Set-orderedness as a generalization of k-orderedness and cyclability

Keishi Ishii

Kenta Ozeki*

National Institute of Informatics,
Tokyo 101-8430, Japan
e-mail: ozeki@nii.ac.jp

Kiyoshi Yoshimoto

Department of Mathematics,
Collage of Science and Technology
Nihon University, Tokyo 101-8308, Japan
e-mail: yosimoto@math.cst.nihon-u.ac.jp

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Abstract

A graph G is called k-ordered if for any sequence of k distinct vertices of G, there exists a cycle in G through these vertices in the order. A vertex set S is called cyclable in G if there exists a cycle passing through all vertices of S. We will define "set-orderedness" which is a natural generalization of k-orderedness and cyclability. We also give a degree sum condition for graphs to satisfy "set-orderedness". This is an extension of well-known sufficient conditions on k-orderedness.

Keywords: k-ordered, cyclable, degree sum

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1 Introduction

A cycle-related property, for instance, a hamilton cycle have been studied for a long time, and as an extension of it, a cycle passing all specified vertices is also widely studied. Many researchers study this type cycle from two aspects; one of them is a cycle passing specified vertices in a given order, another is that without considering the order.

The first one is the notion of *k-orderedness*, which was first introduced by Chartrand. A graph G is called *k-ordered* if for any sequence of k distinct vertices of G, there exists a cycle in G passing through these specified vertices in the order. The second one is the notion of *cyclability*. For any subset S of $V(G)$, S is called *cyclable in G* if there exists a cycle through all vertices of S. Many results of these two concepts are known, see [4, 5, 6, 7, 8, 10, 11, 12, 13] for k-orderedness and [1, 3, 9, 14, 15] for cyclability.

Note that k-orderedness is a stronger concept than cyclability. In this sense, there seems to exist a close relationship between these two concepts, however, this relationship was not studied. The purpose of this paper is to interpolate these concepts. In Section 2, we introduce a new concept *set-orderedness*, which is a natural generalization of k-orderedness.

2 Set-orderedness

In this paper, we consider only finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [2].

The following result is a classical one on k-orderedness by Ng and Schultz. Note that they proved that the same condition as Theorem 1 guarantees the existence of a hamilton cycle passing through specified k vertices in the given order.

Theorem 1 (Ng and Schultz [12]) *Let G be a graph of order $n \geq 3$ and let k be an integer with $3 \leq k \leq n$. If*

$$d_G(u) + d_G(v) \geq n + 2k - 6$$

for any two non-adjacent vertices u and v, then G is k-ordered.

The bound of the degree condition was improved for small k with respect to n by Faudree et al. [6]

Theorem 2 (Faudree et al. [6]) *Let k be an integer with $3 \le k \le n/2$ and let G be a graph of order n. If*

$$d_G(u) + d_G(v) \ge n + 3(k-3)/2$$

for any two non-adjacent vertices u and v, then G is k-ordered.

Let $v_1, v_2, \ldots, v_k$ be k distinct vertices of G. A graph G is called $(v_1, v_2, \ldots, v_k)$-*ordered* if there exists a cycle containing these k vertices in this order. (See Figure 1 (i).) Definitely, a graph G is k-ordered if and only if G is $(v_1, v_2, \ldots, v_k)$-ordered for any distinct vertices $v_1, v_2, \ldots, v_k$. In order to show k-orderedness of a given graph, we need the degree sum condition for all non-adjacent vertices. because we must consider all combinations and orders of k distinct vertices. However, considering only given k vertices $v_1, v_2, \ldots, v_k$ and a cycle through them in such a order, we may be able to restrict the vertices on which we must deal with the degree sum condition to the given k vertices. In fact, Ng and Schultz [12] found the degree sum condition on given k vertices which guarantees the existence of a path passing them in the given order. As a corollary of it, we obtain the following result.

Theorem 3 *Let G be a graph of order $n \ge 3$ and let $v_1, v_2, \ldots, v_k$ be k distinct vertices of G with $k \ge 3$. If*

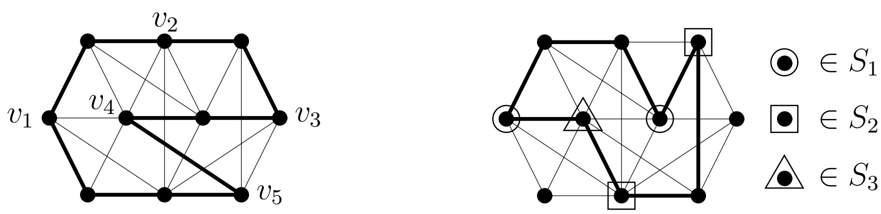
$$d(v_i) + d(v_{i+1}) \ge n + 2k - 6$$

for any $1 \le i \le k$ (regarding v_{k+1} as v_1), then G is $(v_1, v_2, \ldots, v_k)$-ordered.

