# A 2-factor in which each cycle has long length in claw-free graphs*

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**Abstract**

For a graph $G$, we denote by $\delta(G)$ the minimum degree of $G$. A graph $G$ is said to be claw-free if $G$ has no induced subgraph isomorphic to $K_{1,3}$. In this article, we prove that every claw-free graph $G$ with minimum degree at least 4 has a 2-factor in which each cycle contains at least $\lceil \frac{\delta(G)-1}{2} \rceil$ vertices and every 2-connected claw-free graph $G$ with minimum degree at least 3 has a 2-factor in which each cycle contains at least $\delta(G)$ vertices. For the case where $G$ is 2-connected, the lower bound on the length of a cycle is best possible.

*Keywords*: 2-factor, Claw-free graph, Minimum degree
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## 1 Introduction

In this paper, we consider finite graphs. For terminology and notation not defined in this paper, we refer the readers to [6]. A *simple graph* means an undirected graph without loops or multiple edges. A *multigraph* may contain multiple edges but no loops. Let $G$ be a graph. For a vertex $v$ of $G$, the *degree* of $v$ in $G$ is the number of edges incident with $v$. Let $V(G)$, $E(G)$ and $\delta(G)$ be the vertex set, the edge set and the minimum degree of $G$, respectively. We refer to the number of vertices of $G$ as the *order* of $G$ and denote it by $|G|$. A graph $G$ is said to be *claw-free* if $G$ has no induced subgraph isomorphic to $K_{1,3}$ (here $K_{1,3}$ denotes the complete bipartite graph

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with partite sets of cardinalities 1 and 3, respectively). We denote by $L(G)$ the line graph of $G$. Obviously a line graph is claw-free. A 2-*factor* of $G$ is a spanning subgraph of $G$ in which every component is a cycle.

It is a well-known conjecture that every 4-connected claw-free graph is Hamiltonian due to Matthews and Sumner [14]. Since a Hamilton cycle is a connected 2-factor, there are many results on 2-factors of claw-free graphs. For instance, results of both Choudum and Paulraj [5] and Egawa and Ota [7] imply that a moderate minimum degree condition already guarantees that a claw-free graph has a 2-factor.

**Theorem A ([5, 7])** *Every claw-free graph with minimum degree at least* 4 *has a* 2*-factor.*

Broersma, Kriesel and Ryjáček [2] showed that if there exists a function $f(n)$ of $n$ such that $\lim_{n\to\infty} f(n)/n = 0$ and every 4-connected claw-free graph of order $n$ has a 2-factor with at most $f(n)$ components, then every 4-connected claw-free graph is Hamiltonian. Thus, to solve Matthews and Sumner's conjecture, it suffices to show the existence of a 2-factor with small number of components (not necessarily 1 component). Concerning the upper bound on the number of components, Broersma, Paulusma and the third author [3] and Jackson and the third author [11] proved the following, respectively (other related results can be found in [8, 12, 13, 15, 18]).

**Theorem B ([3])** *Every claw-free graph $G$ with minimum degree at least* 4 *has a* 2*-factor with at most* $\max\left\{\left\lfloor\frac{|G|-3}{\delta(G)-1}\right\rfloor, 1\right\}$ *components.*

**Theorem C ([11])** *Every* 2*-connected claw-free graph $G$ with minimum degree at least* 4 *has a* 2*-factor with at most* $\left\lfloor\frac{|G|+1}{4}\right\rfloor$ *components.*

It could be also another possible approach to study about the lengths of cycles in 2-factors of claw-free graphs. Our first main result is the following.

**Theorem 1** *Every* 2*-connected claw-free graph $G$ with minimum degree at least* 3 *has a* 2*-factor in which each component contains at least $\delta(G)$ vertices.*

The proof of the above is given in Section 4. As a corollary of Theorem 1, we can get the following which improve the both of Theorems B and C for 2-connected claw-free graphs.

**Corollary 2** *Every* 2*-connected claw-free graph $G$ with minimum degree at least* 3 *has a* 2*-factor with at most* $\left\lfloor\frac{|G|}{\delta(G)}\right\rfloor$ *components.*

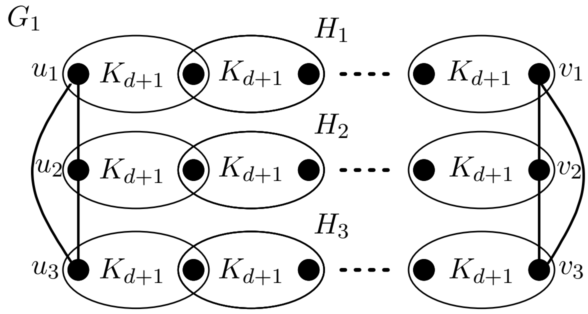


Figure 1: The graph $G_1$

In Theorem 1, the lower bound on the length of a cycle is best possible because we can construct a 2-connected claw-free graph $G$ such that for each 2-factor of $G$, the minimum length of cycles in it is at most $\delta(G)$ as follows: let $d \geq 3$ be an integer, and let $H_1, H_2$ and $H_3$ be graphs with the connectivity one and exactly two end blocks such that each block of $H_i$ $(1 \leq i \leq 3)$ is a complete graph of order $d+1$. For each $1 \leq i \leq 3$, let $u_i$ and $v_i$ be vertices contained in distinct end blocks of $H_i$, respectively, and $u_i$ and $v_i$ are not cut vertices of $H_i$. Let $G_1$ be the graph obtained from $H_1 \cup H_2 \cup H_3$ by joining $u_i$ and $u_{i+1}$ for $1 \leq i \leq 3$ and joining $v_i$ and $v_{i+1}$ for $1 \leq i \leq 3$, where let $u_4 = u_1$ and $v_4 = v_1$, see Figure 1 (here $K_m$ denotes the complete graph of order $m$). Then $G_1$ is a 2-connected claw-free graph which satisfies $\delta(G_1) = d$, and for each 2-factor of $G_1$, the minimum length of cycles in it is at most $d$.

**Remark.** A *path-factor* is a spanning subgraph in which every component is a path. Ando et al. [1] proved that a claw-free graph $G$ has a path-factor in which each component contains at least $\delta(G)+1$ vertices. Moreover they conjectured that if $G$ is 2-connected, then there exists a path-factor in which each component contains at least $3\delta(G)+3$ vertices, but this conjecture is still open unlike the case of 2-factors on 2-connected claw-free graphs.

For claw-free graphs with cut vertices, we can construct an infinite family of examples $G$ in which every 2-factor contains a cycle of length at most $\lceil \frac{\delta(G)+1}{2} \rceil$ (see Section 6). We conjecture that the length is also the lower bound.

**Conjecture 3** *Every claw-free graph $G$ with minimum degree at least 4 has a 2-factor in which each component contains at least $\lceil \frac{\delta(G)+1}{2} \rceil$ vertices.*

In this paper, we will show a slightly weaker statement.

**Theorem 4** *Every claw-free graph $G$ with minimum degree at least 4 has a 2-factor in which each component contains at least $\lceil \frac{\delta(G)-1}{2} \rceil$ vertices.*

An essential part of the proof of Theorem 4 is that we can divide a claw-free graph into mutually vertex-disjoint 2-connected claw-free graphs which has large minimum degree, i.e., we prove the following in Section 5.

