Equivalence of Jackson's and Thomassen's conjectures

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Abstract

A graph G is said to be claw-free if G has no induced subgraph isomorphic to $K_{1,3}$. For a cycle C in a graph G, C is called a Tutte cycle of G if C is a Hamilton cycle of G, or the order of C is at least 4 and every component of $G - C$ has at most three neighbors on C. In [On a closure concept in claw-free graphs, J. Combin. Theory Ser. B **70** (1997), 217–224], Ryjáček proved that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is Hamiltonian) and by Thomassen (every 4-connected line graph is Hamiltonian) are equivalent. In this paper, we show the above conjectures are equivalent with the conjecture by Jackson in 1992 (every 2-connected claw-free graph has a Tutte cycle).

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1 Introduction

In this paper, we consider finite graphs without loops. For terminology and notation not defined in this paper, we refer the readers to [5]. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. The *degree* of a vertex v of G is the number of edges incident with v in G, and we denote by $\delta(G)$ the minimum degree of G. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph induced by X in G, and let $G - X = G[V(G) - X]$. For

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a subgraph H of G, let $G - H = G - V(H)$. A graph G is said to be *Hamiltonian* if G has a *Hamilton cycle*, i.e., a cycle containing all vertices of G, and *Hamilton-connected* if G has a *Hamilton path* between any pair of vertices, i.e., a path containing all vertices of G. A graph G is said to be *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$. For a cycle C of G, C is said to be *maximal* if there exists no cycle C' such that $V(C) \subsetneq V(C')$.

In this paper, we will deal with many statements which are unknown to be true or not. We call two statements *equivalent* if the correctness of one statement implies that of the other and vice versa. Most of the results in this paper are motivated by the following two conjectures due to Matthews and Sumner [16] and Thomassen [22], respectively.

Conjecture A (Matthews and Sumner [16], Thomassen [22]) *The following statements are true.*

(A1) *Every* 4*-connected claw-free graph is Hamiltonian.*

(A2) *Every* 4*-connected line graph is Hamiltonian.*

Since every line graph is claw-free, statement (A2) is a special case of statement (A1). However it is known that a result on closures due to Ryjáček [17] implies that statements (A1) and (A2) are even equivalent.

Theorem B (Ryjáček [17]) *Statements* (A1) *and* (A2) *are equivalent.*

Like Theorem B, many statements that are seemingly stronger or weaker than statements (A1) and (A2) have been proven to be equivalent to it as follows (see a survey [4] for more details). Note that statements (A5) and (A6) were conjectured by Ash and Jackson [1] and Fleischner [7], respectively.

Theorem C *All of the following statements are equivalent to statements* (A1) *and* (A2).

(A3) *Every* 4*-connected claw-free graph is Hamilton-connected* [18].

(A4) *Every* 4*-connected line graph is* 1*-Hamilton-connected* (2*-edge-Hamilton-connected*) [14].

(A5) *Every essentially* 4*-edge-connected graph has a dominating closed trail* [8].

(A6) *Every cyclically* 4*-edge-connected cubic graph has a dominating cycle* [8].

(A7) *Every cyclically* 4*-edge-connected cubic graph that is not* 3*-edge-colorable has a dominating cycle* [11].

(A8) *Every snark has a dominating cycle* [2].

Recently, as a positive result related to Conjecture A, Kaiser and the fourth author [15] proved that every 5-connected claw-free graph with minimum degree at least 6 is Hamilton-connected.

On the other hand, it is known that considering “Tutte cycles” is an effective approach to some problems on Hamiltonicity, where a cycle C of a graph G is called a *Tutte cycle* of G if (i) C is a Hamilton cycle of G, or (ii) $|V(C)| \geq 4$ and every component of $G - C$ has at most three neighbors on C. Note that every Tutte cycle C of a 4-connected graph G is a Hamilton cycle, since otherwise the neighbors of a component of $G - C$ form a cut set of order at most three, contradicting 4-connectedness of G. One can show that every 4-connected planar graphs are Hamiltonian by proving assertions on the existence of certain Tutte cycles in 2-connected planar graphs (see [21, 23]). Starting with this result, many researchers have studied about the existence of certain Tutte cycles not only in planar graphs but also in projective planar graphs or graphs on other surfaces in order to show Hamiltonicity of such graphs, (for example, see [19, 20, 24]). Thus, it has succeeded to show Hamiltonicity of 4-connected planar graphs or graphs on surfaces, considering stronger concept “Tutte cycles”.

Motivated by the above situation for planar graphs, in this paper, we concentrate on Tutte cycles in claw-free graphs. As a possible approach to solve Conjecture A, Jackson [10] proposed the following conjecture (also see a survey [6, Conjecture 2a.5]).

Conjecture D (Jackson [10]) *The following statement is true.*
(A9) *Every 2-connected claw-free graph has a Tutte cycle.*

As mentioned above, Tutte cycles in 4-connected graphs are Hamilton cycles, and hence statement (A9) implies statement (A1). The main result of this paper is to show that the converse also holds. In fact, we prove the following theorem.

Theorem 1 *Statements* (A1) *and* (A9) *are equivalent.*

On the other hand, if a graph has a Tutte cycle, then we can expect that it is long since it can avoid only vertices in a component of the graph after deleting a cut set of order at most three. Actually, Tutte cycles in 4-connected graphs are Hamilton cycles, i.e., Tutte cycles in 4-connected graphs are longest cycles of the graphs. How about 2-connected (or 3-connected) claw-free graphs? In view of Theorem 1, it would be natural to ask that every 2-connected (or 3-connected) claw-free graph has a Tutte cycle which is longest. As an answer of this problem, in Section 6, we will give a 3-connected claw-free graph in which any Tutte cycle is not longest. Thus it is not always true that a 2-connected (or 3-connected) claw-free graph has a longest one. However, the following theorem, which is also our main theorem, implies that if every 2-connected claw-free graph has a Tutte cycle, then we can always take it so that it is maximal.

Theorem 2 *Statement* (A9) *is equivalent to the following statement.*
(A10) *Every 2-connected claw-free graph has a Tutte cycle which is a maximal cycle of the graph.*

In Sections 3 and 4, we prove Theorems 1 and 2 by using closure concepts and other related results, some of which are also new.

2 Notation and terminology

In this section, we prepare terminology and notation which we use subsequent sections. Let G be a graph. For a vertex v of G, we denote by $d_G(v)$ and $N_G(v)$ the degree and the neighborhood of v in G, respectively, and let $N_G[v] = N_G(v) \cup \{v\}$. For an integer l, let $V_l(G) = \{v \in V(G) \mid d_G(v) = l\}$, and let $V_{\geq l}(G) = \bigcup_{m \geq l} V_m(G)$ and $V_{\leq l}(G) = \bigcup_{m \leq l} V_m(G)$. For a subgraph H of G and a vertex v in $G - H$, let $N_H(v) = N_G(v) \cap V(H)$. For subgraphs H and F of G with $V(F) \cap V(H) = \emptyset$, we define $N_H(F) = \bigcup_{v \in V(F)} N_H(v)$. We use $L(G)$ for the line graph of G. Let $e \in E(G)$. We denote by v_e a vertex in $L(G)$ corresponding to e. Let $V(e)$ be the set of end vertices of e, and we define $E_G(e) = \{f \in E(G) \mid V(f) \cap V(e) \neq \emptyset\}$. The *edge degree* of e in G is defined by the number of elements of $E_G(e) - \{e\}$, i.e., the number of edges incident with e. Note that for a graph G, the minimum edge degree of G is d if and only if the minimum degree of $L(G)$ is d. For subsets X and Y of $V(G)$ with $X \cap Y = \emptyset$, let $E_G(X, Y)$ denote the set of edges between X and Y, and let $e_G(X, Y) = |E_G(X, Y)|$. We often identify a subgraph H of G with its vertex set $V(H)$. For example, we write $E_G(H, F)$ instead of $E_G(V(H), V(F))$ for two disjoint subgraphs H and F of G. For a graph H and an edge set X, $H + X$ means the graph with vertex set $V(H) \cup \left(\bigcup_{e \in X} V(e)\right)$ and the edge set $E(H) \cup X$. For a subgraph H of G, let $E_G(H) = E(G[V(H)]) \cup E_G(H, G - H)$. A *star* is a graph consisting of a vertex and edges incident with the vertex (note that a star is not necessary a tree in this paper).