While Theorem 2 shows that Theorem 1 is not sharp, the following example shows the sharpness of Theorem 3. Let k be even integer and n be an odd integer. Consider the graph G which is obtained from k vertices $v_1, v_2, \ldots, v_k$ with all possible edges between them except for $v_i v_{i+1}$ for $1 \le i \le k$ by adding $n-k$ vertices and joining $k-1$ vertices of them to $v_1, v_2, \ldots, v_{k-1}, v_k$, half of remaining $n-2k+1$ vertices to $v_1, v_3, \ldots, v_{k-1}$ and another half to $v_2, v_4, \ldots, v_k$.

As an extension of $(v_1, v_2, \ldots, v_k)$-orderedness, we will define the concept of set-orderedness. In the concept of k-orderedness or $(v_1, v_2, \ldots, v_k)$-orderedness, we must find a cycle passing through the specified vertices in the prescribed order. In this sense, we consider a relaxation of cycles, that is, a cycle passes specified vertices in a "partially desired" order.

Let $S_1, S_2, \ldots, S_l$ be disjoint nonempty vertex sets in a graph G with $\sum_{i=1}^{l} |S_i| = k$. A graph G is called $(S_1, S_2, \ldots, S_l)$-*ordered* if there exists a cycle in G through all vertices of $S_1 \cup S_2 \cup \cdots \cup S_l$ in the order, that is, any vertex of S_j appears in the cycle after any vertex of S_i if $1 \leq i < j \leq l$. (See Figure 1 (ii).)



(i) a $(v_1, v_2, \cdots, v_5)$-ordered graph (ii) an (S_1, S_2, S_3)-ordered graph

Figure 1:

By the definition of $(S_1, S_2, \ldots, S_l)$-orderedness, in case of $l = k$, $(S_1, S_2, \ldots, S_l)$-orderedness means $(v_1, v_2, \ldots, v_k)$-orderedness where $S_i = \{v_i\}$ for $1 \leq i \leq l$. On the other hand, in case of $l = 1$, (S_1)-orderedness is equivalent to cyclability of S_1. In this sense, the concept of $(S_1, S_2, \ldots, S_l)$-orderedness connects k-orderedness and cyclability.

We define *a path cover of* G as a set of paths which are pairwise disjoint and contain all vertices of G. *The path cover number,* denoted by $\mathrm{pc}(G)$, is the minimum number of $|\mathcal{P}|$ among all path covers $\mathcal{P}$ of G. Let $S \subset V(G)$. For convenience, we call a *path cover of* S instead of a path cover of $G[S]$ and denote $\mathrm{pc}(S)$ instead of $\mathrm{pc}(G[S])$. Throughout this paper, the index i is always taken modulo l.

Theorem 4 *Let G be a graph on n vertices and $l \geq 2$ and let $S_1, S_2, \ldots, S_l$ be disjoint nonempty vertex sets. Let $s_i := |S_i|$, $p_i := \mathrm{pc}(S_i)$, $k := \sum_{i=1}^{l} s_i$, $p := \sum_{i=1}^{l} p_i$, $\overline{s_i} := \sum_{j \neq i,i+1} s_j$ and $\overline{p_i} := \sum_{j \neq i,i+1} p_j$. Suppose that for each i $(1 \leq i \leq l)$,*

$$d_G(u) + d_G(v) \geq n + k + p - (s_i + p_i + l)$$

for every pair of non-adjacent vertices $u, v \in S_i$, and

$$d_G(u) + d_G(v) \geq n + \overline{s_i} + \overline{p_i} - 2 - \varepsilon_i$$

for every pair of non-adjacent vertices u, v such that $u \in S_i$ and $v \in S_{i+1}$, where

$$\varepsilon_i := \begin{cases} -2 & \text{if } l = 2, \\ 0 & \text{if } l = 3, 4, \\ & \text{or if } l \geq 5 \text{ and } s_{i-1} = s_{i+2} = 1, \\ 1 & \text{if } l = 5, 6 \text{ and at least one of } s_{i-1} \text{ and } s_{i+2} \text{ is at least } 2, \\ & \text{or if } l \geq 7 \text{ and exactly one of } s_{i-1} \text{ and } s_{i+2} \text{ is at least } 2, \\ 2 & \text{if } l \geq 7 \text{ and both } s_{i-1} \text{ and } s_{i+2} \text{ are at least } 2. \end{cases}$$

Then G is $(S_1, S_2, \ldots, S_l)$-ordered.

Consider the case $l = k \geq 3$. Then each S_i consists of only one vertex, say v_i, and hence we have $s_i = p_i = 1$, $\overline{s_i} = \overline{p_i} = k-2$ and $\varepsilon_i = 0$. Then we can not take any pair of non-adjacent vertices $u_i, v_i \in S_i$ because $s_i = 1$, and hence the first degree condition is vacuous. The second degree condition in Theorem 4 is

$$\begin{aligned} d(u) + d(v) &\geq n + \overline{s_i} + \overline{p_i} - 2 - \varepsilon_i \\ &= n + 2k - 6 \end{aligned}$$

for all pair of non-adjacent vertices $u \in S_i$ and $v \in S_{i+1}$. Therefore, we obtain theorem 3 as a corollary.

In Section 3, we will prove Theorem 4, and in Section 4, we will explain the sharpness of Theorem 4.