**Theorem 5** *Every claw-free graph $G$ with minimum degree at least 4 has a set $\mathscr{G}^*$ of mutually vertex-disjoint subgraphs such that $\bigcup_{G^* \in \mathscr{G}^*} V(G^*) = V(G)$ and each $G^* \in \mathscr{G}^*$ is a 2-connected claw-free graph with $\delta(G^*) \geq \lceil \frac{\delta(G)-1}{2} \rceil$.*

A *clique-factor* is a spanning subgraph in which every component is a clique. Faudree et al. [9] showed that a line graph with minimum degree at least 7 has a clique-factor in which each component contains at least 3 vertices. It is known that if $H$ is a tree, then the line graph $L(H)$ has a clique-factor in which each component contains at least $\lceil \frac{\delta(L(H))+1}{2} \rceil$ vertices ([4]). This supports Conjecture 3 in some sense.

It would be natural to consider the case where the connectivity is at least 3 as the next step. Concerning the number of components of 2-factors in 3-connected claw-free graphs, Kužel et al. [13] proved that every 3-connected claw-free graph $G$ has a 2-factor with at most $\max\{ \lfloor \frac{|G|}{\delta(G)+2} \rfloor, 1\}$ components. Recently, Ozeki et al. [15] improved the result as follows : every 3-connected claw-free graph $G$ has a 2-factor with at most $\lfloor \frac{4|G|}{5(\delta(G)+2)} + \frac{2}{5} \rfloor$ components. In view of this result, one might expect that the coefficient of $\delta(G)$ in the lower bound on the length of a cycle would be greater than 1 for 3-connected claw-free graphs $G$.

*Problem. Determine $f(d) = \max\{m \mid$ every 3-connected non-hamiltonian claw-free graph with minimum degree $d$ has a 2-factor in which each component contains at least $m$ vertices$\}$. In particular, is there a constant $c > 1$ such that $f(d) \geq cd$ holds?*

## 2 Terminology and notation

In this section, we prepare terminology and notation which we use in subsequent sections. Let $G$ be a graph. We denote the number of edges of $G$ by $e(G)$. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph induced by $X$ in $G$, and let $G - X = G[V(G) \setminus X]$. If $H$ is a subgraph of $G$, then let $G - H = G - V(H)$. A subset $X$ of $V(G)$ is called an *independent set* of $G$ if $G[X]$ is edgeless. Let $H_1$ and $H_2$ be subgraphs of $G$ or subsets of $V(G)$, respectively. If $H_1$ and $H_2$ have no common vertex in $G$, we define $E_G(H_1, H_2)$ to be the set of edges of $G$ between $H_1$ and $H_2$, and let $e_G(H_1, H_2) = |E_G(H_1, H_2)|$. For a vertex $v$ of $G$, we denote by $N_G(v)$ and $d_G(v)$ the neighborhood and the degree of $v$ in $G$, respectively. For a positive integer $l$, we define $V_l(G) = \{v \in V(G) \mid d_G(v) = l\}$, and let $V_{\geq l}(G) = \bigcup_{m \geq l} V_m(G)$ and $V_{\leq l}(G) = \bigcup_{m \leq l} V_m(G)$.

Let $e = uv \in E(G)$. We denote by $V(e)$ the set of end vertices of $e$, i.e., $V(e) = \{u, v\}$. The *edge degree* of $e$ in $G$ is defined by the number of edges incident with $e$, and is denoted by $\xi_G(e)$, i.e., $\xi_G(e) = |\{f \in E(G) \mid f \neq e, V(f) \cap V(e) \neq \emptyset\}|$. Note that if $G$ is a simple graph, then $\xi_G(e) = e_G(V(e), G - V(e)) = d_G(u) + d_G(v) - 2$. Let $\xi(G)$ be the minimum edge degree of $G$. For $X \subseteq E(G)$, $G - X$ means the graph with the vertex set $V(G)$ and the edge set $E(G) \setminus X$.

If a graph $S$ consists of a vertex (called a *center*) and edges incident with the center, $S$ is called a *star*. So a star in this paper is not necessary a tree. A connected graph is called a *closed trail* if all the vertices have even degree. A closed trail $T$ in a graph $H$ is called a *dominating closed trail* if $H - T$ is edgeless.

## 3 Preparation for the proof of Theorem 1

To prove Theorem 1, we use Ryjáček closure. In [16], Ryjáček introduced the concept of a closure for claw-free graphs as follows. Let $G$ be a claw-free graph. We call a vertex $v$ of $G$ *locally connected* (resp. *locally disconnected*) if $G[N_G(v)]$ is connected (resp. disconnected). Note that if a vertex $v$ of $G$ is locally disconnected, then $G[N_G(v)]$ is a union of two vertex-disjoint complete graphs (otherwise, $G$ contains a $K_{1,3}$ as an induced subgraph). For a locally connected vertex $v$ of $G$, we add edges joining all pairs of nonadjacent vertices in $N_G(v)$. The *closure* $\mathrm{cl}(G)$ of $G$ is a graph obtained by recursively repeating this operation, as long as this is possible. In [16], it is shown that the closure of a graph has the following property. (Here a graph $H$ is said to be *triangle-free* if $H$ contains no $K_3$.)

**Theorem D ([16])** *If $G$ is a claw-free graph, then the following hold.*

(i) $\mathrm{cl}(G)$ *is well-defined, (i.e., uniquely defined).*

(ii) *There exists a triangle-free simple graph $H$ such that $L(H) = \mathrm{cl}(G)$.*

On the other hand, in [17, Theorem 4], Ryjáček, Saito and Schelp proved that for any vertex-disjoint cycles $D_1, \ldots, D_q$ in $\mathrm{cl}(G)$, $G$ has vertex-disjoint cycles $C_1, \ldots, C_p$ with $p \leq q$ such that $\bigcup_{i=1}^{q} V(D_i) \subseteq \bigcup_{i=1}^{p} V(C_i)$. By modifying the proof, we can improve this result as follows.

**Lemma E ([13])** *Let $G$ be a claw-free graph. If $D_1, \ldots, D_q$ are vertex-disjoint cycles in $\mathrm{cl}(G)$, then $G$ has vertex-disjoint cycles $C_1, \ldots, C_p$ with $p \leq q$ such that for each $j$ with $1 \leq j \leq q$, there exists $i$ with $1 \leq i \leq p$ such that $V(D_j) \subseteq V(C_i)$.*

As a corollary of Lemma E, we can easily obtain the following.

