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An Extremal Problem Resulting in Many Paths

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Abstract

For a bipartite graph the extremal number for the existence of a specific odd (even) length path was determined in J. Graph Theory 8 (1984), 83-95. In this article, we conjecture that for a balanced bipartite graph with partite sets of odd order the extremal number for an even order path guarantees many more paths of differing lengths. The conjecture is proved for a linear portion of the conjectured paths.

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The second author dedicates this article to Sasa Yoshimoto.

1 Introduction

In [2] the extremal number is given for a path to be embeddable in a bipartite graph. We first describe a specific bipartite graph that determines the extremal number for the path P_{2k+2} of order $2k+2$, k a positive integer.

Let $K_{A,B}$ be bipartite with partite sets A and B, $|A| = |B| = 2k+1$, k a positive integer. Further partition both A and B into two sets of order k and $k+1$. Joining all vertices in the k $(k+1)$ element set of A to the $k+1$ (k) element set of B gives a graph G with $2k^2 + 2k$ edges composed of two vertex disjoint copies of $K_{k,k+1}$. This graph G clearly contains no path P_{2k+2}. Surely G is extremal for P_{2k+2}, since

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the addition of any edge gives a graph with $2k^2+2k+1$ edges which contains the path P_{2k+2}. In [2] this is proved, that for a balanced bipartite graph with parts of order $2k+1$, the path P_{2k+2} has extremal number $2k^2+2k$, i.e., any such balanced bipartite graph with $2k^2+2k+1$ edges contains a P_{2k+2}.

Thus let G be as described and let G' denote the graph obtained from G by adding an additional edge (there are two such nonisomorphic graphs). Interestingly this graph G' satisfies a much stronger property. Let C be any $k+1$ element subset of A in G'. Then for each fixed l, $2 \leq l \leq k+1$, it is easily checked that G' contains $k+1$ distinct paths with one end vertex in C and $k+1$ different end vertices in B. This example suggests the following conjecture.

Conjecture 1. *Let G be a subgraph of the complete bipartite graph $K_{2k+1,2k+1}$ of size $e(G) \geq 2k^2+2k+1$ with partite sets A and B. Then for each $k+1$ element subset $C \subset A$ and $2 \leq l \leq k+1$, there exist $k+1$ paths of order $2l$ with one end vertex in C and each of the $k+1$ paths with a different end vertex in B.*

This conjecture, if true, is interesting in that an extremal number for a fixed P_{2k+2} implies the existence of many different P_{2k+2}'s, starting in an arbitrary $k+1$ element set in A and ending at different $k+1$ elements in B. In addition the truth of the conjecture would imply that the same is true for all P_{2l}'s, $2 \leq l \leq k-1$.

The objective of this article is to give credibility to the conjecture by proving it holds for at least $k/9$ values of l. In addition the truth of the conjecture would appear to be applicable, for example, in Ramsey questions involving the existence of cycles. At this point there seems to be no comparable extremal result which forces the existence of many similar well defined paths from the extremal number of a single path.

All notation and terminology not explained here is given in [1].

2 Main result and the Proof

The remainder of this article is devoted to the proof of the following theorem. Its proof is somewhat technical and after some introductory notation and basic observations is broken into three separate cases. For a vertex subset W of a graph G, we denote the maximal degree $\max\{d_G(w) : w \in W\}$ by $\Delta_G(W)$, the minimal degree by $\delta_G(W)$ and $|N_G(W)|$ by $d_G(W)$

Theorem 2. *Conjecture 1 holds for at least $k/9$ values of l.*

Proof. Let $a_0, a_1, \ldots, a_{2k}$ be the vertices in A such that $d_G(a_i) \geq d_G(a_{i+1})$ for all $i \leq 2k-1$. Let $\Delta_G(A) = d_G(a_0) = k+r$, $A_j = \{a_i : d_G(a_i) \geq j\}$, and $A_j^* = A_j \setminus \{a_0\}$. Since the degree of $a_{|A_{k+1}|}$ is at most k,

$$\begin{aligned}
&(k+r)|A_{k+1}| + k(2k+1-|A_{k+1}|) \geq \sum_{i=0}^{|A_{k+1}|-1} d_G(a_i) + \sum_{i=|A_{k+1}|}^{2k} d_G(a_i) \\
&\geq 2k^2 + 2k + 1 \\
&\Longleftrightarrow |A_{k+1}| \geq \frac{k+1}{r}. \qquad (1)
\end{aligned}$$