3 Proofs

Theorem 4 for the case $l = 2$ can be proved by the same argument as the case $l \geq 3$, and hence we omit the proof. In this section, we only prove Theorem 4 for the case $l \geq 3$.

Throughout this section, the index j is also taken modulo l. Let $S \subset V(G)$. We call a path P *an S-dense path* if $S \subset V(P)$ and $|V(P)| = |S| + \mathrm{pc}(S) - 1$, that is, an S-dense path is a shortest possible path through S, given $\mathrm{pc}(S)$.

Lemma 5 *Let G be a graph of order $n \geq 3$ and let $S \subset V(G)$. If $d_G(u) + d_G(v) \geq n - 1$ for every pair of non-adjacent vertices $u, v \in S$, then there exists an S-dense path P.*

Proof of Lemma 5. Let $\mathcal{P} := \{P_1, P_2, \ldots, P_l\}$ be a path cover of S such that $l = \mathrm{pc}(S)$ and let u_i and v_i be the end-vertices of P_i. Possibly $u_i = v_i$. We give an orientation to each path P_i from u_i to v_i and write the oriented path P_i by $\overrightarrow{P_i}$. In addition, the reverse orientation of $\overrightarrow{P}$ is denoted by $\overleftarrow{P}$. Since $\mathcal{P}$ is a minimum path cover of S, $u_iu_j, u_iv_j, v_iv_j \notin E(G)$ for $i \neq j$. Let $T := V(G) - \bigcup_{i=1}^{l} V(P_i)$. Now we will show that $|N_T(u_i) \cap N_T(v_j)| \geq l-1$ for $i \neq j$.

Fix i and j with $1 \leq i \neq j \leq l$. Suppose that $N_{P_i}(u_i)^- \cap N_{P_i}(v_j) \neq \emptyset$, say $w \in N_{P_i}(u_i)^- \cap N_{P_i}(v_j)$. Let $P := v_i \overleftarrow{P_i} w^+ u_i \overrightarrow{P_i} w v_j \overleftarrow{P_j} u_j$. Then $\mathcal{Q} := \{P_h : h \neq i, j\} \cup \{P\}$ is also a path cover of S with $|\mathcal{Q}| < |\mathcal{P}|$, contradicting the minimality of $\mathcal{P}$. Thus, $N_{P_i}(u_i)^- \cap N_{P_i}(v_j) = \emptyset$. Since $N_{P_i}(u_i)^- \cup N_{P_i}(v_j) \subset V(P_i) - \{v_i\}$, we have

$$d_{P_i}(u_i) + d_{P_i}(v_j) \leq |V(P_i)| - 1. \tag{1}$$

By symmetry of i and j, we obtain

$$d_{P_j}(u_i) + d_{P_j}(v_j) \leq |V(P_j)| - 1. \tag{2}$$

Next, suppose that $N_{P_h}(u_i)^- \cap N_{P_h}(v_j) \neq \emptyset$ for $h \neq i, j$, say $w \in N_{P_h}(u_i)^- \cap N_{P_h}(v_j)$. Let $P := v_i \overleftarrow{P_i} u_i w^+ \overrightarrow{P_h} v_h$ and $P' := u_j \overrightarrow{P_j} v_j w \overleftarrow{P_h} u_h$. Then $\mathcal{Q} := \{P_t : t \neq h, i, j\} \cup \{P, P'\}$ is also a path cover of S, a contradiction. Thus, $N_{P_h}(u_i)^- \cap N_{P_h}(v_j) = \emptyset$. Since $N_{P_h}(u_i)^- \cup N_{P_h}(v_j) \subset V(P_h) - \{v_h\}$, we have

$$d_{P_h}(u_i) + d_{P_h}(v_j) \leq |V(P_h)| - 1. \tag{3}$$

By the inequalities (1) – (3), we reduce

$$\sum_{h=1}^{l} \bigl(d_{P_h}(u_i) + d_{P_h}(v_j)\bigr) \leq \sum_{h=1}^{l} \bigl(|V(P_h)| - 1\bigr) = \sum_{h=1}^{l} |V(P_h)| - l.$$

Then by the degree condition,

$$\begin{aligned} d_T(u_i) + d_T(v_j) &\geq n - 1 - \Bigl(\sum_{h=1}^{l} |V(P_h)| - l\Bigr) \\ &= |T| + l - 1, \end{aligned}$$

and hence $|N_T(u_i) \cap N_T(v_j)| \geq l - 1$.

Therefore, we can find $l-1$ distinct vertices $w_1, w_2, \ldots, w_{l-1} \in T$ such that $w_i \in N_T(u_{i+1}) \cap N_T(v_i)$ for $1 \leq i \leq l-1$. Then $P = u_1 \overrightarrow{P_1} v_1 w_1 u_2 \overrightarrow{P_2} v_2 w_2 \ldots w_{l-1} u_l \overrightarrow{P_l} v_l$ is a path such that $S \subset V(P)$ and

$$|V(P)| = \sum_{h=1}^{l} |V(P_h)| + l - 1 = |S| + \mathrm{pc}(S) - 1. \quad \Box$$

For the proof of our main theorem, we need the following lemma. This follows from Lemma 5 by a straight forward induction on l.