**Corollary 6** *Let $m$ be an integer. For a claw-free graph $G$, $G$ has a 2-factor in which each cycle contains at least $m$ vertices if and only if* $\mathrm{cl}(G)$ *has a 2-factor in which each cycle contains at least $m$ vertices.*

**Proof of Corollary 6.** The necessity is clear, and so we prove only sufficiency. Suppose that $\mathrm{cl}(G)$ has a 2-factor in which each cycle contains at least $m$ vertices, and let $D_1, \dots, D_q$ are vertex-disjoint cycles in $\mathrm{cl}(G)$ such that $\bigcup_{j=1}^{q} V(D_j) = V(\mathrm{cl}(G))$ ( $= V(G)$) and $|D_j| \geq m$ for $1 \leq j \leq q$. Then by Lemma E, $G$ has vertex-disjoint cycles $C_1, \dots, C_p$ with $p \leq q$ such that

$$\text{for each } j \text{ with } 1 \leq j \leq q, \text{ there exists } i \text{ with } 1 \leq i \leq p \text{ such that } V(D_j) \subseteq V(C_i), \tag{3.1}$$

in particular, $\bigcup_{j=1}^{q} V(D_j) \subseteq \bigcup_{i=1}^{p} V(C_i)$. Since $\bigcup_{j=1}^{q} V(D_j) = V(G)$, we have that $\bigcup_{i=1}^{p} V(C_i) = V(G)$, i.e., $\bigcup_{i=1}^{p} C_i$ forms a 2-factor of $G$. Since $\bigcup_{j=1}^{q} V(D_j) = \bigcup_{i=1}^{p} V(C_i)$, it follows from (3.1) that for each $i$ with $1 \leq i \leq p$, there exists $j$ with $1 \leq j \leq q$ such that $V(C_i) \supseteq V(D_j)$ (otherwise, $\bigcup_{j=1}^{q} V(D_j) \subsetneq \bigcup_{i=1}^{p} V(C_i)$, a contradiction). Since $|D_j| \geq m$ for $1 \leq j \leq q$, we have that $|C_i| \geq m$ for $1 \leq i \leq p$. Thus $\bigcup_{i=1}^{p} C_i$ is a desired 2-factor of $G$. □

Now we are ready to state new statement which is equivalent to Theorem 1 (see Proposition 8). Here a multigraph $H$ is called *essentially $k$-edge-connected* if $e(H) \geq k+1$ and $H - X$ has at most one component which contains an edge for every $X \subseteq E(H)$ with $|X| < k$. It is easy to see that for a graph $H$, $H$ is essentially $k$-edge-connected if and only if $L(H)$ is $k$-connected.

**Theorem 7** *Let $H$ be an essentially 2-edge-connected triangle-free simple graph. If $\delta(L(H)) \geq 3$, then $L(H)$ has a 2-factor in which each cycle contains at least $\delta(L(H))$ vertices.*

By Theorem D and Corollary 6, we can obtain the following proposition.

**Proposition 8** *Theorems 1 and 7 are equivalent.*

**Proof of Proposition 8.** It is clear that Theorem 1 implies Theorem 7 because line graphs are claw-free. So we prove the converse.

Suppose that Theorem 7 is true, and let $G$ be a 2-connected claw-free graph with $\delta(G) \geq 3$. We show that $G$ has a 2-factor in which each cycle contains at least $\delta(G)$ vertices. By Theorem D (ii) and since $G$ is 2-connected, there exists an essentially 2-edge-connected triangle-free simple graph $H$ such that $L(H) = \mathrm{cl}(G)$. Note that $\delta(L(H)) \geq \delta(G)$, and hence $\delta(L(H)) \geq 3$. Therefore, by Theorem 7, $L(H)$ ( $= \mathrm{cl}(G)$) has a 2-factor in which each cycle contains at least $\delta(L(H))$ vertices. This together with Corollary 6 implies that $G$ has a 2-factor in which each cycle contains at least $\delta(L(H))$ ( $\geq \delta(G)$) vertices. □

## 4 Proof of Theorem 1

By Proposition 8, it is enough to show Theorem 7 for Theorem 1. Before proving Theorem 7, we define a few terminologies.

It is well known that for a connected multigraph $H$ with $e(H) \geq 3$, $L(H)$ is Hamiltonian if and only if $H$ is a star or $H$ has a dominating closed trail (see [10]). Obviously if $H$ is a star, then $L(H)$ is a clique. A set $\mathscr{D}$ is called a *cover set* of $E(H)$ if (i) $\mathscr{D}$ is a set of edge-disjoint connected subgraphs in $H$ such that $\bigcup_{D\in\mathscr{D}} E(D) = E(H)$, and (ii) for each $D \in \mathscr{D}$, $D$ is a star or $D$ has a dominating closed trail. If all members in a cover set $\mathscr{D}$ are stars, then we call $\mathscr{D}$ a *star cover set* of $E(H)$.

The following fact implies Theorem 7.

**Theorem 9** *Let $H$ be an essentially 2-edge-connected triangle-free simple graph. If $\xi(H) \geq 3$, then $H$ has a cover set $\mathscr{D}$ of $E(H)$ such that $e(D) \geq \xi(H)$ for all $D \in \mathscr{D}$.*

**Proof of Theorem 7.** Let $H$ be an essentially 2-edge-connected triangle-free simple graph, and suppose that $\delta(L(H)) \geq 3$. Since $\xi(H) = \delta(L(H)) \geq 3$, by Theorem 9, $H$ has a cover set $\mathscr{D}$ of $E(H)$ such that $e(D) \geq \xi(H)$ ( $\geq 3$) for all $D \in \mathscr{D}$. As $D$ is a star or a connected subgraph which has a dominating closed trail for each $D$ in $\mathscr{D}$, we have that $L(D)$ has a Hamilton cycle $C_D$ for each $D$ in $\mathscr{D}$ (note that $|C_D| = |L(D)| = e(D) \geq \xi(H) = \delta(L(H))$). Since $\bigcup_{D\in\mathscr{D}} E(D) = E(H)$ and $\mathscr{D}$ is a set of edge-disjoint connected subgraphs, it follows that $\bigcup_{D\in\mathscr{D}} C_D$ forms a desired 2-factor of $L(H)$. $\square$

Hence in the rest of this section, we prove Theorem 9. The following lemma will be used.

**Lemma F ([18])** *Let $H$ be an essentially 2-edge-connected graph. If $\xi(H) \geq 3$, then there exists a set $\mathscr{T}$ of vertex-disjoint closed trails in $H$ such that $V_{\geq 3}(H - V_1(H)) \subseteq \bigcup_{T\in\mathscr{T}} V(T)$.*

**Proof of Theorem 9.** If $H$ is a star, then $\{H\}$ is a desired cover set of $E(H)$. Thus we may assume that $H$ is not a star. Then since $H$ is essentially 2-edge-connected, we have $\delta(H - V_1(H)) \geq 2$. By Lemma F, there exists a set $\mathscr{T}$ of vertex-disjoint closed trails in $H$ such that $V_{\geq 3}(H - V_1(H)) \subseteq \bigcup_{T\in\mathscr{T}} V(T)$. Since $\delta(H - V_1(H)) \geq 2$ and $V_{\geq 3}(H - V_1(H)) \subseteq \bigcup_{T\in\mathscr{T}} V(T)$, it follows that $v \in V_2(H - V_1(H))$ for all $v \in V_{\geq 3}(H) \setminus \bigcup_{T\in\mathscr{T}} V(T)$. This implies that for each $v \in V_{\geq 3}(H) \setminus \bigcup_{T\in\mathscr{T}} V(T)$,

$$\begin{array}{c}\text{there exist exactly two vertices } u_1 \text{ and } u_2 \text{ in } N_H(v) \text{ with } u_1 \neq u_2 \\ \text{such that } u_i \in V_{\geq 2}(H) \text{ for } i = 1, 2 \text{ and } N_H(v) \setminus \{u_1, u_2\} \subseteq V_1(H)\end{array} \tag{4.1}$$

(see the left of Figure 2). In particular, $N_H(v) \setminus \{u_1, u_2\} \neq \emptyset$ since $v \in V_{\geq 3}(H)$, that is, $N_H(v) \cap V_1(H) \neq \emptyset$, and hence $|N_H(v) \setminus \{u_1, u_2\}| \geq d_H(v) - 2 \geq \xi(H) - 1$.