3 Closure

In this and the next sections, we will prove Theorems 1 and 2. In order to prove them, here we consider a new statement and divide the proof into two theorems. Before mentioning those, we need some definitions.

A connected graph T is called a *closed trail* (abbreviated as CT) if all vertices of T have even degree in T. Let H be a multigraph, and let T be a CT of H. We call T a *dominating closed trail* of H if $H - T$ is edgeless (in case that T is a cycle, we call T a *dominating cycle*), and T is said to be *edge-maximal* if there exists no closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. Note that a dominating CT of H is an edge-maximal CT of H. In [9], it is shown that for a connected multigraph H with $|E(H)| \geq 3$, H has a dominating CT if and only if $L(H)$ is Hamiltonian. Hence by the definition of an edge-maximal CT, we can easily obtain the following lemma.

Lemma 1 *Let H be a graph, and let T be an edge-maximal CT of H and $H^* = H[V(T)] + E_H(T, H - T)$. Then $L(H^*)$ has a Hamiltonian cycle which is a maximal cycle of $L(H)$.*

Let H be a graph with $|E(H)| \geq 3$. For a closed trail T of H, T is called a *Tutte closed trail* of H if (i) $E_H(T) = E(H)$, or (ii) $|E_H(T)| \geq 4$ and $e_H(F, T) \leq 3$ for every component F of $H - T$, and T is called a *weakly Tutte closed trail* of H if (i) $E_H(T) = E(H)$, or (ii)

$|E_H(T)| \geq 4$ and $e_H(F, T) \leq 3$ for all $F \in \mathcal{F}_H(T)$, where let $\mathcal{F}_H(T) = \{F \mid F$ is a component of $H - T$ with $|V(F)| \geq 2\}$. If T is a Tutte closed trail (resp. a weakly Tutte closed trail) and an edge-maximal closed trail of H, then we call T a *Tutte* (resp. a *weakly Tutte*) *edge-maximal closed trail* of H. Furthermore, we need the following terminology and notation. Now let H be a connected multigraph. For an edge-cut set X of H, X is called an *essential k-edge-cut set* of H if $|X| = k$ and $G - X$ has exactly two components of orders at least 2. We define $\mathcal{E}_k(H) = \{X \subseteq E(H) \mid X$ is an essential k-edge-cut set of $H\}$. For an integer $k \geq 2$, H is called *essentially k-edge-connected* if $|E(H)| \geq k + 1$ and $\mathcal{E}_l(H) = \emptyset$ for all $l < k$. It is known that for a multigraph H such that $L(H)$ is not complete, H is essentially k-edge-connected if and only if $L(H)$ is k-connected and that if H is essentially 2-edge-connected and H is not a star, then $H - V_1(H)$ is 2-edge-connected.

We are ready to state a new statement that plays a crucial role in the proofs of Theorems 1 and 2. We also give two theorems.

(A11) *Every essentially* 2*-edge-connected multigraph has a weakly Tutte edge-maximal CT.*

Theorem 3 *If statement* (A1) *is true, then statement* (A11) *is also true.*

Theorem 4 *If statement* (A11) *is true, then statement* (A10) *is also true.*

Here we prove Theorems 1 and 2 assuming Theorems 3 and 4.

Proof of Theorem 1. It is clear that statement (A10) implies statement (A9) and statement (A9) implies statement (A1). On the other hand, if statement (A1) is true, then by Theorem 3, statement (A11) is true, and by Theorem 4, statement (A10) is also true. This completes the proofs of Theorems 1 and 2. $\square$

Thus, to prove Theorems 1 and 2, it suffices only to show Theorems 3 and 4. We will prove Theorems 3 and 4 in the next section and in the rest of this section, respectively. Notice that by Theorems 3 and 4, we have that statement (A11) is also equivalent to statement (A1).

Before preparing some results to prove Theorem 4, we also state other statements and a theorem as follows.

(A12) *Every essentially* 2*-edge-connected multigraph has a weakly Tutte CT.*

(A13) *Every essentially* 2*-edge-connected multigraph has a Tutte CT.*

Theorem 5 *If statement* (A12) *is true, then statement* (A13) *is also true.*

We can easily see that statement (A11) implies statement (A12). Moreover, by the definition of a Tutte CT, it is easy to check that statement (A13) implies statement (A5) "every essentially 4-edge-connected graph has a dominating CT". Therefore, combining this with Theorems C, 3 and 5, we have that statement (A1) is also equivalent to statements (A12) and (A13). Note that it is not necessary to prove Theorem 5 for the proofs of Theorems 3 and 4, but we prove it since it may itself be interesting (we will prove Theorem 5 in Section 5).

Now we introduce some concept to prove Theorem 4. In [17], Ryjáček introduced the concept of a closure for claw-free graphs as follows. For a vertex v of a graph G, we call v a *locally connected vertex* of G if $G[N_G(v)]$ is connected. For a locally connected vertex v of a graph G, we call v an *eligible vertex* of G if $G[N_G(v)]$ is not compete. Let G be a claw-free graph. For an eligible vertex v of G, the operation of adding all possible edges between vertices in $N_G(v)$ is called *local completion* at v. In [17], it is shown that this operation preserves the claw-freeness of the original graph. Iterating local completions as long as possible, we obtain the graph G^* in which $G^*[N_{G^*}(v)]$ is a complete graph for every locally connected vertex v, i.e., there is no eligible vertex in G^*. We call this graph the *closure* of G, and denote it $\mathrm{cl}(G)$. In [17], it is shown that the closure of a graph has the following property.

Theorem E (Ryjáček [17]) *Let G be a claw-free graph. Then the following hold.*

(i) $\mathrm{cl}(G)$ *is well-defined, (i.e., uniquely defined).*

(ii) *There exists a triangle-free simple graph H such that $L(H) = \mathrm{cl}(G)$.*

(iii) *The length of a longest cycle in G and in $\mathrm{cl}(G)$ is the same.*

To obtain Theorem E (iii), Ryjáček actually proved the following, where for an eligible vertex v of a claw-free graph G, let G_v be the graph obtained from G by local completion at v.

Proposition F (Ryjáček [17]) *Let G be a claw-free graph and v be an eligible vertex of G. If C' is a longest cycle of G_v, then G has a cycle C such that $V(C) = V(C')$.*

Proposition F might not hold for a cycle C' which is not a longest cycle of G_v. However, in the proof of Proposition F, the maximality of $|V(C')|$ is only used for the fact that $N_{G_v}[v] \subseteq V(C')$ if $E(G_v[N_{G_v}[v]]) \cap E(C') \neq \emptyset$. Therefore, the same argument can work in the proof of the following proposition.

Proposition 6 *Let G be a claw-free graph and v be an eligible vertex of G. If C' is a maximal cycle of G_v, then G has a maximal cycle C such that $V(C) = V(C')$.*

As a corollary of Proposition 6, we can obtain the following, where for convenience, we call a cycle C of a graph G a *Tutte maximal cycle* of G if C is a Tutte cycle and a maximal cycle of G. Note that if C' is a Tutte cycle of G_v, then C is a Tutte cycle of G for any cycle C in G such that $V(C) = V(C')$.

Corollary 7 *Let G be a claw-free graph. If $\mathrm{cl}(G)$ has a Tutte maximal cycle, then G has a Tutte maximal cycle.*

By the definition of a weakly Tutte edge-maximal CT, the following holds.