Lemma 6 *Let G be a graph of order $n \geq 3$ and let $l \geq 1$. Let $S_1, S_2, \dots, S_l$ be disjoint nonempty vertex sets. Let $s_i := |S_i|$, $p_i := \mathrm{pc}(S_i)$, $k := \sum_{i=1}^{l} s_i$ and $p := \sum_{i=1}^{l} p_i$. Suppose that for each i $(1 \leq i \leq l)$ and for every pari of non-adjacent vertices $u, v \in S_i$,*

$$d_G(u) + d_G(v) \geq n + k + p - (s_i + p_i + l).$$

There exist l disjoint paths $P_1, P_2, \dots, P_l$ such that P_i is an S_i-dense path for each $1 \leq i \leq l$.

Proof of Theorem 4. By Lemma 6, there exist l disjoint paths $P_1, P_2, \dots, P_l$ such that P_i is an S_i-dense path for each $1 \leq i \leq l$. Let u_i and v_i be the end-vertices of P_i and let $T := V(G) - \bigcup_{i=1}^{l} V(P_i)$. Note that $v_i = u_i$ if $s_i = 1$. Now we will connect P_i and P_{i+1}. First, if v_i and u_{i+1} are adjacent, then using the edge $v_i u_{i+1}$, we can join two paths P_i and P_{i+1}. We call this operation *Operation* 1 *on* (v_i, u_{i+1}).

Next, suppose that v_i and u_{i+1} are not adjacent and $N_T(v_i) \cap N_T(u_{i+1}) \neq \emptyset$, say $w_i \in N_T(v_i) \cap N_T(u_{i+1})$. If w_i is not in use for other pairs, then we can connect v_i and u_{i+1} by using w_i. After connecting v_i and u_{i+1}, we obtain the path $u_i \overrightarrow{P_i} v_i w_i u_{i+1} \overrightarrow{P_{i+1}} v_{i+1}$. We call this operation *Operation* 2 *on* (v_i, u_{i+1}). (See Figure 2.)

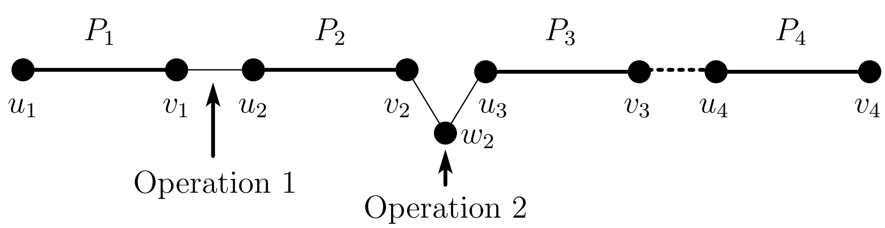


Figure 2: Operations 1 and 2.

By repeating Operations 1 and 2 for $\bigcup_{i=1}^{l} P_i$, we obtain a cycle or a union of paths, denoted by P. Note that P_i is contained in P as a subpath, and P_{i+1} lies on P next to P_i if v_i and u_{i+1} are connected by one of the operations. Let $T' := V(G) - V(P)$. If P is a cycle, then there is nothing to prove. Thus we may assume that there exists a pair (v_i, u_{i+1}) on which we can perform neither Operation 1 nor 2. Then $v_i u_{i+1} \notin E(G)$ and $N_{T'}(v_i) \cap N_{T'}(u_{i+1}) = \emptyset$. We also give an orientation to P and for $x \in V(P)$, we define x^+ as the successor of x along $\overrightarrow{P}$. Note that $v_j^+ := w_j$

if Operation 2 is performed on (v_j, u_{j+1}), and we define $v_j^+ := u_{j+1}$ even if neither Operation 1 nor 2 are performed on (v_j, u_{j+1}).

Choose such dense paths $P_1, P_2, \dots, P_l$, such a union P of paths, which is obtained by repeating Operations 1 and 2, and a pair (v_i, u_{i+1}) on which we can perform neither Operation 1 nor 2 so that

(P1) Operation 1 is performed as many times as possible,

(P2) Operation 2 is performed as many times as possible; subject to (P1).

In addition to (P1) and (P2), we choose $P_1, .., P_l, P$ and (v_i, u_{i+1}) so that

(P3) The number of performing Operation 2 on (v_{i-1}, u_i) and (v_{i+1}, u_{i+2}) is as large as possible; subject to (P2).

Without loss of generality, we may assume that $i = l$. Thus, $u_1v_l \notin E(G)$ and $N_{T'}(u_1) \cap N_{T'}(v_l) = \emptyset$. Since $N_{T'}(u_1) \cap N_{T'}(v_l) = \emptyset$, we have

$$d_{T'}(u_1) + d_{T'}(v_l) \leq |T'|. \tag{4}$$

Let Q_1 and Q_l be parts of P from u_1 to u_2 and from v_{l-1} to v_l, respectively. (See Figure 3.)