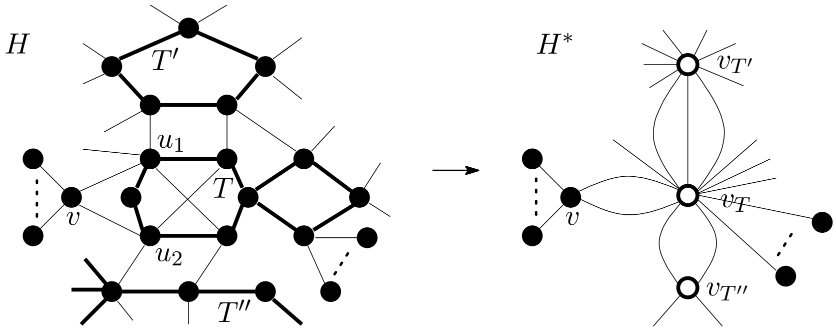


Figure 2: The graph $H^*$

Let $H^*$ be the graph obtained from $H$ by contracting an induced subgraph $H[V(T)]$ of $H$ to a vertex $v_T$ for each $T \in \mathscr{T}$ (note that $H^*$ may be a multigraph, see Figure 2). Let $X = \{v_T \mid T \in \mathscr{T}\}$. Note that $|X| = |\mathscr{T}|$ since $\mathscr{T}$ is a set of vertex-disjoint closed trails in $H$. By the definition of $H^*$,

$$d_{H^*}(v) = d_H(v) \text{ for all } v \in V(H^*) \setminus X. \tag{4.2}$$

By (4.2) and since $V_1(H) \cap \left(\bigcup_{T \in \mathscr{T}} V(T)\right) = \emptyset$,

$$V_1(H) = V_1(H^*) \setminus X. \tag{4.3}$$

By the definition of $H^*$ and (4.2) and since $\xi(H) \geq 3$, we also have that $V_{\leq 2}(H^*) \setminus X$ is an independent set of $H^*$.

To show the existence of a cover set $\mathscr{D}$ of $E(H)$, we find a mapping $\varphi : E(H^*) \to V(H^*)$ so that

(M1) $\varphi(e) = u$ or $\varphi(e) = v$ for all $e = uv \in E(H^*)$,

(M2) $|\varphi^{-1}(v)| = 0$ for all $v \in V_{\leq 2}(H^*) \setminus X$,

(M3) $|\varphi^{-1}(v)| \geq \xi(H)$ for all $v \in V_{\geq 3}(H^*) \setminus X$,

(M4) $|\varphi^{-1}(v_T)| + e(H[V(T)]) \geq \xi(H)$ for all $v_T \in X$.

If there exists such a mapping $\varphi$, then we can construct a desired cover set as follows. Suppose that there exists such a mapping $\varphi$. Let $\mathscr{S} = \{S_v \mid v \in V_{\geq 3}(H^*) \cup X\}$ be a set of stars such that for each $S_v \in \mathscr{S}$, $S_v$ is a star consisting of a vertex $v$ (as the center) and the edges in $\varphi^{-1}(v)$. Then by the conditions (M1) and (M2), $\mathscr{S}$ is a star cover set of $E(H^*)$. Furthermore by the conditions (M3) and (M4), $e(S_v) = |\varphi^{-1}(v)| \geq \xi(H)$ for each $v \in V_{\geq 3}(H^*) \setminus X$ and $e(S_{v_T}) = |\varphi^{-1}(v_T)| \geq \xi(H) - e(H[V(T)])$ for each $v_T \in X$. Hence by the definition of $H^*$, and by considering a subset of $E(H)$ corresponding to $E(S_v)$ for each $v \in V_{\geq 3}(H^*) \cup X$ and replacing

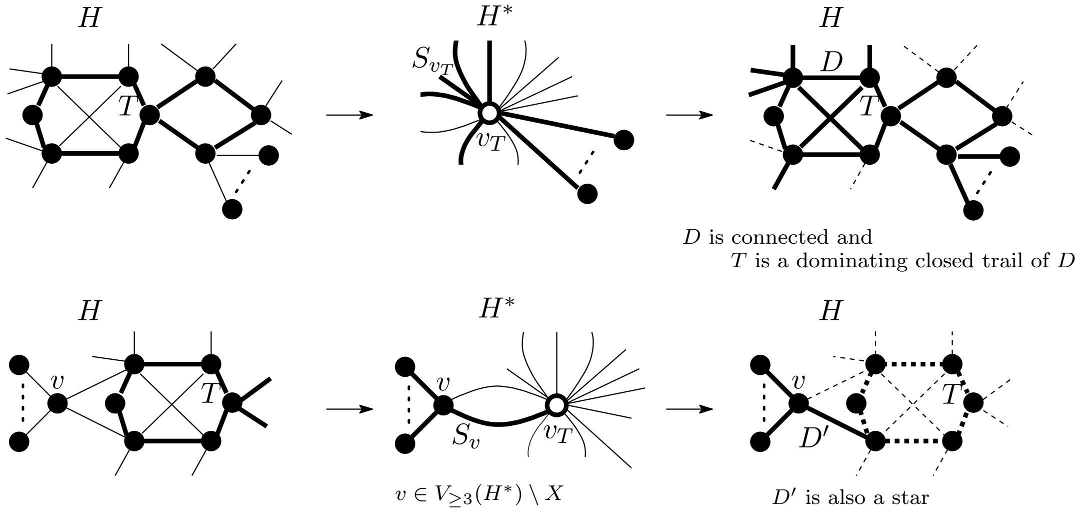


Figure 3: The desired cover set of $E(H)$

$v_T$ with $H[V(T)]$ for each $v_T \in X$, we can find a cover set $\mathscr{D}$ of $E(H)$ such that $e(D) \geq \xi(H)$ for all $D \in \mathscr{D}$ (see Figure 3). Thus it suffices to show the existence of the above mapping $\varphi$.

Let $H' = H^* - (V_1(H^*) \setminus X)$. To show the existence of a mapping $\varphi$ satisfying the conditions (M1)–(M4), we define a mapping $\varphi' : E(H') \to V(H')$ as follows. Since the number of vertices of odd degree is even in $H'$, there exists a collection of paths $P_1, \dots, P_l$ in $H'$ such that each vertex in $o(H')$ appears in the set of end vertices of them exactly ones (note that $l = |o(H')|/2$), where for a graph $G$, $o(G)$ denotes the set of vertices with odd degree in $G$. By considering the symmetric difference of them, we may assume that $P_1, \dots, P_l$ are pairwise edge-disjoint. For each $1 \leq i \leq l$, write $P_i = x_1^i x_2^i x_3^i \dots x_{|P_i|-1}^i x_{|P_i|}^i$, and let $e_j^i = x_j^i x_{j+1}^i$ for each $1 \leq j \leq |P_i|-1$, and we define $\varphi'(e_j^i) = x_{j+1}^i$ for each $1 \leq j \leq |P_i| - 1$. Let $H'' = H' - \bigcup_{i=1}^{l} E(P_i)$. By the definitions of $P_1, \dots, P_l$, we have $o(H'') = \emptyset$. Hence the edges of each component of order at least 2 in $H''$ can be covered by pairwise edge-disjoint cycles. For each cycle, written by $y_1 y_2 y_3 \dots y_{m-1} y_m$ $(= y_1)$, we define $\varphi'(e_j) = y_{j+1}$, where $e_j = y_j y_{j+1}$ for each $1 \leq j \leq m-1$. Then by the definition of $\varphi'$, for each $v \in V(H')$,

$$|\varphi'^{-1}(v)| \geq (d_{H'}(v) - 1)/2 = \Big( \big( d_{H^*}(v) - |(N_{H^*}(v) \cap V_1(H^*)) \setminus X| \big) - 1 \Big)/2. \tag{4.4}$$

