + 2 < n/3$ and $\delta(G) \geq n/2 + 1$ imply that $G - S$ has two connected components H_1, H_2. Since $\delta(H_1), \delta(H_2) \geq n/2 + 1 - s$, we have $n/2 + 2 - s \leq |H_1| \leq |H_2| \leq n/2 - 2$.

Consider the case where $x, y \in H_2$. The other cases for locations of x and y, such as both in H_1 or in S or split between the sets H_1, H_2, and S can be handled in the same way with the same results.

Since $s \leq k + 2$ and $k \leq n/6$, it follows that

$$\delta(H_2) \geq n/2 + 1 - s \geq n/2 - k - 1 \geq (n/2 - 2)/2 + 1 \geq |H_2|/2 + 1$$

is true. Thus H_2 is panconnected by Theorem 2 (and so is the possibly denser H_1). Let P be a good x, y-path in H_2.

Note that since S is a minimum cut, there is a matching with s edges between S and H_1 and between S and H_2. Note also that each vertex $v \in H_1 \cup H_2$ has at least 4 adjacencies in S, since $d_{H_2}(v) \leq (n/2 + 1) - 4$. Now it follows easily that $G' = G - V(P)$ is 2-connected. To see this observe first that H_1 has a Hamiltonian cycle C. Then for any $v_1, v_2 \in H_2$, there are four pairwise internally vertex disjoint paths, two from v_1 and two from v_2, into four distinct vertices of C. Using appropriate subpaths of C we obtain two internally vertex disjoint paths from v_1 to v_2. Similar argument applies for the variations when v_1 and v_2 are located anywhere in G'. Thus the 2-connectivity of G' follows by Whitney's theorem (see [10]).

In each case, since $|G'| = n - k - 1$, we also have $\delta(G') \geq n/2 - k \geq ((n - k - 1) + 2)/3 = (|G'| + 2)/3$. Then Theorem 1 implies that every longest cycle in G' is a dominating cycle. Note that a longest cycle of G' has at least $n - 2k$ vertices, by Dirac's theorem (see [2]).

Step 2. Our next objective is to insert the good x, y-path P obtained in Step 1 into a longest dominating cycle C of $G' = G - V(P)$. Assume first that $\{x, y\}$ has no neighbor in the independent set $G' - V(C)$, and hence $d_C(v) \geq n/2 + 1 - k$, for $v \in \{x, y\}$. If there exists a neighbor of x and a neighbor of y which are consecutive on C, then P and C form a good cycle of length at least $n - k + 1$ that misses the independent set $G' - V(C)$, thus is dominating. If there does not exist a neighbor of x and a neighbor of y that are consecutive on C, then a good cycle can be formed by selecting a neighbor of y closest to a neighbor of x on C, which will yield a cycle of length at least

$$\begin{aligned}&2|N_C(x) \cap N_C(y)| + |N_C(x) \setminus N_C(y)| + |N_C(y) \setminus N_C(x)| - 1 + |P| \\ &= d_C(x) + d_C(y) - 1 + (k + 1) \geq 2(n/2 - k + 1) + k = n - k + 2.\end{aligned}$$

Assuming that x or y has an adjacency in $G' - V(C)$, say x' or y', then the path P' with a new end vertex x' or y' can be used as in the previous argument to insert P' into C to obtain a good cycle of the same length or longer. This completes the proof of Claim 1.

Claim 2: *If C is a longest good cycle for x, y and it has length at least $n - k + 1$, then C is a dominating good cycle.*

Let $C = P \cup Q$ be a good cycle of maximum length $m \geq n - k + 1$, where P is the good path on $k + 1$ vertices from x to y, and Q is the path from y to x with $m - k + 1$ vertices.

Assume on the contrary that $H = G - V(C)$ is not independent. Let u and v be endvertices of a longest path in H with $\ell \geq 2$ vertices. By the maximality of C, neither u nor v can be adjacent to consecutive vertices of Q . Also by the maximality of C, any adjacency of u on Q implies that v is not adjacent to any vertex of Q within a distance $\ell + 1$ of this adjacency.