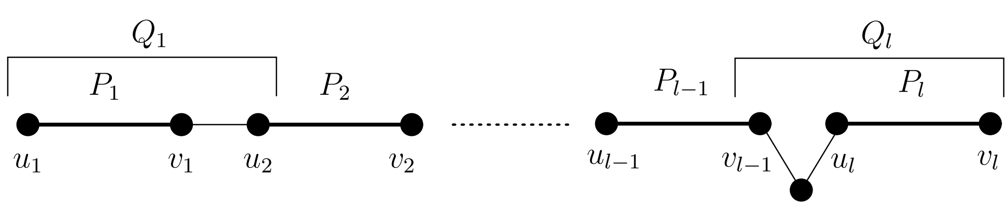


Figure 3: Q_1 and Q_l.

Claim 1 *$d_{Q_1}(u_1) + d_{Q_1}(v_l) \leq |V(Q_1)|$ and $d_{Q_l}(u_1) + d_{Q_l}(v_l) \leq |V(Q_l)|$.*

Proof. Suppose that $N_{Q_1}(u_1)^- \cap N_{Q_1}(v_l) \neq \emptyset$, say $w \in N_{Q_1}(u_1)^- \cap N_{Q_1}(v_l)$. Then we can replace $\overrightarrow{Q_l} \cup \overrightarrow{Q_1}$ with $v_{l-1}\overrightarrow{Q_l}v_lw\overleftarrow{Q_1}u_1w^+\overrightarrow{Q_1}u_2$, contradicting the choice of P. Hence $N_{Q_1}(u_1)^- \cap N_{Q_1}(v_l) = \emptyset$. This implies that $d_{Q_1}(u_1) + d_{Q_1}(v_l) \leq |V(Q_1)|$. By considering the reverse orientation $\overleftarrow{P}$, we have $d_{Q_l}(u_1) + d_{Q_l}(v_l) \leq |V(Q_l)|$. $\square$

Let w_j be the vertex connecting v_j and u_{j+1} under Operation 2 on (v_j, u_{j+1}), if Operation 2 is performed on (v_j, u_{j+1}). Let

$$W := \{w_j : \text{Operation 2 is performed on } (v_j, u_{j+1}), \text{ and } j \neq 1, l-1, l\}$$

and let $\eta := |W|$.

Let $P' := P - (Q_1 \cup Q_l)$. Since $|V(P_j)| = s_j + p_j - 1$, we have

$$|V(P')| \;=\; \overline{s_l} + \overline{p_l} - l + \eta. \tag{5}$$

Let $r := 2|V(P')| - \big(d_{P'}(u_1) + d_{P'}(v_l)\big) - \varepsilon_l$. By the definition of ε_l, note that $r \geq -2$. The following claim holds.

Claim 2 *$l - 3 \geq \eta \geq l - 2 + r$. In particular, $r \leq -1$.*

Proof. It is clear that $l - 3 \geq \eta$ by the definition of η. Suppose that $\eta \leq l - 3 + r$. Then by (4), (5) and Claim 1, we have

$$\begin{aligned}
&d_G(u_1) + d_G(v_l)\\
&= d_{T'}(u_1) + d_{T'}(v_l) + d_{Q_1}(u_1) + d_{Q_1}(v_l)\\
&\qquad + d_{Q_l}(u_1) + d_{Q_l}(v_l) + d_{P'}(u_1) + d_{P'}(v_l)\\
&\leq |T'| + |V(Q_1)| + |V(Q_l)| + 2|V(P')| - r - \varepsilon_l\\
&= n + |V(P')| - r - \varepsilon_l\\
&= n + \overline{s_l} + \overline{p_l} - l + \eta - r - \varepsilon_l\\
&\leq n + \overline{s_l} + \overline{p_l} - 2 - \varepsilon_l - 1,
\end{aligned}$$

a contradiction. Thus, $\eta \geq l-2+r$. Moreover, since $\eta \leq l-3$, we have $l-3 \geq l-2+r$, or $r \leq -1$. $\square$

Since $2|V(P')| \geq d_{P'}(u_1) + d_{P'}(v_l)$, we have $r \geq -\varepsilon_l$. Therefore by Claim 2, the case $\varepsilon_l = 0$ is done. Thus, we may assume that $\varepsilon_l \geq 1$, in particular, $l \geq 5$.

We also have the following claim. The proof of them is obvious, and hence we leave it to the reader.

Claim 3 (i) *If $v_2 \in N_G(u_1)$, then Operation 1 is performed on at least one of the pairs (v_1, u_2) and (v_2, u_3).*

(i) *If $u_{l-1} \in N_G(v_l)$, then Operation 1 is performed on at least one of the pairs (v_{l-2}, u_{l-1}) and (v_{l-1}, u_l).*

We divide the rest of the proof into three cases.

Case 1. $v_2 \notin N_G(u_1)$ and $u_{l-1} \notin N_G(v_l)$.

If $s_2 \geq 2$, then $v_2 \in V(P')$, and if $s_{l-1} \geq 2$, then $u_{l-1} \in V(P')$. Thus, by the definition of ε_l, we have $d_{P'}(u_1) + d_{P'}(v_l) \leq 2|V(P')| - \varepsilon_l$. This implies that $r \geq 0$, contradicting Claim 2. $\square$

Case 2. $v_2 \notin N_G(u_1)$ and $u_{l-1} \in N_G(v_l)$, or $v_2 \in N_G(u_1)$ and $u_{l-1} \notin N_G(v_l)$.