Proposition 8 *Let G be a claw-free graph, and let H be a graph with $L(H) = \mathrm{cl}(G)$. If H has a weakly Tutte edge-maximal CT, then $L(H)$ has a Tutte maximal cycle.*

Proof of Proposition 8. Let T be a weakly Tutte edge-maximal CT of H and $H^* = H[V(T)] + E_H(T, H - T)$. Then by Lemma 1, $L(H^*)$ has a Hamilton cycle C which is a maximal cycle of $L(H)$. On the other hand, by the definition of a weakly Tutte CT, $e_H(F,T) \leq 3$ for all $F \in \mathcal{F}_H(T)$. Since $E_H(F) \cap E(H^*) = E_H(F,T)$ for each $F \in \mathcal{F}_H(T)$, we have that $|N_C(L(F))| = |E_H(F) \cap E(H^*)| = e_H(F,T) \leq 3$ for each $F \in \mathcal{F}_H(T)$. Moreover, by again the definition of a weakly Tutte CT, $V(C) = E(H^*) = E_H(T) = E(H)$ or $|V(C)| = |E(H^*)| = |E_H(T)| \geq 4$ holds. These imply that C is a Tutte cycle of $L(H)$. Thus C is a Tutte maximal cycle of $L(H)$. $\square$

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Suppose that statement (A11) is true. Let G be a 2-connected claw-free graph. By Theorem E (ii), there exists a triangle-free simple graph H such that $L(H) = \mathrm{cl}(G)$. If $L(H)$ is complete, then $L(H)$ clearly has a Hamilton cycle, and hence by Theorem E (iii), G has a Hamilton cycle, that is, G has a Tutte maximal cycle. Thus we may assume that $L(H)$ is not complete, and hence H is essentially 2-edge-connected. Since we assumed that statement (A11) is true, H has a weakly Tutte edge-maximal CT. Then, by Proposition 8, $L(H)$ has a Tutte maximal cycle. Hence by Corollary 7, G has a Tutte maximal cycle. Thus statement (A10) is also true and this completes the proof of Theorem 4. $\square$

4 Proof of Theorem 3

4.1 Set up for the proof of Theorem 3

In the end of this section, we will prove Theorem 3, that is, prove statement (A11) assuming (A1), by induction on the number of elements of $\mathcal{E}_2(H) \cup \mathcal{E}_3(H)$, where H is a given essentially 2-edge-connected multigraph. In order to do that, we need the following for the first step of the induction. Here for a graph H and a subset S of $E(H) \cup V(H)$, a closed trail T of H is a called an *S-closed trail* (abbreviated as S-CT) if $S \subseteq E(T) \cup V(T)$. Furthermore, if T is a dominating closed trail (resp. a weakly Tutte closed trail) and an S-closed trail of H, we call T a *dominating* (resp. a weakly Tutte) *S-closed trail* of H.

Lemma 2 *Statement* (A1) *is equivalent to the following statement.*

(A14) *Every essentially 4-edge-connected multigraph H has a dominating $V_{\geq 4}(H)$-CT, i.e., H has a Tutte edge-maximal CT.*

Proof of Lemma 2. By Theorem C, it is easy to see that statement (A14) implies statement (A1). So it suffices to show the converse. Assume that statement (A1) is true. Then by Theorem C, every essentially 4-edge-connected graph has a dominating CT. Let H be an essentially 4-edge-connected multigraph. Let H^* be the graph obtained from H by adding a pendant edge to each vertex in $V_{\geq 4}(H)$. Then H^* is also essentially 4-edge-connected and $V_{\geq 4}(H^*) = V_{\geq 4}(H)$. By the assumption, H^* has a dominating closed trail T. Since each vertex in $V_{\geq 4}(H^*)$ is incident with a pendant edge, $V_{\geq 4}(H^*) \subseteq V(T)$. Therefore by the definition of H^*, since $V_{\geq 4}(H^*) = V_{\geq 4}(H)$, we have that T is a dominating $V_{\geq 4}(H)$-CT of H. $\square$

We next prepare some results to prove the case of $\mathcal{E}_2(H) = \emptyset$ and $\mathcal{E}_3(H) \neq \emptyset$. To show this case, we actually consider about weakly Tutte closed trails passing through specified vertices and edges. Before mentioning the statement, we prepare the following terminology. Let H be a multigraph. For three distinct edges e_1, e_2 and e_3 in H, (e_1, e_2, e_3) is called a 3-*star* of H if there exists a vertex u of H such that $d_H(u) = 3$, $u \in V(e_1) \cap V(e_2) \cap V(e_3)$ and $V(e_3) - \{u\} \subseteq V_{\geq 3}(H)$, and u is called the *center* of (e_1, e_2, e_3).

(A15) *Let H be an essentially 4-edge-connected multigraph, and let (e_1, e_2, e_3) be a 3-star of H. Then H has a dominating $\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)$-CT.*

In order to consider statement (A15), we need the concept called "$V_2(H)$-*dominated*". A graph H is said to be $V_2(H)$-*dominated* if for any distinct four vertices u_1, u_2, v_1 and v_2 in H with $\{u_1, u_2, v_1, v_2\} = V_2(H)$, the graph $H + \{u_1u_2, v_1v_2\}$ has a dominating $\{u_1u_2, v_1v_2\}$-CT. The following was proven by Kužel [13].

Theorem G (Kužel [13]) *Statement* (A1) *is equivalent to the following statement.*

(A16) *Any subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.*

Actually, we show the following theorem in this section.

Theorem 9 *If statement* (A16) *is true, then statement* (A15) *is also true.*

We prove Theorem 9 in the next subsection and prove Theorem 3 in Subsections 4.3 and 4.4. At the end of this subsection, we give another theorem as follows.

Theorem 10 *If statement* (A15) *is true, then statement* (A1) *is also true.*

Combining Theorem 10 with Theorems G and 9, statement (A1) is also equivalent to statement (A15). Note that it is not necessary to prove Theorem 10 for the proof of Theorem 9, but we prove it since it may itself be interesting (we will prove Theorem 10 in Section 5).

4.2 Proof of Theorem 9

We first prove Theorem 9. We need some concepts and results.

Let $k \geq 3$ be an integer, and let H be an essentially 3-edge-connected graph such that $L(H)$ is not complete. Note that $V_{\leq 2}(H)$ is an independent set of H. The *core* of a graph H denoted by $\mathrm{core}(H)$, is the graph obtained by recursively deleting all vertices of degree 1, recursively deleting a vertex z with degree 2 in H and adding the edge xy with $N_H(z) = \{x, y\}$, and recursively deleting the created loops. It is easy to see that if H is an essentially k-edge-connected graph such that $L(H)$ is not complete, then $\mathrm{core}(H)$ is a 3-edge-connected essentially k-edge-connected multigraph (in particular, $\delta(\mathrm{core}(H)) \geq 3$). Moreover we can see that the following holds.

Lemma 3 *Let H be an essentially 4-edge-connected graph such that $L(H)$ is not complete, and let $H^* = \mathrm{core}(H)$. Suppose that H^* has a dominating $V_{\geq 4}(H^*)$-closed trail T^*. Then H has a dominating $V_{\geq 4}(H)$-closed trail T which satisfies the following:*

- *If $xy \in E(T^*)$, then $xy \in E(T)$ or $xz, yz \in E(T)$ for some $z \in V_2(H)$.*

Proof of Lemma 3. By the definition of a core, for each $xy \in E(H^*)$, $xy \in E(H)$ or there exists a vertex z in $V_2(H)$ such that $xz, yz \in E(H)$. Let $X = \{e \in E(H^*) \mid e \notin E(H)\}$. For each $e = xy \in X$, let z_e be a vertex in $V_2(H)$ such that $N_H(z_e) = \{x, y\}$. Then by replacing e with a path xz_ey for each $e = xy \in E(T^*) \cap X$, we can obtain a closed trail T of H such that $V(T) = V(T^*) \cup \{z_e \mid e \in E(T^*) \cap X\}$ and $E(T) = \{xz_e, yz_e \mid e = xy \in E(T^*) \cap X\} \cup (E(T^*) - X)$. Moreover, since $V_{\geq 4}(H^*) = V_{\geq 4}(H)$ by the definition of a core and the assumption, $V_{\geq 4}(H) = V_{\geq 4}(H^*) \subseteq V(T^*) \subseteq V(T)$. Therefore, to complete the proof, we have only to prove that T is a dominating CT of H. Note that $|E(H)| \geq 5$ because H is essentially 4-edge-connected. Let $x \in V(H-T)$. Since $V(T^*) \subseteq V(T)$, $x \notin V(T^*)$. Suppose that $N_H(x) \not\subseteq V(T)$, and let $z \in N_H(x) - V(T)$. If $\{x, z\} \subseteq V_{\geq 3}(H)$, then by the definition of a core, $\{x, z\} \subseteq V(H^*)$ and $xz \in E(H^*)$. Since $x, z \notin V(T^*)$, this contradicts that T^* is a dominating CT of H^*. Thus $\{x, z\} \cap V_{\leq 2}(H) \neq \emptyset$. Since H is essentially 4-edge-connected and $L(H)$ is not complete, we also have that $\{x, z\} \cap V_{\geq 3}(H) \neq \emptyset$. Since $x, z \in V(H - T)$ and $xz \in E(H)$, we may assume that $x \in V_{\geq 3}(H)$ and $z \in V_{\leq 2}(H)$. Since $V_{\geq 4}(H) \subseteq V(T)$, $x \in V_3(H)$. Then $E_H(xz) - \{xz\} \in \mathcal{E}_2(H) \cup \mathcal{E}_3(H)$, a contradiction. Thus $N_H(x) \subseteq V(T)$. Since x is an arbitrary vertex in $H - T$, this implies that T is a dominating CT of H. $\square$