Let U be a subset of A and γ a positive real number. Let $H(U, \gamma)$ be the graph whose vertex set is U and edge set is $\{uv : |N_G(u) \cap N_G(v)| \geq \gamma\}$. Suppose U contains three vertices u, v, w such that all of $|N_G(u) \cap N_G(v)|, |N_G(v) \cap N_G(w)|$ and $|N_G(w) \cap N_G(u)|$ are smaller than $k/9$. Then,

$$\begin{aligned}
d_G(U) &\geq |N_G(u) \cup N_G(v) \cup N_G(w)| \\
&\geq |N_G(u)| + |N_G(v)| + |N_G(w)| \\
&\quad -(|N_G(u) \cap N_G(v)| + |N_G(v) \cap N_G(w)| + |N_G(w) \cap N_G(u)|) \\
&> 3\delta_G(U) - 3k/9 \\
&\Longleftrightarrow \delta_G(U) < d_G(U)/3 + k/9.
\end{aligned}$$

Conversely, if $\delta_G(U) \geq d_G(U)/3 + k/9$, then for any three vertices in U, there are two vertices which are adjacent in $H(U, k/9)$. Therefore, the following claim holds.

Claim 1. *Let U be a vertex subset of A. If $\delta_G(U) \geq d_G(U)/3 + k/9$, then the stability of $H(U, k/9)$ is at most two.*

In particular, since $d_G(U) \le |B| = 2k+1$,

$$\text{if } \delta_G(U) \ge \frac{7}{9}k + \frac{1}{3}, \text{ then the stability of } H(U, k/9) \text{ is at most two.} \tag{2}$$

We denote the vertices in C by $c_0, c_1, \ldots, c_k$ where $d_G(c_i) \ge d_G(c_{i+1})$ for all $i \le k-1$. If $\Delta_G(C) \le k-r+1$, then

$$2k^2+2k+1 \le (k-r+1)|C| + (k+r)(|A|-|C|) = 2k^2+2k-r+1 < 2k^2+2k+1.$$

Therefore, since $|N_G(a_0) \cup N_G(c_0)| \le |B| = 2k+1$,

$$\Delta_G(C) \ge k-r+2 \text{ and } |N_G(a_0) \cap N_G(c_0)| \ge 1. \tag{3}$$

We divide the remainder of the proof into three cases.

Case 1. $\Delta_G(A) = k+1$, *i.e.*, $r = 1$.
From (1), $|A_{k+1}| \ge k+1$, and so A_{k+1} contains a vertex of C. From (2), the stability of $H(A_{k+1}, k/9)$ is at most two. Therefore, $H(A_{k+1}, k/9)$ has a hamilton path or is the union of two cliques.

1. Suppose that there is a component X in $H(A_{k+1}, k/9)$ containing a vertex z of C such that $|X| \ge k/9$. Since X is a clique or $X = H(A_{k+1}, k/9)$, obviously for any $2 \le l \le k/9$, there is a path $P = x_1x_2\cdots x_l$ in X where $x_l = z$. Since $|N_G(x_i) \cap N_G(x_{i+1})| \ge k/9$, for any $y \in N_G(x_1)$, there exists a path

$$P_y = yx_1y_1x_2\cdots x_{l-1}y_{l-1}x_l$$

where $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y_1, y_2, \ldots, y_{i-1}\}$ for $1 \le i \le l-1$. Since $d_G(x_1) = k+1$ and $x_l = z \in C$, the set $\{P_y : y \in N_G(x_1)\}$ gives the desired set of $k+1$ paths.