By symmetry, we may assume that $v_2 \in N_G(u_1)$ and $u_{l-1} \notin N_G(v_l)$. Then $u_{l-1} \notin N_G(v_l)$ implies that $d_{P'}(u_1) + d_{P'}(v_l) \leq 2|V(P')| - 1$ or $\varepsilon_l = 1$. In each case, we have $r \geq -1$, and hence $r = -1$ and $\eta = l - 3$ by Claim 2.

Since $\eta = l - 3$, Operation 2 is performed on (v_j, u_{j+1}) for every $2 \leq j \leq l - 2$. By Claim 3 (i), Operation 1 is performed on (v_1, u_2).

Suppose that $w_{l-2} \in N_G(u_1) \cap N_G(v_l)$. Then using w_{l-2} in order to connect between u_1 and v_l, we can take a union of paths $P - \{v_{l-2}w_{l-2}, w_{l-2}u_{l-1}\} \cup \{u_1 w_{l-2}, w_{l-2} v_l\}$, contradicting the choice (P3), because Operation 1 is performed on (v_1, u_2) and Operation 2 is performed on (v_{l-3}, u_{l-2}). (See Figure 4.) Thus, we obtain $w_{l-2} \notin N(u_1) \cap N(v_l)$, and this implies that $d_{P'}(u_1) + d_{P'}(v_l) \leq 2|V(P')| - 2$, or $d_{P'}(u_1) + d_{P'}(v_l) \leq 2|V(P')| - 1$ and $\varepsilon_l = 1$. Then $r \geq 0$, which contradicts Claim 2. $\square$

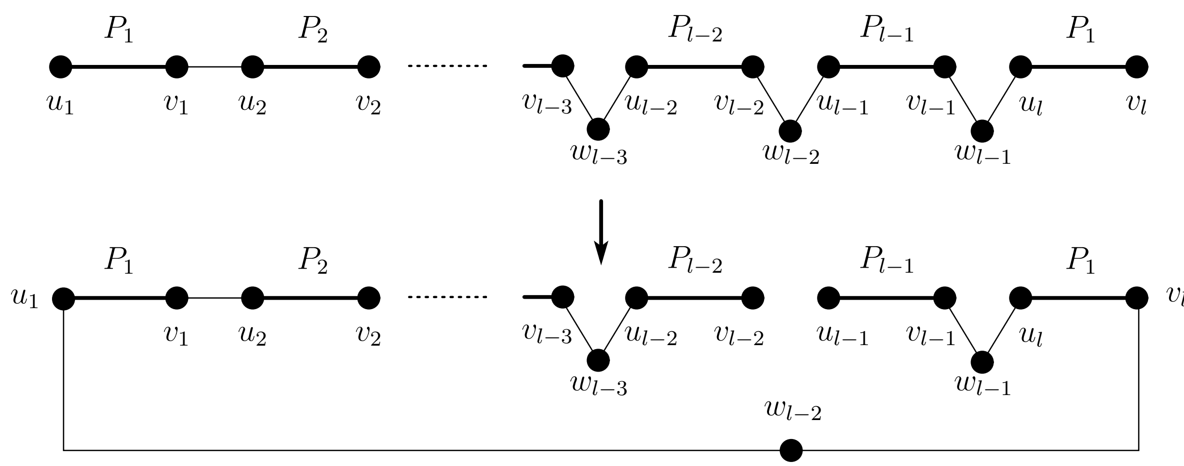


Figure 4:

Case 3. $v_2 \in N_G(u_1)$ and $u_{l-1} \in N_G(v_l)$.

Case 3.1. $l \geq 7$.

Since $r \geq -2$, we have $\eta \geq l - 4$ by Claim 2. Therefore on at least one of the pairs (v_2, u_3) and (v_{l-2}, u_{l-1}), Operation 2 is performed. By symmetry, we may assume that Operation 2 is performed on (v_2, u_3). This implies that on (v_1, u_2), Operation 1 is performed by Claim 3 (i).

Suppose that Operation 1 is not performed on (v_{l-2}, u_{l-1}). Then on (v_{l-1}, u_l), Operation 1 is also performed, by Claim 3 (ii). Since $\eta \geq l-4$ and $l \geq 7$, there exist consecutive pairs (v_j, u_{j+1}) and (v_{j+1}, u_{j+2}) such that Operation 2 is performed on both pairs. If $w_j \in N_G(u_1) \cap N_G(v_l)$, then we can change P with $P - \{v_{j-1}w_j, w_ju_j\} \cup \{v_lw_j, w_ju_1\}$, which contradicts the choice (P3), because Operation 1 is performed on both (v_1, u_2) and (v_{l-1}, u_l). Thus, $w_j \notin N_G(u_1) \cap N_G(v_l)$ and by symmetry, $w_{j+1} \notin N_G(u_1) \cap N_G(v_l)$. Therefore $d_{P'}(u_1) + d_{P'}(v_l) \leq 2|V(P')| - 2$, and hence $r \geq 0$, which contradicting Claim 2.