We also need the following operation (see [8] for more details). Let H be a graph and $z \in V_{\geq 4}(H)$, and let $u_1, u_2, \ldots, u_d$ ($d = d_H(z)$) be an ordering of neighbors of z (we allow repetition in case of parallel edges). Then the graph H_z obtained from the disjoint union of $G - z$ and the cycle $C_z = z_1z_2 \ldots z_dz_1$ by adding the edges u_iz_i for each $1 \leq i \leq d$ is called an *inflation* of H at z. If $\delta(H) \geq 3$, then, by successively taking an inflation at each vertex of degree greater than 3, we can obtain a cubic graph H^I, called a *cubic inflation* of H. An inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of z) and the operation may decrease the edge-connectivity. However, the following was proven in [8].

Theorem H (Fleischner and Jackson [8]) *Let H be an essentially 4-edge-connected graph with $\delta(H) \geq 3$. Then some cubic inflation of H is also essentially 4-edge-connected.*

Let H^I be a cubic inflation of a graph H and for each $z \in V(H)$, set $I(z) = V(C_z)$ if $z \in V_{\geq 4}(H)$; otherwise, set $I(z) = \{z\}$. Observing that a dominating cycle in H^I must contain at least one vertex in $I(z)$ for each $z \in V_{\geq 4}(H)$, we immediately have the following fact (which is implicit in [8]).

Lemma I (Fleischner and Jackson [8]) *Let H be a graph with $\delta(H) \geq 3$, and let H^I be a cubic inflation of H. Suppose that H^I has a dominating cycle C. Then H has a dominating $V_{\geq 4}(H)$-closed trail T which satisfies the following*:

- *If $uv \in E(C)$ with $u \in I(x)$ and $v \in I(y)$ for some $x, y \in V(H)$ $(x \neq y)$, then $xy \in E(T)$.*

Proof of Theorem 9. Suppose that statement (A16) is true. Let H be an essentially 4-edge-connected multigraph, and let (e_1, e_2, e_3) be a 3-star of H (note that $V(e_3) \subseteq V_{\geq 3}(H)$ and that $V(e_1) \cup V(e_2) \subseteq V_{\geq 2}(H)$ because H is essentially 4-edge-connected). We will find a dominating $\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)$-CT of H.

If $L(H)$ is complete, then we can easily see that (i) H is a star such that $V(e_1) = V(e_2) = V(e_3)$, or (ii) H is a triangle such that e_3 is an unique simple edge in H or $V(e_i) = V(e_3)$ and $V(e_{3-i}) \neq V(e_3)$ for some $i = 1$ or 2. In either case, clearly H has a spanning closed trail T such that $\{e_1, e_2\} \subseteq E(T)$, that is, H has a desired closed trail.

Thus we may assume that $L(H)$ is not complete. Let u be the center of (e_1, e_2, e_3). Let $H^* = \mathrm{core}(H)$. Then H^* is an essentially 4-edge-connected graph with $\delta(H^*) \geq 3$. Note that $e_3 \in E(H^*)$ since $V(e_3) \subseteq V_{\geq 3}(H)$. Let e_1^* and e_2^* be two distinct edges incident with u in H^* such that $e_i^* \neq e_3$ for each $i = 1, 2$, and let $e_3^* = e_3$. Note that (e_1^*, e_2^*, e_3^*) is a 3-star with center u of H^*.

By Theorem H, there exists a cubic inflation H^I of H^* such that H^I is essentially 4-edge-connected. Note that H^I is a simple graph. Note also that by the definition of a 3-star, $I(u) = \{u\}$. For each i with $1 \leq i \leq 3$, let $v_i \in V(e_i^*) - \{u\}$, and let $v_i' \in I(v_i)$ such that $uv_i' \in E(H^I)$. We claim that H^I has a dominating cycle containing uv_1', uv_2' and v_3'. Since H^I is essentially 4-edge-connected, if $v_k'v_l' \in E(H^I)$ for some k and l with $1 \leq k < l \leq 3$, then it is easy to check that $H^I \cong K_4$, and hence H^I has a desired dominating cycle. Thus we may assume that $v_k'v_l' \notin E(H^I)$ for each k and l with $1 \leq k < l \leq 3$.

Let $\{w_1^{(3)}, w_2^{(3)}\} = N_{H^I}(v_3') - \{u\}$. Then since $H' := H^I - \{u, v_3'\}$ is a subgraph of H^I such that $\delta(H') = 2$ and $V_2(H') = \{v_1', v_2', w_1^{(3)}, w_2^{(3)}\}$ and we assumed that statement (A16) is true,

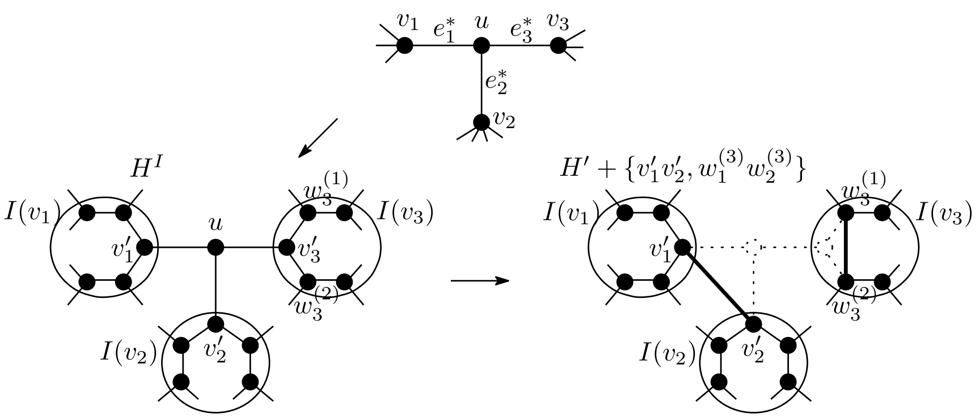

Figure 1: The subgraph H' of H^I

$H' + \{v_1'v_2', w_1^{(3)}w_2^{(3)}\}$ has a dominating cycle C' containing $v_1'v_2'$ and $w_1^{(3)}w_2^{(3)}$ (see Figure 1). Hence $(C' - \{v_1'v_2', w_1^{(3)}w_2^{(3)}\}) + \{uv_1', uv_2', v_3'w_1^{(3)}, v_3'w_2^{(3)}\}$ is a desired dominating cycle of H^I. Thus the assertion holds. Then by Lemma I, H^* has a dominating $\{e_1^*, e_2^*\} \cup V(e_3^*) \cup V_{\geq 4}(H^*)$-CT. Hence by Lemma 3 and the definition of e_1^*, e_2^* and e_3^*, H has a dominating $\{e_1, e_2\} \cup V(e_3) \cup V_{\geq 4}(H)$-CT. Therefore, statement (A15) is true, and this completes the proof of Theorem 9. $\square$

4.3 Preparation for the proof of Theorem 3

In this subsection, we prepare some technical lemmas to prove Theorem 3.

In the proof of Theorem 3, we will restrict maximal cycles on H to some component. To show that the resulting graph is a weakly Tutte CT, we use the following lemma.