Consider the case when $\ell \leq 3$. Hence, $d_Q(v) \leq (m - (k - 1) - 2(d_Q(u) - 1)$, and so $2d_Q(u) + d_Q(v) \leq m - k + 3$. Also, the roles of u and v can be interchanged, and so $d_Q(u) + d_Q(v) \leq 2(m - k + 3)/3$. Clearly, $d_{P-\{x,y\}}(u) + d_{P-\{x,y\}}(v) \leq 2(k - 1)$ and $d_H(u) + d_H(v) \leq 2(\ell - 1)$. This results in the following inequality:

$$2(n/2 + 1) \leq d(u) + d(v) \leq 2(m - k + 3)/3 + 2(k - 1) + 2(\ell - 1).$$

This implies $n \geq m + \ell \geq 3n/2 - 2k - 2\ell + 6 \geq 3n/2 - 2k > n$, a contradiction.

Next we assume that $\ell > 3$. Observe that $d_H(u), d_H(v) \leq \ell - 1$, since u and v are the end vertices of a maximum path of length $\ell - 1$ of H. Thus we have

$$d_Q(u) = d_G(u) - d_{P-\{x,y,\}}(u) - d_H(u) \geq (n/2 + 1) - (k - 1) - (l - 1) = n/2 - k - \ell + 3,$$

and the same bound is valid for $d_Q(v)$.

Let t be the number of vertices of Q adjacent to both u and v. Then Q has $d_Q(u) - t$ vertices adjacent to u and not v (and $d_Q(v) - t$ vertices of Q adjacent to v and not u). Traversing Q from y towards x there are t vertices followed by at least ℓ consecutive non-neighbors of $\{u, v\}$, and each of the further $(d_Q(u) - t) + (d_Q(v) - t)$ neighbors of u or v must be followed by at least one non-neighbor of $\{u, v\}$. Hence for some $r \leq \ell$,

$$|Q| \geq t(\ell+1)+2(d_Q(u)+d_Q(v)-2t)-r \geq 4(n/2-k-\ell+3)+t(\ell-3)-\ell \geq 4(n/2-k-\ell+3)-4.$$

Thus we obtain $n = |P-\{x,y\}|+|Q|+|H| \geq |Q|+\ell+k-1 \geq 4(n/2-k-\ell+3)-4+\ell+k-1 = 2n - 3k - 3\ell + 7$ implying $n \leq 3k + 3\ell - 7$. Since $\ell \leq |H| = n - m \leq k - 1$, we obtain $n \leq 3k + 3(k - 1) - 7 < 6k < n$, a contradiction. This completes the proof of Claim 2.

Claim 3: *If P is a good x, y-path with $\alpha(G) \geq n/2 - k + 2$, then P can be inserted into a good cycle that is dominating.*

Since $\alpha(G) \geq n/2 - k + 2 \geq 6k/2 - k + 2 = 2k + 2 > k + 3$, Lemma 1 implies that $\kappa(G) > k + 3$. In particular, $G - V(P)$ is 2-connected.

By Claim 1, as described in Step 2, P can be inserted into a good cycle of length at least $n - k + 2$. Then by Claim 2, a maximum length good cycle containing P is a dominating good cycle. This concludes the proof of Claim 3.

By Claim 1 and 2, G has a dominating good cycle $C = P \cup Q$ of maximum length $m \geq n - k + 1$, where P is a good x, y-path, Q is a path from y to x, and $H = G - V(C)$ is an independent set.

Given any $w \in H$, the maximality of C implies that w can not be adjacent to two consecutive vertices of Q. Moreover, $A(w) = N^+_{Q-x}(w) \cup \{w\}$ is an independent set, since any adjacency within $A(w)$ would result in a longer good cycle including w.

Observe that every $w' \in H - A(w)$ has at most one adjacency in $A(w)$, for otherwise a good cycle could be formed including w and w'. Therefore each w' can be either added to $A(w)$ or can replace its only neighbor in $A(w)$. In this way we obtain an independent set $A(H)$ containing H such that $|A(H)| \geq |A(w)| = |N_{Q-x}(w)|+1 \geq (d_G(w)-d_{P-y}(w))+1 \geq n/2-k+2$, so now Claim 3 can be used.

For $w \in H$, let $U(w) = N^+_Q(w) \cap N^-_Q(w)$. If $u \in U(w)$, then w is interchangeable with u to obtain a good cycle C' that includes w and excludes u. This C' is dominating, provided $H' = (H-w)+u$ is independent. Any edge between u and $H-w$ results in C', a cycle of the same maximum length that is not dominating, contradicting Claim 2. Thus we conclude that $U(w) \subseteq A(H)$. Since $d_Q(w) \geq n/2+1-(k-1) = n/2-k+2$, and $|Q| \leq n-k$, we have $|U(w)| \geq 3(d_Q(w)-1)-|Q| > 3(n/2-k)-(n-k) = n/2-2k$. Consequently there are more than $n/2-2k$ vertices of C that might play the role of a given $w \in H$ in the independent set $A(H)$. For a given maximum length dominating good cycle C, let $A = A(C)$ be an independent set of maximum order containing $H = G - V(C)$. Note that $|A(C)| \geq |A(H)| \geq n/2-k+2$.