Now we define a mapping $\varphi : E(H^*) \to V(H^*)$ as follows: for each $e = uv \in E(H^*)$, we let

$$\varphi(e) = \begin{cases} u & \text{if } v \in V_{\leq 2}(H^*) \setminus X \\ \varphi'(e) & \text{otherwise} \end{cases} .$$

Since $V_{\leq 2}(H^*) \setminus X$ is an independent set of $H^*$, $\varphi$ is well defined. By the definitions of $\varphi$ and $\varphi'$, we can easily see that $\varphi$ satisfies the conditions (M1) and (M2). So we show that $\varphi$ satisfies the conditions (M3) and (M4). Since $(V_1(H^*) \setminus X) \cap V(H') = \emptyset$ by the definition of $H'$, it follows from the definition of $\varphi$ that

$$|\varphi^{-1}(v)| \geq |\varphi'^{-1}(v)| + |(N_{H^*}(v) \cap V_1(H^*)) \setminus X| \text{ for all } v \in X. \tag{4.5}$$

We first show that $\varphi$ satisfies the condition (M3). Let $v \in V_{\geq 3}(H^*) \setminus X$. Then by (4.2) and the definitions of $H^*$ and $X$, $v \in V_{\geq 3}(H) \setminus \bigcup_{T \in \mathscr{T}} V(T)$. Hence by (4.1), there exist exactly two vertices $u_1, u_2 \in N_H(v)$ with $u_1 \neq u_2$ such that $u_i \in V_{\geq 2}(H)$ for $i = 1, 2$, $N_H(v) \setminus \{u_1, u_2\} \subseteq V_1(H)$ and $|N_H(v) \setminus \{u_1, u_2\}| \geq \xi(H) - 1$. Therefore by (4.2), (4.3) and the definition of $H^*$, there exist distinct two edges $vu'_1, vu'_2 \in E(H^*)$ such that $u'_i \in V_{\geq 2}(H^*) \cup X$ for $i = 1, 2$, $N_{H^*}(v) \setminus \{u'_1, u'_2\} \subseteq V_1(H^*) \setminus X$ and $|N_{H^*}(v) \setminus \{u'_1, u'_2\}| \geq \xi(H) - 1$ ($vu'_1$ and $vu'_2$ may be parallel edges in $H^*$, e.g., $v_T = u'_1 = u'_2$ in Figure 2). If $u'_i \in V_2(H^*) \setminus X$ for some $i = 1$ or 2, then by the definition of $\varphi$, we have $|\varphi^{-1}(v)| \geq |(N_H(v) \cap V_1(H^*)) \setminus X| + |(N_H(v) \cap V_2(H^*)) \setminus X| \geq |N_{H^*}(v) \setminus \{u'_1, u'_2\}| + 1 \geq (\xi(H) - 1) + 1 = \xi(H)$. Thus we may assume that $u'_i \in V_{\geq 3}(H^*) \cup X$ for $i = 1, 2$. Note that by the definition of $H'$, $u'_i \in V(H')$ for $i = 1, 2$. Since $N_{H^*}(v) \setminus \{u'_1, u'_2\} \subseteq V_1(H^*) \setminus X$, we have $v \in V_2(H')$ (the degree of $v$ in $H'$ is even) and $vu'_i \in E(H')$ for $i = 1, 2$. Hence by the definition of $\varphi'$, $\varphi'(vu'_j) = v$ and $\varphi'(vu'_{3-j}) = u'_{3-j}$ for some $j$ with $j \in \{1, 2\}$. We may assume that $j = 1$. Then by the definition of $\varphi$ and since $\{v, u'_1\} \subseteq V_{\geq 3}(H^*) \cup X$, it follows that $\varphi(vu'_1) = \varphi'(vu'_1) = v$. Therefore $|\varphi^{-1}(v)| \geq |(N_H(v) \cap V_1(H^*)) \setminus X| + |\{vu'_1\}| \geq |N_{H^*}(v) \setminus \{u'_1, u'_2\}| + 1 \geq (\xi(H) - 1) + 1 = \xi(H)$. Thus $\varphi$ satisfies the condition (M3).

We next show that $\varphi$ satisfies the condition (M4). Let $T \in \mathscr{T}$. Since $H$ is a triangle-free simple graph, all cycles which are contained in $T$ have order at least 4. Hence there exist two independent edges $e_1$ and $e_2$ in $T$. For each $i = 1, 2$, let

$$\epsilon_i = \big|\{e \in E(H[V(T)]) \mid e \neq e_i, V(e) \cap V(e_i) \neq \emptyset\}\big|.$$

Since $H$ is triangle-free, $\big|\{e \in E(H[V(T)]) \mid V(e) \cap V(e_i) \neq \emptyset \text{ for } i = 1, 2\}\big| \leq 2$. Hence by the definition of $\epsilon_i$,

$$e(H[V(T)]) \geq (\epsilon_1 + \epsilon_2) - 2 + |\{e_1, e_2\}| = \epsilon_1 + \epsilon_2. \tag{4.6}$$

We also have $e_H(V(e_i), H - T) \geq \xi(H) - \epsilon_i$ for each $i = 1, 2$. Hence since $d_{H^*}(v_T) = e_H(T, H - T)$ by the definition of $H^*$ and since $E_H(V(e_1), H - T) \cap E_H(V(e_2), H - T) = \emptyset$, we obtain

$$d_{H^*}(v_T) = e_H(T, H - T) \geq e_H(V(e_1), H - T) + e_H(V(e_2), H - T) \geq 2\xi(H) - (\epsilon_1 + \epsilon_2). \tag{4.7}$$

Then by (4.4) through (4.7),

$$\begin{aligned}
|\varphi^{-1}(v_T)| &\geq |\varphi'^{-1}(v_T)| + |(N_{H^*}(v_T) \cap V_1(H^*)) \setminus X| \\
&\geq \big(d_{H^*}(v_T) - |(N_{H^*}(v_T) \cap V_1(H^*)) \setminus X| - 1\big)/2 + |(N_{H^*}(v_T) \cap V_1(H^*)) \setminus X| \\
&\geq \big(2\xi(H) - (\epsilon_1 + \epsilon_2) - 1\big)/2 \\
&= \xi(H) - \big((\epsilon_1 + \epsilon_2) + 1\big)/2 \\
&\geq \xi(H) - \big(e(H[V(T)]) + 1\big)/2.
\end{aligned}$$

Since $e(H[V(T)]) \geq 4$, we get $|\varphi^{-1}(v_T)| + e(H[V(T)]) \geq \xi(H) - \big(e(H[V(T)])+1\big)/2 + e(H[V(T)]) \geq \xi(H) + \big(e(H[V(T)]) - 1\big)/2 > \xi(H)$. Thus $\varphi$ satisfies the condition (M4).

This completes the proof of Theorem 9. $\square$

## 5 Proof of Theorem 4

As mentioned in Section 1, the essential part of the proof of Theorem 4 is Theorem 5, and hence at first we give the proof. In order to prove this, we use the following lemma.