Lemma 4 *Let H be a graph, and let T be a weakly Tutte CT of H. If T' is a CT of H such that $E_H(T') = E_H(T)$, then T' is also a weakly Tutte CT of H.*

Proof of Lemma 4. Let T' be a CT of H such that $E_H(T') = E_H(T)$, and suppose that T' is not a weakly Tutte CT of H. Then there exists $F' \in \mathcal{F}_H(T')$ with $e_H(F', T') \geq 4$. Write $E_H(F', T') = \{e_1, \ldots, e_l\}$ $(l \geq 4)$. Since $E_H(F', T') \subseteq E_H(T') = E_H(T)$, $V(T) \cap \left(\bigcup_{i=1}^{l} V(e_i)\right) \neq \emptyset$. Let $S = V(T) \cap \left(\bigcup_{i=1}^{l} V(e_i)\right)$, and suppose that $S \subseteq V(T') \cap \left(\bigcup_{i=1}^{l} V(e_i)\right)$. Then $\{e_1, \ldots, e_l\} = E_H(F', T') \subseteq E_H(T, H - T)$ and there exists a component F of $H - T$ such that $V(F') \subseteq V(F)$, which contradicts the assumption that T is a weakly Tutte CT of H. Thus $S \cap V(F') \cap \left(\bigcup_{i=1}^{l} V(e_i)\right) \neq \emptyset$, and hence $E(F') \cap E_H(T) \neq \emptyset$. Since $E(F') \cap E_H(T') = \emptyset$, this contradicts the assumption that $E_H(T') = E_H(T)$. $\square$

In the rest of this subsection, we fix the following notation. Let k be an integer with $2 \leq k \leq 3$, and let H be an essentially k-edge-connected graph.

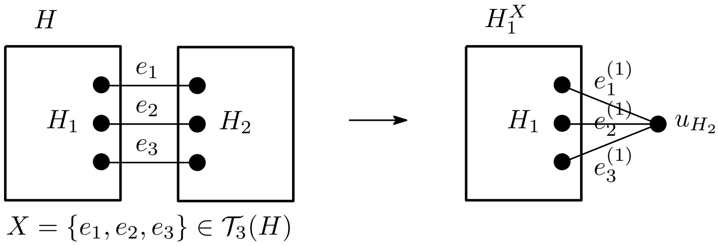

Figure 2: The graph H_1^X

To prove Theorem 3, we prepare the following terminology and notation. Let $\mathcal{T}_k(H) = \{(X, H_1, H_2) \mid X \in \mathcal{E}_k(H)$ and, H_1 and H_2 are distinct components of $G-X\}$. Let $(X, H_1, H_2) \in \mathcal{T}_k(H)$. We define two graphs H_1^X and H_2^X as follows. For each $i = 1, 2$, let H_i^X be the graph obtained from H by contracting H_{3-i} to a vertex $u_{H_{3-i}}$. Note that H_i^X is also an essentially k-edge-connected multigraph. If $X = \{e_1, \ldots, e_k\}$, then for each i, j with $1 \le i \le 2$ and $1 \le j \le k$, let $e_j^{(i)}$ be the edge in H_i^X corresponding to e_j (see Figure 2).

Now we fix the following notation. Let $(X, H_1, H_2) \in \mathcal{T}_k(H)$, and write $X = \{e_1, \ldots, e_k\}$.

Lemma 5 *Let $1 \le i \le 2$. If H_i^X has a weakly Tutte edge-maximal closed trail T_i such that $E(T_i) \cap \{e_1^{(i)}, \ldots, e_k^{(i)}\} = \emptyset$, then T_i is a weakly Tutte edge-maximal closed trail of H, or H_i^X has a weakly Tutte edge-maximal closed trail R_i such that $E(R_i) \cap \{e_1^{(i)}, \ldots, e_k^{(i)}\} \neq \emptyset$.*

Proof of Lemma 5. We may assume that $i = 1$. Note that T_1 is a weakly Tutte CT of H because $E(T_1) \cap \{e_1^{(1)}, \ldots, e_k^{(1)}\} = \emptyset$. Suppose that T_1 is not a weakly Tutte edge-maximal CT of H. Then there exists an edge-maximal closed trail T' of H such that $E_H(T_1) \subsetneq E_H(T')$. Note that $E(T') \cap X \neq \emptyset$ because T_1 is an edge-maximal CT of H_1^X such that $E(T_1) \cap \{e_1^{(1)}, \ldots, e_k^{(1)}\} = \emptyset$. Note also that $|E(T') \cap X| = 2$ because $2 \le k \le 3$. We may assume that $E(T') \cap X = \{e_1, e_2\}$, and let $R_1 = (T' - V(H_2)) + \{e_1^{(1)}, e_2^{(1)}\}$. Then R_1 is a CT of H_1^X. Since $E_H(T_1) \subseteq E_H(T')$, $E_{H_1^X}(T_1) - \{e_1^{(1)}, \ldots, e_k^{(1)}\} = E_H(T_1) \cap E(H_1) \subseteq E_H(T') \cap E(H_1)$. Moreover, by the definition of R_1 and since $\{e_1^{(1)}, \ldots, e_k^{(1)}\} \subseteq E_{H_1^X}(R_1)$ because $u_{H_2} \in V(R_1)$, $(E_H(T') \cap E(H_1)) \cup \{e_1^{(1)}, \ldots, e_k^{(1)}\} = E_{H_1^X}(R_1)$. This implies that $E_{H_1^X}(T_1) \subseteq E_{H_1^X}(R_1)$. Since T_1 is an edge-maximal CT of H_1^X, we have that $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$, and hence R_1 is also an edge-maximal CT of H_1^X. Furthermore, since T_1 is a weakly Tutte CT of H_1^X and $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$, it follows from Lemma 4 that R_1 is also a weakly Tutte CT of H_1^X. Thus R_1 is a weakly Tutte edge-maximal CT of H_1^X such that $E(R_1) \cap \{e_1^{(1)}, \ldots, e_k^{(1)}\} \neq \emptyset$. $\square$

We further fix the following notation in the following three lemmas (Lemmas 6 through 8). Let $e_i = v_i^{(1)} v_i^{(2)}$ with $v_i^{(1)} \in V(H_1)$ and $v_i^{(2)} \in V(H_2)$ for each $1 \le i \le k$. Let l_1 and l_2 be integers with $1 \le l_1 < l_2 \le k$, and for each $i = 1, 2$, let T_i be a $\{e_{l_1}^{(i)}, e_{l_2}^{(i)}\}$-CT of H_i^X and $T = \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_{l_1}, e_{l_2}\}$.

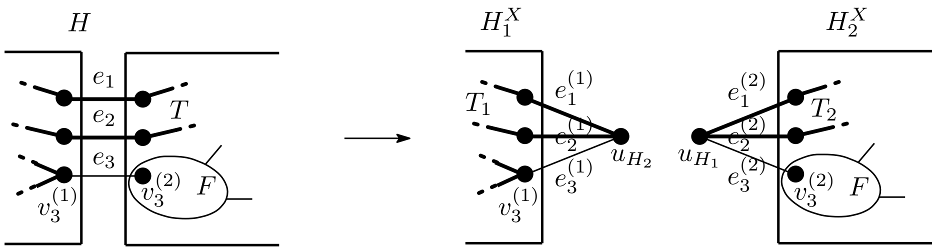

Figure 3: The component F of $H - T$

Lemma 6 *If T_i is a weakly Tutte CT of H_i^X for each $i = 1, 2$ and $\{v_1^{(i)}, \dots, v_k^{(i)}\} \subseteq V(T_i)$ for some $i = 1$ or 2, then T is a weakly Tutte CT of H.*

Proof of Lemma 6. We may assume that $l_1 = 1$ and $l_2 = 2$, and hence $\{v_1^{(i)}, v_2^{(i)}\} \subseteq V(T_i)$ for each $i = 1, 2$. By the symmetry of T_1 and T_2, we also may assume that $\{v_1^{(1)}, \dots, v_k^{(1)}\} \subseteq V(T_1)$. Let F be a component of $H - T$. Since $\{v_1^{(1)}, \dots, v_k^{(1)}\} \subseteq V(T_1) - \{u_{H_2}\} \subseteq V(T)$ and $\{e_1, e_2\} \subseteq E(T)$, we have that if $v_k^{(2)} \notin V(F)$, then F is a component of $H_i^X - T_i$ for some $i = 1$ or 2, and hence $E_H(F, T) = E_{H_i^X}(F, T_i)$ for some $i = 1$ or 2; if $v_k^{(2)} \in V(F)$ (note that in this case, $k = 3$), then F is a component of $H_2^X - T_2$ and $e_k^{(2)} \in E_{H_2^X}(F, T_2)$, and hence $E_H(F, T) = \big(E_{H_2^X}(F, T_2) - \{e_k^{(2)}\}\big) \cup \{e_k\}$ (see Figure 3). Since T_i is a weakly Tutte CT of H_i^X for each $i = 1, 2$, this implies that T is a weakly Tutte CT of H. $\Box$