Next we shall find a good x,y-path P^* saturated by A, that is an x,y-path of length k containing at most one pair of consecutive vertices not in A.

The path P^* will be obtained by alternating between A and $G-A$. For any $s < k$, let $P' = (a_1, z_1, a_2, \dots, a_s, z_s, a_{s+1})$ be a path with $a_i \in A, z_i \in G-A$. Then P' will be extended by appending a path $(a_{s+1}, z_{s+1}, a_{s+2})$, where $a_{s+2} \in A-P'$ and $z_{s+1} \in (N(a_{s+1}) \cap N(a_{s+2})) - P'$. To see that this can be done, observe that $2(n/2+1) \leq d(a_{s+1})+d(a_{s+2}) \leq (n/2+k-2)+|N(a_{s+1}) \cap N(a_{s+2})|$. Then, using $s<k$, it follows that $|N(a_{s+1}) \cap N(a_{s+2})| \geq n/2-k+4 \geq s+1$, and therefore the set $(N(a_{s+1}) \cap N(a_{s+2}))-P'$ is not empty.

Obviously we can start and terminate P' at predetermined vertices of A, in particular at x and y, provided $x,y \in A$. If $\{x,y\} \not\subset A$, then we use any neighbors, $x' \in N_A(x)$ or $y' \in N_A(y)$ or both, and build an alternating x',y'-path of length shorter by one or two as needed. We might also need to adjust the alternating path for the parity of k. It is enough to include an edge from $G-A$ at the beginning of the procedure, by inserting a path (a_1, z, z', a_2) such that $z, z' \in G-A$.

Let C^* be a maximum length dominating good cycle containing P^*, that is given by Claim 3. Set $C^* = P^* \cup Q^*$ and $H^* = G - C^*$. Assume that $|C^*| = m < n$ and let $w \in H^* \cap A$. Observe that the neighbors of w belong to C^*-A, furthermore C^* has at most one pair of consecutive vertices on P^* that might both be adjacent to w. Then it follows $d_G(w) \leq m/2+1 < n/2+1$, a contradiction. Thus we conclude that $w \notin A$. Let $B = A(C^*)$ be a maximum independent set containing H^*, which also has at least $n/2-k+2$ vertices.

Since there are more than $n/2-2k$ vertices of C^* that might play the role of a given $w \in H^*$, we have $|B \setminus A| \geq n/2-2k$, thus $|A \cup B| \geq n/2-2k+(n/2-k+2) = n-3k+2$. For any $v \in A \cap B$, we would have $d_G(v) \leq n-|A \cup B| \leq 3k-2 < n/2+1$, a contradiction. Thus A and B are disjoint.

We will now build a good x,y-path P^{**} containing as many vertices from $A \cup B$ as follows. Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be such that $|A_0| = |B_0| = n_0 = \lceil n/2 \rceil - k$. Let $G_0 \subseteq G$ be the $n_0 \times n_0$ bipartite subgraph induced by $A_0 \cup B_0$. Clearly $\delta(G_0) \geq \lceil n/2 \rceil + 1 - (n - 2n_0) \geq n_0 - k + 1$. Furthermore, since $\delta(G_0) \geq n_0 - k + 1 \geq (n_0 + 1)/2$, G_0 has a Hamiltonian cycle C_0, by a theorem of Moon and Moser [7].

Let $x^*, y^* \in (A \cup B) - G_0$ be distinct vertices in the same partition class, say $x^*, y^* \in A - C_0$, and let k^* be an even integer, $k - 5 \leq k^* \leq k - 2$. We show first that the subgraph induced by $G_0 \cup \{x^*, y^*\}$ contains an x^*, y^*-path of length $k^* + 2$.