**Lemma 10** *Let $G$ be a claw-free graph and $B$ be a block of $G$. Let $u$ be a vertex in $B$, and let $x_1$ and $x_2$ be distinct two cut vertices in $G$ such that $x, y \in N_B(u)$. If $x_1x_2 \notin E(G)$, then $G[N_B(u)]$ has a clique-factor with 2 components.*

**Proof of Lemma 10.** Let $G_i = G[N_B(x_i) \cup \{x_i\}]$ for $i = 1, 2$. Since $x_1$ and $x_2$ are cut vertices in $G$, each $x_i$ is locally disconnected. This implies that $G_1$ and $G_2$ are complete graphs, respectively. Suppose that there exists a vertex $y$ in $N_B(u) \setminus (V(G_1) \cup V(G_2))$. Then by the definitions of $G_1$ and $G_2$, it follows that $yx_i \notin E(G)$ for $i = 1, 2$. Since $x_1x_2 \notin E(G)$, this implies that $G[\{u, x_1, x_2, y\}]$ is isomorphic to $K_{1,3}$, a contradiction. Thus $N_B(u) \subseteq V(G_1) \cup V(G_2)$, and hence $G_1' := G[N_B(u) \cap V(G_1)]$ and $G_2' := G[N_B(u) \setminus V(G_1')]$ forms a clique-factor with 2 components in $G[N_B(u)]$. $\square$

**Proof of Theorem 5.** Let $G$ be a claw-free graph with $\delta(G) \geq 4$. It suffices to consider the case where $G$ is connected. If $G$ is 2-connected, then $\mathscr{G}^* = \{G\}$ is a desired set. Thus we may assume that $G$ has a cut vertex. Let $\mathscr{B}$ be the set of blocks of $G$ and $X$ be the set of cut vertices of $G$. We consider the block cut tree $T$ of $G$, i.e., $V(T) = \mathscr{B} \cup X$ and $E(T) = \{Bv \mid B \in \mathscr{B}, v \in V(B) \cap X\}$. Since every cut vertex of $G$ is locally disconnected, it follows that for each $v \in X$,

$$G[N_G(v)] \text{ is a union of two vertex-disjoint complete graphs.} \tag{5.1}$$

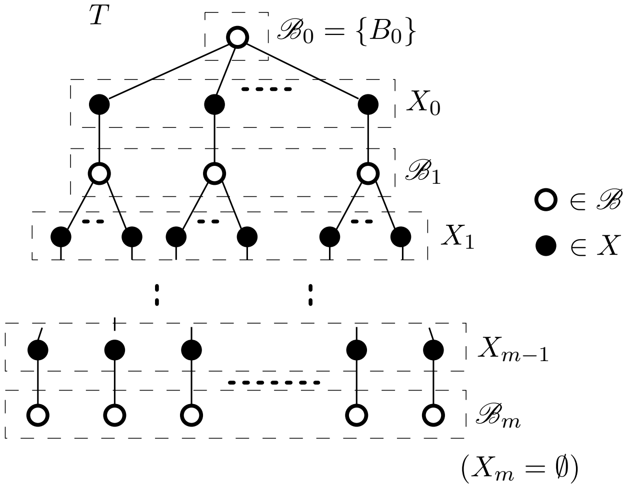


Figure 4: Subsets $\mathscr{B}_0, \dots, \mathscr{B}_m$ and subsets $X_0, \dots, X_m$

Hence $d_T(v) = 2$ for all $v \in X$. Let $B_0 \in \mathscr{B}$ be the root of $T$. We consider $T$ as an oriented tree from $B_0$ to leaves, and denote it $\overrightarrow{T}$. For each $v \in X$, we let $B_{v^+}$ (resp. $B_{v^-}$) denote a successor (resp. a predecessor) of $v$ along $\overrightarrow{T}$ (note that $B_{v^+}$ and $B_{v^-}$ are uniquely defined because $d_T(v) = 2$, and note also that $B_{v^+}, B_{v^-} \in \mathscr{B}$). For each $B \in \mathscr{B} \backslash \{B_0\}$, we let $v_B$ denote a predecessor of $B$ along $\overrightarrow{T}$ (note that $v_B$ is uniquely defined). We define subsets $\mathscr{B}_0, \dots, \mathscr{B}_m$ of $\mathscr{B}$ and subsets $X_0, \dots, X_m$ of $X$ inductively by the following procedure. First let $\mathscr{B}_0 = \{B_0\}$, and let $X_0 = \{v \in X \mid B_{v^-} = B_0\}$. Now let $i \geq 1$, and assume that we have defined $\mathscr{B}_0, \dots, \mathscr{B}_{i-1}$ and $X_0, \dots, X_{i-1}$. If $X_{i-1} \neq \emptyset$, then let $\mathscr{B}_i = \{B_{v^+} \mid v \in X_{i-1}\}$, and let $X_i = \{v \in X \mid B_{v^-} \in \mathscr{B}_i\}$; if $X_{i-1} = \emptyset$, we let $m = i - 1$ and terminate the procedure (see Figure 4). Then it follows from the definitions of $\mathscr{B}_0, \dots, \mathscr{B}_m$ and $X_0, \dots, X_m$ that $\mathscr{B}$ is disjoint union of $\mathscr{B}_0, \dots, \mathscr{B}_m$ and $X$ is disjoint union of $X_0, \dots, X_{m-1}$.

We define a mapping $\varphi : X \to \mathscr{B}$ inductively as follows. First for each $v \in X_0$, let

$$\varphi(v) = \begin{cases} B_{v^-} \ ( = B_0) & \text{if } |N_G(v) \cap V(B_{v^-})| \geq \frac{\delta(G)}{2} \\ B_{v^+} & \text{otherwise} \end{cases} .$$

Now let $i \geq 1$, and assume that we have defined $\varphi(v)$ for each $v \in X_l$ with $1 \leq l \leq i - 1$. If $i \leq m - 1$, then for each $v \in X_i$, let

$$\varphi(v) = \begin{cases} B_{v^-} & \text{if } \varphi(v_{B_{v^-}}) = B_{v^-} \text{ and } |N_G(v) \cap V(B_{v^-})| \geq \frac{\delta(G)}{2} \\ B_{v^-} & \text{if } \varphi(v_{B_{v^-}}) \neq B_{v^-} \text{ and } |N_G(v) \cap V(B_{v^-})| \geq \frac{\delta(G)+1}{2} \\ B_{v^-} & \text{if } \varphi(v_{B_{v^-}}) \neq B_{v^-},\ |N_G(v) \cap V(B_{v^-})| = \frac{\delta(G)}{2} \text{ and } vv_{B_{v^-}} \notin E(G) \\ B_{v^+} & \text{otherwise} \end{cases} ;$$

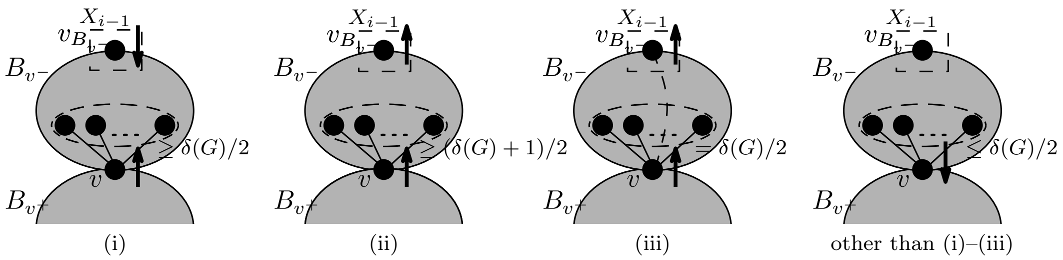


Figure 5: The definition of $\varphi$

if $i = m$, then we terminate the procedure (see Figure 5). Let

$$\mathscr{G}^* = \{G[(V(B) \setminus X) \cup \varphi^{-1}(B)] \mid B \in \mathscr{B}\}.$$

We show that $\mathscr{G}^*$ is a desired set. By the definition of $\mathscr{G}^*$, it is easy to check that $\mathscr{G}^*$ is a set of vertex-disjoint subgraphs in $G$ such that $\bigcup_{G^* \in \mathscr{G}^*} V(G^*) = V(G)$ and $G^*$ is a claw-free graph for all $G^* \in \mathscr{G}^*$. Let $G^* \in \mathscr{G}^*$, and let $G^* = G[(V(B) \setminus X) \cup \varphi^{-1}(B)]$ for some $B \in \mathscr{B}_i$ with $0 \leq i \leq m$.