Lemma 7 *If T_i is an edge-maximal CT of H_i^X for each $i = 1, 2$ and $\{v_1^{(i)}, \dots, v_k^{(i)}\} \subseteq V(T_i)$ for some $i = 1$ or 2, then T is an edge-maximal CT of H.*

Proof of Lemma 7. If $\{v_1^{(1)}, \dots, v_k^{(1)}\} \subseteq V(T_1)$, then let $A = \{v_1^{(1)}, \dots, v_k^{(1)}\}$; otherwise, let $A = \{v_1^{(2)}, \dots, v_k^{(2)}\}$. Suppose that T is not an edge-maximal CT of H. Then there exists an edge-maximal closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. Note that $E(T') \cap X \neq \emptyset$. Let m_1 and m_2 be integers with $1 \leq m_1 < m_2 \leq k$ such that $E(T') \cap X = \{e_{m_1}, e_{m_2}\}$. For each $i = 1, 2$, let $R_i = (T' - V(H_{3-i})) + \{e_{m_1}^{(i)}, e_{m_2}^{(i)}\}$. Then R_i is a CT of H_i^X for each $i = 1, 2$. Let $1 \leq i \leq 2$. Since $E_H(T) \subseteq E_H(T')$, we have that $E_{H_i^X}(T_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\} = E_H(T) \cap E(H_i) \subseteq E_H(T') \cap E(H_i) = E_{H_i^X}(R_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\}$. Since $\{e_1^{(i)}, \dots, e_k^{(i)}\} \subseteq E_{H_i^X}(T_i) \cap E_{H_i^X}(R_i)$ because $u_{H_{3-i}} \in V(T_i) \cap V(R_i)$, this implies that $E_{H_i^X}(T_i) \subseteq E_{H_i^X}(R_i)$. Since T_i is an edge-maximal CT of H_i^X, we obtain $E_{H_i^X}(T_i) = E_{H_i^X}(R_i)$, i.e., $E_{H_i^X}(T_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\} = E_{H_i^X}(R_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\}$. Since i is an arbitrary integer with $1 \leq i \leq 2$, $E_{H_i^X}(T_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\} = E_{H_i^X}(R_i) - \{e_1^{(i)}, \dots, e_k^{(i)}\}$ holds for each $i = 1, 2$. On the other hand, since $A \subseteq (V(T_1) - \{u_{H_2}\}) \cup (V(T_2) - \{u_{H_1}\}) = V(T)$, $X \subseteq E_H(T)$, and hence $X \subseteq E_H(T')$. Thus we obtain $E_H(T) = (E_{H_1^X}(T_1) - \{e_1^{(1)}, \dots, e_k^{(1)}\}) \cup (E_{H_2^X}(T_2) - \{e_1^{(2)}, \dots, e_k^{(2)}\}) \cup X = (E_{H_1^X}(R_1) - \{e_1^{(1)}, \dots, e_k^{(1)}\}) \cup (E_{H_2^X}(R_2) - \{e_1^{(2)}, \dots, e_k^{(2)}\}) \cup$

$X = E_H(T')$, a contradiction. $\square$

We call $(X, H_1, H_2) \in \mathcal{T}_k(H)$ a *minimal* 3-*tuple* of H if there exists no $X' \in \mathcal{E}_k(H)$ such that $H - X'$ has a component H_2' such that $V(H_2') \subsetneq V(H_2)$. Then by the definition of a minimal 3-tuple, we can obtain the following.

Lemma 8 *Suppose that $k = 3$ and (X, H_1, H_2) is a minimum 3-tuple of H. If $d_H(v_j^{(2)}) = 2$ for some j with $1 \leq j \leq 3$, then H_2 is isomorphic to K_2.*

Proof of Lemma 8. We may assume that $j = 3$. Since H is essentially 3-edge-connected, $X \in \mathcal{E}_3(H)$ and $d_H(v_3^{(2)}) = 2$, it follows that there exists an unique vertex v' in $N_H(v_3^{(2)}) \cap V(H_2)$. Note that $v' \in V_{\geq 3}(H)$ and $H_2 - v_3^{(2)}$ is connected. Then $X' := \{e_1, e_2, v_3^{(2)}v'\}$ is an edge-cut set of H, and $H_1 + \{e_3\}$ and $H_2 - v_3^{(2)}$ are components of $H - X'$. Therefore, since (X, H_1, H_2) is a minimal 3-tuple of H, we have $|V(H_2 - v_3^{(2)})| = 1$. $\square$

4.4 Proof of Theorem 3

We finally prove Theorem 3.

Proof of Theorem 3. Assume that statement (A1) is true. Let H be an essntially 2-edge-connected multigraph. We will prove that H has a weakly Tutte edge-maximal CT by induction on $g(H) := |\mathcal{E}_2(H) \cup \mathcal{E}_3(H)|$. If $g(H) = 0$, then H is essentially 4-edge-connected. By the assumption that statement (A1) is true and Lemma 2, H has a desired CT, and we are done. Hence we may assume that $g(H) \geq 1$.

By way of a contradiction, suppose that

$$H \text{ has no weakly Tutte edge-maximal CT.} \tag{4.1}$$

Suppose first that $\mathcal{E}_2(H) \neq \emptyset$, let $(X, H_1, H_2) \in \mathcal{T}_2(H)$ and write $X = \{e_1, e_2\}$. Then H_i^X is also essentially 2-edge-connected and $g(H_i^X) < g(H)$ for each $i = 1, 2$. Hence by the induction hypothesis, H_i^X has a weakly Tutte edge-maximal closed trail T_i for each $i = 1, 2$. By Lemma 5 and (4.1), we may assume that $E(T_i) \cap \{e_1^{(i)}, e_2^{(i)}\} \neq \emptyset$ for each $i = 1, 2$, and hence $\{e_1^{(i)}, e_2^{(i)}\} \subseteq E(T_i)$ for each $i = 1, 2$. Then by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ is a weakly Tutte edge-maximal CT of H, which contradicts (4.1) again. Thus $\mathcal{E}_2(H) = \emptyset$.

Then H is essentially 3-edge-connected. Let (X, H_1, H_2) be a minimal 3-tuple of H in $\mathcal{T}_3(H)$. Write $X = \{e_1, e_2, e_3\}$ and $e_i = v_i^{(1)}v_i^{(2)}$ with $v_i^{(1)} \in V(H_1)$ and $v_i^{(2)} \in V(H_2)$ for each $1 \leq i \leq 3$. Note that H_i^X is also essentially 3-edge-connected, and $g(H_i^X) < g(H)$ for each $i = 1, 2$, and hence by the induction hypothesis, H_1^X has a weakly Tutte edge-maximal CT. We define $\mathcal{T} = \{T_1 \mid T_1 \text{ is a weakly Tutte edge-maximal CT of } H_1^X \text{ such that } E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} \neq \emptyset\}$. By Lemma 5 and (4.1), $\mathcal{T} \neq \emptyset$ (note that $|E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}| = 2$ for all $T_1 \in \mathcal{T}$).

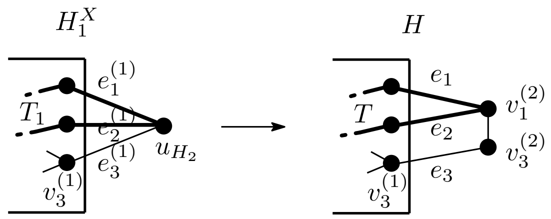

Figure 4: The closed trail T of H

We divide the proof of Theorem 3 into two cases.

Case 1. $d_{H_2^X}(v_j^{(2)}) \geq 3$ for each j with $1 \leq j \leq 3$.