Assume that such a path does not exist. Then each vertex $u \in N_{C_0}(x^*)$ "knocks out" a possible adjacency of y^* on C_0, i.e. if $N^* = \{v \in B_0 \mid d_{C_0}(u, v) = k^* \text{ for some } u \in N_{C_0}(x^*)\}$, then $N^* \cap N_{C_0}(y^*) = \emptyset$. Observing that $N^*, N_{C_0}(y^*) \subseteq B_0$, and since $d_{C_0}(x^*), d_{C_0}(y^*) \geq n_0 - k + 1$, it follows that $2(n_0 - k + 1) \leq d_{C_0}(x^*) + d_{C_0}(y^*) = |N^*| + |N_{C_0}(y^*)| \leq n_0$. This is a contradiction, since $n_0 - k + 1 > n_0/2$.

The x^*, y^*-path obtained above will be used to join two disjoint paths $P_x = (x, x_1, x^*)$ and $P_y = (y, y^*)$. Selecting these short paths and the value k^* depend on the parity of k and the position of x and y with respect to A and B as follows:

Case (a). $k \geq 4$ and even. If $x, y \in A$ (the case $x, y \in B$ is exactly the same), then set $P_x = (x), P_y = (y)$ and $k^* = k - 2$. If $x, y \in G - (A \cup B)$, then we choose $x^* \neq y^*$ in A (or in B), we set $P_x = (x, x^*), P_y = (y, y^*)$, and $k^* = k - 4$. To see that there are such independent edges xx^* and yy^* note that $|A|, |B| \geq n/2 - k + 2$, hence each vertex in $G - (A \cup B)$ has at least $n/4 - k + 2 > 2$ adjacencies in A or in B.

If $y \in A, x \in G - A$, then we choose an arbitrary $x^* \in A - y$ and a vertex $x_1 \in N(x) \cap N(x^*)$. Note that $x_1 \neq y$ exists, since any two vertices of G have at least two common neighbors. Then we set $P_x = (x, x_1, x^*)$, $P_y = (y)$ and $k^* = k - 4$.

Case (b). $k \geq 5$ and odd. If $x \in B, y \in A$, then we choose a vertex $x^* \in N_A(x) - y$, set $P_x = (x, x^*), P_y = (y)$, and $k^* = k - 3$. If $x, y \in G - (A \cup B)$, then y has an adjacency $y^* \in A$, and x has an adjacency $x_1 \in B$, by the maximality of A and B. Now choose an arbitrary $x^* \in N_A(x_1) - y^*$. Thus we set $P_x = (x, x_1, x^*), P_y = (y, y^*)$, and $k^* = k - 5$.

If $x, y \in B$ or $y \in B, x \in G - B$, then let $y^* \in N_A(y)$ be an arbitrary vertex, and choose any $x^* \in A - y^*$. As before, there is a vertex $x_1 \in N_G(x) \cap N_G(x^*)$ disjoint from $\{y, y^*\}$. Then we set $P_x = (x, x_1, x^*)$, $P_y = (y, y^*)$ and $k^* = k - 5$.

In each case after P_x and P_y are specified, we define A_0, B_0 to be any sets $A_0 \subseteq A - (P_x \cup P_y)$, $B_0 \subseteq B - (P_x \cup P_y)$ such that $|A_0| = |B_0| = \lceil n/2 \rceil - k$. Now P_x and P_y are joined in $A_0 \cup B_0$ into a good x, y-path $P^{**} = (x, x_1, x^*, \ldots, y^*, y)$. The vertices of P^{**} belong to $A \cup B$ with the possible exception of two among x, y and x_1. Thus we obtain $|P^{**} \cap A| \geq |P^{**} \cap B| \geq \lfloor (k-1)/2 \rfloor$.

Let C^{**} be a maximum length dominating good cycle containing P^{**}, that is given by Claim 3. Set $C^{**} = P^{**} \cup Q^{**}$, $H^{**} = G - C^{**}$ and assume that $w \in H^{**} \cap (A \cup B)$. Observe that $d_{P^{**} - \{x,y\}}(w) \leq k - 1 - \lfloor (k-1)/2 \rfloor \leq k/2$, and $d_{Q^{**}}(w) \leq (m - k + 2)/2 \leq (n - k + 1)/2$. Then it follows that $d_G(w) = d_{P^{**} - \{x,y\}}(w) + d_{Q^{**}}(w) < n/2 + 1$, a contradiction. Thus we conclude that $w \notin A \cup B$.

Then there is a largest independent set $A(C^{**})$ with at least $n/2 - k + 2$ vertices and containing $H^{**} = G - C^{**}$, thus disjoint from $A \cup B$. Since $3(n/2 - k + 2) > n$, this leads to a contradiction and completes the proof of Theorem 3.

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Received: July 10, 2012