Thus we may assume that Operation 1 is performed on (v_{l-2}, u_{l-1}). Then $\eta = l-4$. This implies that for any $2 \leq j \leq l-3$, Operation 2 is performed on (v_j, u_{j+1}). Since $\eta = l-4 \geq 3$, there exist three consecutive pairs (v_{j-1}, u_j), (v_j, u_{j+1}) and (v_{j+1}, u_{j+2}) such that Operation 2 is performed on every pair. By the same argument as above, $w_j \notin N_G(u_1) \cap N_G(v_l)$. Therefore $d_{P'}(u_1) + d_{P'}(v_l) \leq 2|V(P')| - 1$, and hence $r \geq -1$. This contradicts Claim 2 together with $\eta = l-4$. □

Case 3.2. $l = 5$ or $l = 6$.

In these cases, note that $\varepsilon_l = 1$. Therefore $r = -1$ and $\eta = l-3$, by the definition of η and Claim 2. This implies that on both (v_2, u_3) and (v_{l-2}, u_{l-1}), Operation 2 is performed. Then by Claims 3 (i) and (ii), Operation 1 is performed on both (v_1, u_2) and (v_{l-1}, u_l). Moreover since $\eta = l-3$, we can find consecutive pairs (v_j, u_{j+1}) and (v_{j+1}, u_{j+2}) such that Operation 2 is performed on both pairs. By the same argument as Case 3.1, $w_j, w_{j+1} \notin N_G(u_1) \cap N_G(v_l)$ and hence $d_{P'}(u_1) + d_{P'}(v_l) \leq 2|V(P')| - 2$, a contradiction again. □

4 Examples

In this section, we will show that almost all of degree sum conditions of Theorem 4 are best possible. (In Examples 1 and 6, the orders of the graph G_1 and G_2 depend on the cardinalities of specified vertices.) Throughout this section, we use S_1, S_2, $\ldots$, S_l as disjoint vertex sets with $|S_i| = s_i$ and $\sum_{i=1}^{l} s_i = k$. The first example shows that the first degree sum condition of Theorem 4 is best possible.

Example 1: Let S_1, S_2, …, S_l be partite sets of some complete l-partite graph. We construct a graph G_1 by adding $(k-l-1)$ new vertices and joining them to all vertices of $S_1 \cup \ldots \cup S_l$. Then $p_i = s_i$ for all i and $k = p$. Note that the second degree condition is vacuously true.

We need $\sum_{i=1}^{l} \big(s_i - 1\big) = k - l$ vertices to obtain l disjoint paths such that the i-th path has all vertices of S_i. Thus, G_1 is not $(S_1, S_2, \ldots, S_l)$-ordered. On the other hand, for every i with $1 \le i \le l$ for every pair of non-adjacent vertices $u, v \in S_i$,

$$\begin{aligned} d_{G_1}(u) + d_{G_1}(v) &= 2\big(|V(G_1)| - s_i\big) \\ &= |V(G_1)| + \big(2k - l - 1\big) - 2s_i \\ &= |V(G_1)| + k + p - \big(s_i + p_i + l\big) - 1, \end{aligned}$$

and hence we cannot decrease the value of the first degree sum condition without breaking the conclusion.

Next we will show that the lower bound of the second condition of Theorem 4 is also sharp. In order to show that, we have to consider some cases depending on the value of l. Note that in Examples 2–5, the first degree sum condition is vacuously true.

Example 2: Let $l = 3$ and let S_i be disjoint cliques for $1 \le i \le 3$. We connect every vertex of S_i and every vertex of S_{i+1} for $i = 1, 2$. Moreover, we add $(n-k)$ new vertices and join some of them to $S_1 \cup S_2$ and others to $S_2 \cup S_3$. Let G_2 be a graph obtained by above construction. Then $|V(G_2)| = n$ and $p_i = 1$ for any $1 \le i \le 3$.

Since we cannot pass a vertex of S_1 after a vertex of S_3 without passing a vertex of S_2, G_2 is not (S_1, S_2, S_3)-ordered. On the other hand, for every pair of $u \in S_1$ and $v \in S_3$,

$$\begin{aligned} d_{G_2}(u) + d_{G_2}(v) &= (s_1 - 1 + s_2) + (s_2 + s_3 - 1) + (n - k) \\ &= n + k - s_1 - s_3 - 2 \\ &= |V(G_2)| + \overline{s_3} + \overline{p_3} - 2 - \varepsilon_i - 1, \end{aligned}$$

and hence when $l = 3$, we cannot decrease the value of ε_i.

Example 3: Let $l = 4$ and let S_i be disjoint cliques for $1 \le i \le 4$. We connect all pairs of S_i and S_j except for S_1 and S_4, and S_2 and S_3. Moreover, we add $n-k$ new vertices and join one vertex of them to $\bigcup_{i=1}^{4} S_i$, some of others to $S_1 \cup S_2$, and the remaining vertices to $S_3 \cup S_4$. Let G_3 be a graph obtained by above construction. Then $|V(G_3)| = n$ and $p_i = 1$ for any $1 \leq i \leq 4$.