Let $T_1 \in \mathcal{T}$, and we may assume that $E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = \{e_1^{(1)}, e_2^{(1)}\}$. Then by the assumption of Case 1, $(e_1^{(2)}, e_2^{(2)}, e_3^{(2)})$ is a 3-star with center u_{H_1} in H_2^X. Moreover, by the definition of a minimal 3-tuple and since $\mathcal{E}_2(H) = \emptyset$, H_2^X is essentially 4-edge-connected. Since we assumed that statement (A1) is true, it follows from Theorems G and 9 that statement (A15) is also true. Thus H_2^X has a dominating $\{e_1^{(2)}, e_2^{(2)}\} \cup V(e_3^{(2)}) \cup V_{\geq 4}(H_2^X)$-closed trail T_2, i.e., T_2 is a weakly Tutte edge-maximal CT of H_2^X, $\{e_1^{(2)}, e_2^{(2)}\} \subseteq E(T_2)$ and $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\} \subseteq V(T_2)$. Hence by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ is a weakly Tutte edge-maximal CT of H, which contradicts (4.1).

Case 2. $d_{H_2^X}(v_j^{(2)}) \leq 2$ for some j with $1 \leq j \leq 3$.

We may assume that $d_{H_2^X}(v_3^{(2)}) \leq 2$. Then by the denition of H_2^X and since $X \in \mathcal{E}_3(H)$, $d_H(v_3^{(2)}) = d_{H_2^X}(v_3^{(2)}) = 2$. Hence by Lemma 8, $H_2 \cong K_2$, i.e., $v_1^{(2)} = v_2^{(2)}$ and $v_1^{(2)} \neq v_3^{(2)}$. Let $T_1 \in \mathcal{T}$. We choose T_1 so that $e_3^{(1)} \in E(T_1)$ or $\{v_1^{(1)}, v_2^{(1)}, v_3^{(1)}\} \subseteq V(T_1)$ if possible.

Suppose that $e_3^{(1)} \in E(T_1)$. By the symmetry of $e_1^{(1)}$ and $e_2^{(1)}$, we may assume that $E(T_1) \cap \{e_1^{(1)}, e_2^{(1)}\} = \{e_1^{(1)}\}$. Let $T_2 = H_2^X - \{e_2^{(2)}\}$. Then T_2 is clearly a weakly Tutte $\{e_1^{(2)}, e_3^{(2)}\} \cup V(e_2^{(2)})$-CT of H_2^X such that $E_{H_2^X}(T_2) = E(H_2^X)$, i.e., T_2 is a weakly Tutte edge-maximal CT of H_2^X, $\{e_1^{(2)}, e_3^{(2)}\} \subseteq E(T_2)$ and $\{v_1^{(2)}, v_2^{(2)}, v_3^{(2)}\} \subseteq V(T_2)$. Hence by Lemmas 6 and 7, $T := \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_3\}$ is a weakly Tutte edge-maximal CT of H, which contradicts (4.1). Thus $e_3^{(1)} \notin E(T_1)$, that is, $E(T_1) \cap \{e_1^{(1)}, e_2^{(2)}, e_3^{(2)}\} = \{e_1^{(1)}, e_2^{(2)}\}$.

Let $T_2 = H_2^X - v_3^{(2)}$ and $T = \big((T_1 - u_{H_2}) \cup (T_2 - u_{H_1})\big) + \{e_1, e_2\}$ $(= (T_1 - u_{H_2}) + \{e_1, e_2\})$. Then T_2 is clearly a weakly Tutte $\{e_1^{(2)}, e_2^{(2)}\}$-CT of H_2^X such that $E_{H_2^X}(T_2) = E(H_2^X)$, i.e., T_2 is a weakly Tutte edge-maximal CT of H_2^X and $\{e_1^{(2)}, e_2^{(2)}\} \subseteq E(T_2)$. Then by Lemma 6, we also have that T is a weakly Tutte CT of H. Hence by Lemma 7 and (4.1), $v_3^{(1)} \notin V(T_1)$ and there exists an edge-maximal closed trail T' of H such that $E_H(T) \subsetneq E_H(T')$. In particular, since $v_3^{(1)} \notin V(T_1)$,

$$E_H(T) = (E_{H_1^X}(T_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}) \cup \{e_1, e_2, v_1^{(2)}v_3^{(2)}\} \text{ (see Figure 4).} \tag{4.2}$$

Note that since T_1 is an edge-maximal closed trail of H_1^X and $E_H(T) \subsetneq E_H(T')$, $E_H(T') \cap$

$X \neq \emptyset$. Let l_1 and l_2 be integers with $1 \leq l_1 < l_2 \leq 3$ such that $E_H(T') \cap X = \{e_{l_1}, e_{l_2}\}$. Let $R_1 = (T' - V(H_2)) + \{e_{l_1}^{(1)}, e_{l_2}^{(1)}\}$. Then R_1 is a CT of H_1^X. Since $E_H(T) \subseteq E_H(T')$, $E_{H_1^X}(T_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = E_H(T) \cap E(H_1) \subseteq E_H(T') \cap E(H_1) = E_{H_1^X}(R_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}$. Since $\{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} \subseteq E_{H_1^X}(T_1) \cap E_{H_1^X}(R_1)$ because $u_{H_2} \in V(T_1) \cap V(R_1)$, this implies that $E_{H_1^X}(T_1) \subseteq E_{H_1^X}(R_1)$. Since T_1 is an edge-maximal CT of H_1^X, we have $E_{H_1^X}(T_1) = E_{H_1^X}(R_1)$. Since T_1 is a weakly Tutte CT of H_1^X, this together with Lemma 4 implies that R_1 is a weakly Tutte CT of H_1^X. Therefore R_1 is a weakly Tutte edge-maximal CT of H_1^X such that $E(R_1) \cap \{e_1^{(1)}, e_2^{(1)}, e_3^{(2)}\} \neq \emptyset$, i.e., $R_1 \in \mathcal{T}$. Then by the choice of T_1, we have that $\{l_1, l_2\} = \{1, 2\}$ and $v_3^{(1)} \notin V(R_1)$. Then by the definition of R_1, $E(T') \cap X = \{e_1, e_2\}$ and $v_3^{(1)} \notin V(T')$. Therefore we obtain

$$E_H(T') = (E_{H_1^X}(R_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}) \cup \{e_1, e_2, v_1^{(2)} v_3^{(2)}\}. \tag{4.3}$$

Since $E_{H_1^X}(T_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\} = E_{H_1^X}(R_1) - \{e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\}$, it follows from (4.2) and (4.3) that $E_H(T) = E_H(T')$, which contradicts the fact that $E_H(T) \subsetneq E_H(T')$.

This completes the proof of Theorem 3. $\square$

5 Proofs of Theorems 5 and 10

As mentioned in the paragraph following Theorem 5 and the paragraph following Theorem 10 in Sections 3 and 4, respectively, we prove Theorems 5 and 10 in this section.

Proof of Theorem 5. Assume that statement (A12) is true. Let H be an essentially 2-edge-connected multigraph. Let H^* be a graph obtained from H by adding a pendant edge to each vertex in $V_{\geq 4}(H)$. Then H^* is also essentially 2-edge-connected and $V_{\geq 4}(H^*) = V_{\geq 4}(H)$. Since we assumed that statement (A12) is true, H^* has a weakly Tutte closed trail T. Then by the definition of H^*, T is also a weakly Tutte CT of H. We show that T is a Tutte CT of H. Suppose that T is not a Tutte CT of H. Since T is a weakly Tutte CT of H, there exists a component F of $H - T$ such that $|V(F)| = 1$, say $V(F) = \{x\}$, and $x \in V_{\geq 4}(H)$. Then by the definition of H^*, there exists a vertex y in $N_{H^*}(x) \cap V_1(H^*)$. Since $x \notin V(T)$ and $V(H - T) \subseteq V(H^* - T)$, we have that xy is a graph in $\mathcal{F}_{H^*}(T)$ such that $e_{H^*}(\{x, y\}, T) = d_H(x) \geq 4$, which contracts that T is a weakly Tutte CT of H^*. Thus T is a Tutte CT of H^*. Hence statement (A13) is also true, and this completes the proof of Theorem 5. $\square$

Proof of Theorem 10. By Lemma 2, it is enough to show that statement (A15) implies statement (A14). Assume that statement (A15) is true. Let H be an essentially 4-edge-connected multigraph. We will find a dominating $V_{\geq 4}(H)$-CT. If $L(H)$ is complete, then H is a star or a triangle, and hence we can easily see that H has a desired dominating CT. Thus, we may assume that $L(H)$ is not complete.