Let $u \in V(G^*)$. We first show that $d_{G^*}(u) \geq \lceil \frac{\delta(G)-1}{2} \rceil$.

*Case 1.* $u \in \varphi^{-1}(B)$ and $B \in \mathscr{B}_0$, i.e., $B = B_0$.

Then by the definition of $\varphi$, $|N_G(u) \cap V(B_0)| \geq \frac{\delta(G)}{2}$. Since $G[N_G(u) \cap V(B_0)]$ is a complete graph by (5.1), $|N_G(v) \cap V(B_0)| \geq |(N_G(u) \cap V(B_0)) \setminus \{v\}| + |\{u\}| \geq \frac{\delta(G)}{2}$ for all $v \in N_G(u) \cap V(B_0)$. Hence by the definition of $\varphi$, $\varphi(v) = B_0$ for all $v \in N_G(u) \cap V(B_0) \cap X$. Since $G^* = G[(V(B_0) \setminus X) \cup \varphi^{-1}(B_0)]$, this implies that $(N_G(u) \cap V(B_0)) \cup \{u\} \subseteq V(G^*)$, and hence $d_{G^*}(u) = |N_G(u) \cap V(B_0)| \geq \frac{\delta(G)}{2}$.

*Case 2.* $u \in \varphi^{-1}(B)$ and $B \in \mathscr{B}_i$ with $1 \leq i \leq m$.

Assume for the moment that $u = v_B$, or $u \neq v_B$ and $\varphi(v_B) = B$. Then by the definition of $\varphi$, $|N_G(u) \cap V(B)| \geq \frac{\delta(G)}{2}$. Since $G[N_G(u) \cap V(B)]$ is a complete graph by (5.1), $|N_G(v) \cap V(B)| \geq \frac{\delta(G)}{2}$ for all $v \in N_G(u) \cap V(B)$. Since $\varphi(v_B) = B$, it follows from the definition of $\varphi$ that $\varphi(v) = B$ for all $v \in N_G(u) \cap V(B) \cap X$. Since $G^* = G[(V(B) \setminus X) \cup \varphi^{-1}(B)]$, this implies that $(N_G(u) \cap V(B)) \cup \{u\} \subseteq V(G^*)$, and hence $d_{G^*}(u) = |N_G(u) \cap V(B)| \geq \frac{\delta(G)}{2}$. Thus we may assume that $u \neq v_B$ and $\varphi(v_B) \neq B$. Then by the definition of $\varphi$, (i) $|N_G(u) \cap V(B)| \geq \frac{\delta(G)+1}{2}$ or (ii) $|N_G(u) \cap V(B)| = \frac{\delta(G)}{2}$ and $uv_B \notin E(G)$. Since $G[N_G(u) \cap V(B)]$ is a complete graph by (5.1), it follows that for each $v \in N_G(u) \cap V(B)$, either (i') $|N_G(v) \cap V(B)| \geq \frac{\delta(G)+1}{2}$ or (ii') $|N_G(v) \cap V(B)| = \frac{\delta(G)}{2}$ and $vv_B \notin E(G)$ holds. Hence by the definition of $\varphi$, $\varphi(v) = B$ for all $v \in N_G(u) \cap V(B) \cap (X \setminus \{v_B\})$. Since $G^* = G[(V(B) \setminus X) \cup \varphi^{-1}(B)]$, this implies that

$\big((N_G(u) \cap V(B)) \cup \{u\}\big) \setminus \{v_B\} \subseteq V(G^*)$ (note that if (ii) holds, then $(N_G(u) \cap V(B)) \cup \{u\} \subseteq V(G^*)$), and hence $d_{G^*}(u) \geq |(N_G(u) \cap V(B)) \setminus \{v_B\}| \geq \frac{\delta(G)-1}{2}$.

*Case 3.* $u \notin \varphi^{-1}(B)$, i.e., $u \in V(B) \setminus X$.

Since $u \in V(B) \setminus X$, $d_B(u) = d_G(u) \geq \delta(G)$. If $|N_B(u) \setminus X| \geq \frac{\delta(G)-1}{2}$, then $G^* = G[(V(B) \setminus X) \cup \varphi^{-1}(B)]$ implies that $d_{G^*}(u) \geq |N_B(u) \setminus X| \geq \frac{\delta(G)-1}{2}$. Thus we may assume that $|N_B(u) \setminus X| \leq \frac{\delta(G)-2}{2}$, that is, $|N_B(u) \cap X| \geq \frac{\delta(G)+2}{2}$ ( $\geq 3$). In the rest of Case 3, we let $v^* = v_B$ if $B \neq B_0$; otherwise, let $v^*$ be an arbitrary vertex in $G - B$.

Suppose that $G[N_B(u) \cap X]$ is a complete graph. Then $|N_G(v) \cap V(B)| \geq |(N_B(u) \cap X) \setminus \{v\}| + |\{u\}| \geq \frac{\delta(G)+2}{2}$ for all $v \in N_B(u) \cap X$. Hence by the definition of $\varphi$, $\varphi(v) = B$ for all $v \in (N_B(u) \cap X) \setminus \{v^*\}$. Since $G^* = G[(V(B) \setminus X) \cup \varphi^{-1}(B)]$, this implies that $(N_B(u) \cap X) \setminus \{v^*\} \subseteq V(G^*)$, and hence $d_{G^*}(u) \geq \frac{\delta(G)}{2}$. Thus we may assume that $G[N_B(u) \cap X]$ is not complete.

Then by Lemma 10, there exists a complete subgraph $F$ of $B[N_B(u) \cup \{u\}]$ such that (i) $u \in V(F)$ and $|F| \geq \frac{\delta(G)+3}{2}$ or (ii) $u \in V(F)$, $|F| = \frac{\delta(G)+2}{2}$ and $v^* \notin V(F)$. Then since $F$ is a complete graph, it follows that for each $v \in V(F)$, either (i') $|N_G(v) \cap V(B)| \geq \frac{\delta(G)+1}{2}$ or (ii') $|N_G(v) \cap V(B)| = \frac{\delta(G)}{2}$ and $vv^* \notin E(G)$ holds. Hence by the definition of $\varphi$, $\varphi(v) = B$ for all $v \in (V(F) \cap X) \setminus \{v^*\}$. Since $G^* = G[(V(B) \setminus X) \cup \varphi^{-1}(B)]$, this implies that $V(F) \setminus \{v^*\} \subseteq V(G^*)$ (note that if (ii) holds, then $V(F) \subseteq V(G^*)$), and hence $d_{G^*}(u) \geq |V(F) \setminus \{u, v^*\}| \geq \frac{\delta(G)-1}{2}$.