Since we can use only one vertex to connect S_2 and S_3, or S_4 and S_1, G_3 is not (S_1, S_2, S_3, S_4)-ordered. On the other hand, for every pair of $u \in S_1$ and $v \in S_4$,

$$\begin{aligned} d_{G_3}(u) + d_{G_3}(v) &= (k - s_4 - 1 + 1) + (k - s_1 - 1 + 1) + (n - k - 1) \\ &= n + k - s_1 - s_4 - 1 \\ &= |V(G_3)| + \overline{s_4} + \overline{p_4} - 2 - \varepsilon_4 - 1, \end{aligned}$$

and hence when $l = 4$, we cannot decrease the value of ε_i.

Example 4: Let $l = 5$ and let S_i be disjoint cliques for $1 \leq i \leq 5$ with $s_4 \geq 2$ or $s_2 \geq 2$. We connect all pairs of S_i and S_j except for S_1 and S_5, and S_3 and S_4. Moreover, we add new $n-k$ vertices and join one vertex of them to $\bigcup_{i=1}^{5} S_i$, some of others to $S_1 \cup S_2 \cup S_3$, and the remaining vertices to $S_2 \cup S_4 \cup S_5$. Let G_4 be a graph obtained by above construction. Then $|V(G_4)| = n$ and $p_i = 1$ for any $1 \leq i \leq 5$.

By the same reason as G_3, G_4 is not $(S_1, \ldots, S_5)$-ordered. On the other hand, for every pair of $u \in S_1$ and $v \in S_5$,

$$\begin{aligned} d_{G_4}(u) + d_{G_4}(v) &= (k - s_5 - 1 + 1) + (k - s_1 - 1 + 1) + (n - k - 1) \\ &= n + k - s_1 - s_5 - 1 \\ &= |V(G_4)| + \overline{s_5} + \overline{p_5} - 2 - \varepsilon_5 - 1, \end{aligned}$$

and hence when $l = 5$, and $s_{i-1} \geq 2$ or $s_{i+2} \geq 2$, we cannot decrease the value of ε_i.

Example 5: Let $l = 6$ and let S_i be disjoint cliques for $1 \leq i \leq 6$ with $s_5 \geq 2$ or $s_2 \geq 2$. We connect all pairs of S_i and S_j except for S_1 and S_6, S_2 and S_3, and S_4 and S_5. We add $n-k$ new vertices, and join two vertices of them to $\bigcup_{i=1}^{6} S_i$, some of others to $S_1 \cup S_3 \cup S_5$, and the remaining vertices to $S_2 \cup S_4 \cup S_6$. Let G_5 be a graph obtained by above construction. Then $|V(G_5)| = n$ and $p_i = 1$ for any $1 \leq i \leq 6$.

Again, G_5 is not $(S_1, \ldots, S_6)$-ordered, and for every pair of $u \in S_1$ and $v \in S_6$,

$$\begin{aligned} d_{G_5}(u) + d_{G_5}(v) &= (k - s_6 - 1 + 2) + (k - s_1 - 1 + 2) + (n - k - 2) \\ &= n + k - s_1 - s_6 \\ &= |V(G_5)| + \overline{s_6} + \overline{p_6} - 2 - \varepsilon_6 - 1, \end{aligned}$$

and hence when $l = 6$, and $s_{i-1} \geq 2$ or $s_{i+2} \geq 2$, we cannot decrease the value of ε_i.

Example 6: Let $l \geq 7$ and $s_i = 2$ or $s_i = 3$ for all $1 \leq i \leq l$. We define H_i as a path $u_i u_i' v_i$ if $s_i = 3$, and in case of $s_i = 2$, define H_i as an edge $u_i v_i$. By connecting all pairs of vertices in H_i and H_j, and removing $4l$ edges $\{u_i u_{i+1}, u_i v_{i+1}, v_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq l\}$, we obtain a graph H. Then by adding $(l-1)$ new vertices to a graph H and joining them to all other vertices, we construct a graph G_6. Then $|V(G_6)| = k+l-1$, $p_i = 1$ and $\overline{p_i} = l-2$ for any $1 \leq i \leq l$.

Because at least one vertex not in S_i is necessary to connect S_i and S_{i+1}, G_6 is not $(S_1, \ldots, S_l)$-ordered, and for every pair of $u_i \in S_i$ and $v_{i+1} \in S_{i+1}$,

$$\begin{aligned} d_{G_6}(u_i) + d_{G_6}(v_{i+1}) &= \Big(k - s_i - 4 + 1 + l - 1\Big) + \Big(k - s_{i+1} - 4 + 1 + l - 1\Big) \\ &= n + k - s_i - s_{i+1} + p - 7 \\ &= |V(G_6)| + \overline{s_6} + \overline{p_6} - 2 - \varepsilon_6 - 1. \end{aligned}$$

Hence when $s_i = 2$ or $s_i = 3$ for all $1 \leq i \leq l$, we cannot decrease the value of ε_i.

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