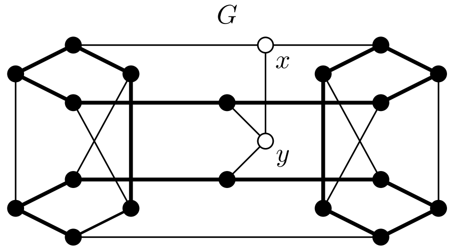

Figure 5: The cubic graph to construct the example

Then $H^* := \text{core}(H)$ is an essentially 4-edge-connected graph with $\delta(H^*) \geq 3$. By Theorem H, there exists a cubic inflation H^I of H^* such that H^I is essentially 4-edge-connected. Since we assumed that statement (A15) is true, taking any vertex in H^I as the center of a 3-star, we can find a dominating cycle of H^I. By Lemma I, H^* has a dominating $V_{\geq 4}(H^*)$-CT. By Lemma 3, H also has a dominating $V_{\geq 4}(H)$-CT. Hence statement (A14) is also true, and this completes the proof of Theorem 10. $\square$

6 Concluding remarks

In 1992, Jackson posed the possible approach to the well-known conjecture on the existence of a Hamilton cycle in 4-connected claw-free graphs (Conjecture A), using a Tutte cycle. Indeed, he conjectured that statement (A9) "every 2-connected claw-free graph has a Tutte cycle" is true (Conjecture D), which directly implies Conjecture A. In this paper, we have concentrated on a Tutte cycle on claw-free graphs and seen that many statements (A1)–(A16) are equivalent (see Theorems B, C, G, 1–5, 9, 10 and Lemma 2).

By the above fact, we have that statement (A10) "every 2-connected claw-free graph has a Tutte maximal cycle" is seemingly stronger than statement (A9), that is, if (A9) is true, then we can always take a Tutte cycle so that it is maximal. However, as mentioned in Section 1, it is not always true that a 3-connected claw-free graph has a Tutte cycle which is longest even if statement (A9) is true. The following is the 3-connected claw-free graph showing this. Let G be the graph illustrated in Figure 5. Then it is easy to check that G is an essentially 3-edge-connected (3-connected) cubic graph which is not Hamiltonian. Moreover, the edges depicted in Figure 5 by bold lines induce a cycle C such that $V(C) = V(G) - \{x, y\}$ and C is a maximal cycle of G. Let $d \geq 3$ be an integer. Let G^* be the graph obtained from G by adding $d - 2$ pendant edges to each vertex in $\{x, y\}$ and at least $2d - 2$ pendant edges to each vertex in $V(G) - \{x, y\}$, and let X be the set of pendant edges which are incident with $\{x, y\}$ in G^*. Note that $|X \cup \{xy\}| = 2d - 3$. Then by the definition of G^* and since G is essentially 3-edge-connected, we have that G^* is also essentially 3-edge-connected and the minimum edge

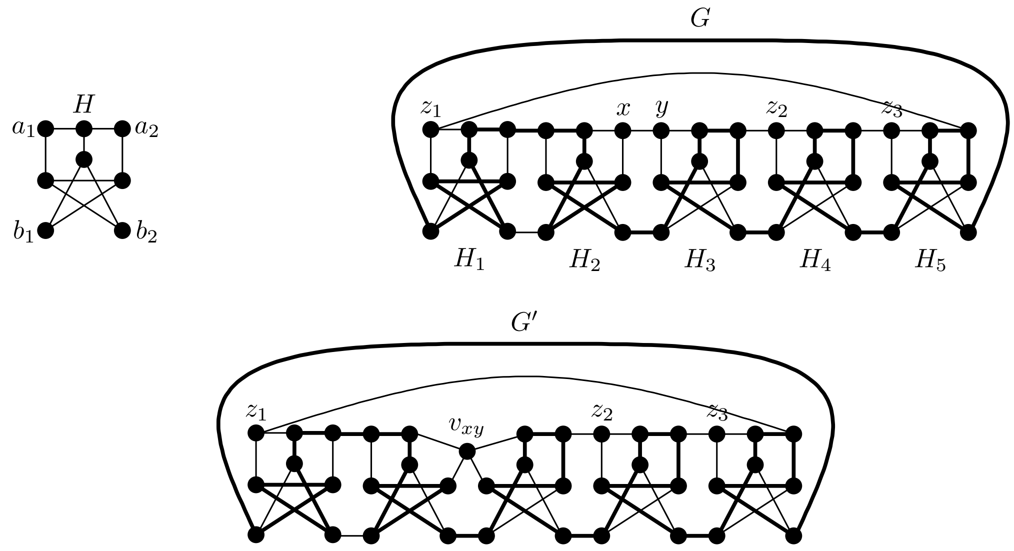

Figure 6: The cubic graph introduced by Kochol [12]

degree of G^* is just d. Furthermore, since G is not Hamiltonian and C is a maximal cycle of G satisfying $V(C) = V(G) - \{x, y\}$, for every closed trail (cycle) T of G^* with $T \neq C$, $|E_{G^*}(T)| < |E_{G^*}(C)|$ holds. These imply that $L(G^*)$ is a 3-connected claw-free graph with $\delta(L(G^*)) = d$, and for any longest cycle D of $L(G^*)$, $V(D) = E_{G^*}(C) = E(G^*) - (X \cup \{xy\})$ holds. Since $|E_{G^*}(C) \cap E_{G^*}(xy)| = e_{G^*}(\{x, y\}, V(G^*) - \{x, y\}) = 4$, every cycle D of $L(G^*)$ with $V(D) = E_{G^*}(C)$ is not a Tutte cycle of $L(G^*)$. Thus any Tutte cycle of $L(G^*)$ is not longest.

In addition, if statement (A9) is true, then we can also take Tutte closed trails (weakly Tutte closed trails, weakly Tutte edge-maximal closed trails) in essentially 2-edge-connected graphs (see statements (A11)–(A13)). Moreover, it is also true that every essentially 4-edge-connected graph has a Tutte edge-maximal CT if statement (A9) is true (see statement (A14)). However, it is not always true that an essentially 3-edge-connected graph has a Tutte edge-maximal CT. We finally give the graph showing this. We use the methods of Kochol [12] for constructions of snarks with a maximal cycle that is not a dominating cycle. (Note that by using this method, we can also construct a 3-connected claw-free graph in which any Tutte cycle is not longest other than the above graph.) Let G be the graph in the right side of Figure 6. It arises from five copies of the graph H (H_1, H_2, H_3, H_4, H_5) illustrated in the left side of Figure 6 after joining the vertices a_i and b_i of degree 2 as in depicted in the figure. Then G is an essentially 3-edge-connected (3-connected) cubic graph and the cycle C depicted by bold lines is a maximal cycle of G such that $V(C) = V(G) - \{x, y, z_1, z_2, z_3\}$. Let G' be the graph obtained from G by contracting xy to a vertex v_{xy} (see Figure 6), and let G^* be the graph obtained from G' by adding a pendant edge to each vertex in $V(G') - \{v_{xy}, z_1, z_2, z_3\}$. Then G^* is also essentially 3-edge-connected and C is a dominating CT of G^*, i.e., C is an edge-maximal CT of G^*. Since each vertex in $V(G') - \{v_{xy}, z_1, z_2, z_3\}$ is incident with a pendant edge in G^* and $E_{G^*}(C) = E(G^*)$, every edge-maximal closed trail of G^* contains $V(G') - \{v_{xy}, z_1, z_2, z_3\}$. On the other hand, since C is a maximal cycle of G satisfying $V(G) - \{x, y, z_1, z_2, z_3\}$ and by the definition of H, G, G' and G^*, we can see that for every closed trail T of G^* with $v_{xy} \in V(T)$, $V(G) - \{x, y, z_1, z_2, z_3\} = V(G') - \{v_{xy}, z_1, z_2, z_3\} \not\subseteq V(T)$ holds (note that there exists no Hamilton path in H from a_1 to $\{a_2, b_2\}$ and H has no two disjoint paths covering $V(H)$ from a_1 to $\{a_2, b_2\}$ and from b_1 to $\{a_2, b_2\}$, respectively, see [12] for more details). Thus C is an unique edge-maximal CT of G^*. But since C is not a Tutte CT of G^*, any Tutte CT of G^* is not an edge-maximal CT of G^*.

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