By Cases 1–3, $d_{G^*}(u) \geq \lceil \frac{\delta(G)-1}{2} \rceil$. Since $u$ is an arbitrary vertex in $G^*$, $\delta(G^*) \geq \lceil \frac{\delta(G)-1}{2} \rceil$.

We next show that $G^*$ is 2-connected. Suppose that $G^*$ is not 2-connected. Since $\delta(G^*) \geq \lceil \frac{\delta(G)-1}{2} \rceil \geq 2$, $|G^*| \geq 3$. Hence there exist distinct two subgraphs $F_1$ and $F_2$ of $G^*$ such that $V(F_j) \setminus V(F_{3-j}) \neq \emptyset$ for $j = 1, 2$, and $|V(F_1) \cap V(F_2)| \leq 1$, $V(G^*) = V(F_1) \cup V(F_2)$ and $E_G(F_1 - F_2, F_2 - F_1) = \emptyset$. Since $|B| \geq |G^*| \geq 3$, we also have that $B$ is 2-connected. Hence $V(B - G^*) = (V(B) \cap X) \setminus \varphi^{-1}(B) \neq \emptyset$. Let $w \in V(F_1) \cap V(F_2)$ if $V(F_1) \cap V(F_2) \neq \emptyset$; otherwise, let $w$ be an arbitrary vertex in $B - G^*$. Let $U = \{u \in V(B - G^*) \mid N_G(u) \cap (V(F_1) \setminus \{w\}) \neq \emptyset\}$. Since $B$ is 2-connected, $U \neq \emptyset$. If there exists $u \in U$ such that $N_G(u) \cap (V(F_2) \setminus \{w\}) \neq \emptyset$, then $G[V(F_1 \cup F_2 \cup B_{u^+}) \cup \{u\}]$ contains $K_{1,3}$ as an induced subgraph of $G$ because $E_G(V(F_1) \setminus \{w\}, V(F_2) \setminus \{w\}) = \emptyset$ and $E_G(B - \{u\}, B_{u^+} - \{u\}) = \emptyset$, a contradiction. Thus $N_G(u) \cap (V(F_2) \setminus \{w\}) = \emptyset$ for all $u \in U$. Hence since $B$ is 2-connected, there exist $u \in U$ and $v \in V(B) \setminus (V(G^*) \cup U)$ such that $uv \in E(G)$. Then $G[V(F_1 \cup B_{u^+}) \cup \{u, v\}]$ contains $K_{1,3}$ as an induced subgraph of $G$ because $N_G(v) \cap (V(F_1) \setminus \{w\}) = \emptyset$ and $E_G(B - \{u\}, B_{u^+} - \{u\}) = \emptyset$, a contradiction again. Thus $G^*$ is 2-connected.

This completes the proof of Theorem 5. □

**Proof of Theorem 4.** Let $G$ be a claw-free graph with $\delta(G) \geq 4$. We show that $G$ has a 2-factor in which each component contains at least $\lceil \frac{\delta(G)-1}{2} \rceil$ vertices. By Theorem A, we may assume that $\delta(G) \geq 6$. Then by applying Theorem 5, we have that there exists a set $\mathscr{G}^*$ of vertex-disjoint subgraphs in $G$ such that $\bigcup_{G^* \in \mathscr{G}^*} V(G^*) = V(G)$, and $G^*$ is a 2-connected claw-free graph and $\delta(G^*) \geq \lceil \frac{\delta(G)-1}{2} \rceil \geq 3$ for all $G^* \in \mathscr{G}^*$. Hence by Theorem 1, each graph $G^*$ in $\mathscr{G}^*$ has a 2-factor in which each cycle contains at least $\delta(G^*)$ $\left( \geq \lceil \frac{\delta(G)-1}{2} \rceil \right)$ vertices. Since $V(G) = \bigcup_{G^* \in \mathscr{G}^*} V(G^*)$ (disjoint union), this implies that $G$ has a desired 2-factor. $\Box$

# 6 Sharpness of Conjecture 3 and Theorem 5

We give a graph showing the sharpness of the lower bound on the length of a cycle in Conjecture 3. The graph $G_2$ illustrated in Figure 6 (here, in Figure 6, "+" means the join of vertices) is a claw-free graph with minimum degree $d$ $(\geq 4)$, and for each 2-factor of $G_2$, the minimum length of cycles in it is at most $\lceil \frac{d+1}{2} \rceil$, and $G_2$ has a 2-factor such that the minimum length of cycles in it is just $\lceil \frac{d+1}{2} \rceil$. To see this, let $\mathscr{C}$ be a set of vertex-disjoint cycles in $G_2$ such that $\bigcup_{C \in \mathscr{C}} C$ is a 2-factor of $G_2$. If $x \in X$ and $y \in Y$ are contained in same cycle in $\mathscr{C}$, then by the definition of $G_2$, the maximum size of the cycle containing both $x$ and $y$ is $\lceil \frac{d+1}{2} \rceil$. Hence by the symmetry, we may assume that for all $x \in X \cup X'$ and all $y \in Y \cup Y'$, the cycle containing $x$ and the cycle containing $y$ in $\mathscr{C}$ are distinct. Then by the definition of $G_2$, there exists a subset $\mathscr{C}'$ of $\mathscr{C}$ such that $\bigcup_{C \in \mathscr{C}'} V(C) = X$ or $\bigcup_{C \in \mathscr{C}'} V(C) = X'$. Since $|X| = |X'| = \lceil \frac{d+1}{2} \rceil$, the minimum length of cycles in $\mathscr{C}$ is at most $\lceil \frac{d+1}{2} \rceil$. Moreover, by these arguments, we can take a set $\mathscr{C}$ so that the minimum length of cycles in $\mathscr{C}$ is just $\lceil \frac{d+1}{2} \rceil$. (Note that by adding $X$, $Y$, $X'$ and $Y'$ iteratively, we can construct a sufficiently large graph.)

Obviously the above example also shows the sharpness of the lower bound of $\delta(G^*)$ in Theorem 5.

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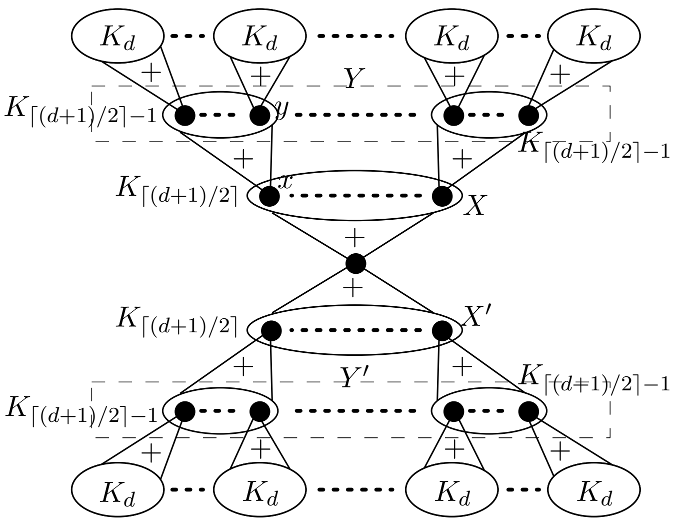


Figure 6: The graph $G_2$


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