2. Assume that any component in $H(A_{k+1}, k/9)$ contains no vertex in C or the order is less than $k/9$. Since A_{k+1} contains a vertex of C, $H(A_{k+1}, k/9)$ is the union of two cliques. Let X be the largest component in $H(A_{k+1}, k/9)$. Since the other component contains a vertex of C, $|X| > 8k/9$. As C contains a vertex of degree $k+1$ and $|B| = 2k+1$, $|N_G(C) \cap N_G(X)| \ge 1$.

2.1. Suppose that $|N_G(C) \cap N_G(X)| \geq 2$. Let $y^1, y^2 \in N_G(C) \cap N_G(X)$ and $z^j \in N_G(y^j) \cap C$ and $x^j \in N_G(y^j) \cap X$ for $j = 1, 2$, i.e., G contains two paths $x^1y^1z^1$ and $x^2y^2z^2$. For any $2 \leq l \leq k/9$, let $P = x_1x_2 \cdots x_{l-1}$ be a path in X where $x_{l-1} = x^1$. For any $y \in N_G(x_1) \setminus \{y^1\}$, there exist a path

$$P_y^1 = yx_1y_1x_2 \cdots x_{l-1}y^1z^1$$

where $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y^1, y_1, y_2, \ldots, y_{i-1}\}$ for $1 \leq i \leq l-2$. Since $d_G(x_1) = k+1$, the set $\{P_y : y \in N_G(x_1)\}$ gives the desired set of at least k. If $y^1 \in N_G(x_1)$, then by using $x^2y^2z^2$, we can obtain one more desired path as above.

2.2. Assume that $|N_G(C) \cap N_G(X)| = 1$. This implies for any $x \in X$, $N_G(X) \setminus N_G(C) = B \setminus N_G(C)$ and $d_G(C) = k+1$, and

$$\text{for any } z \in C,\ |N_G(z) \cap N_G(c_0)| \geq d_G(z). \tag{4}$$

Let $U = \{c_0, c_1, \ldots, c_{\lceil 2k/9-1 \rceil}\}$.

2.2.a. Suppose $\delta_G(U) \geq d_G(U)/3 + k/9$, then from Claim 1 the stability of $H(U, k/9)$ is at most two. Let X_C be a largest component in $H(U, k/9)$. From (4), $c_0 \in X_C$. For any $2 \leq l \leq k/9$, there is a path $z_1z_2 \cdots z_l$ in X_C where $z_1 = c_0$. For any $y \in N_G(c_0)$, there exists a path

$$P_y = yz_1y_1z_2 \cdots z_{l-1}y_{l-1}z_l$$

where $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y_1, y_2, \ldots, y_{i-1}\}$ for $1 \leq i \leq l-1$. Since $d_G(c_0) = k+1$, the set $\{P_y : y \in N_G(c_0)\}$ gives the desired $k+1$ paths of order $2l$.

2.2.b Suppose $\delta_G(U) < d_G(U)/3 + k/9 = (k+1)/3 + k/9 = (4k+3)/9$, then $d_G(c_{\lceil 2k/9 \rceil})$ is also smaller than $(4k+3)/9$. Since $|C \setminus U| = \lceil 7k/9 + 1 \rceil$ and $|A \setminus (C \setminus U)| \geq 2k + 1 - 7k/9 - 2$, the number of non-adjacent pairs between A and B is:

$$\begin{aligned}
& (2k+1)^2 - (2k^2 + 2k + 1) = 2k^2 + 2k \\
\geq\ & (2k+1-(k+1))(|A \setminus (C \setminus U)| - 1) + (2k + 1 - \frac{4k+3}{9})(|C \setminus U| + 1) \\
>\ & (2k+1-(k+1))(2k - \frac{7k}{9} - 2) + (2k + 1 - \frac{4k+3}{9})(\frac{7}{9}k + 2) \\
=\ & \frac{197}{81}k^2 + \frac{44}{27}k + \frac{4}{3},
\end{aligned}$$

a contradiction.

Therefore, for the remainder of the proof

$$\Delta_G(A) \geq k+2, \text{ i.e., } r \geq 2.$$

Case 2. $|A^*_{k+1}| \geq 2k/9 - 2$.
From (2), the stability of $H(A^*_{k+1}, k/9)$ is at most two. Since $k+1+k+r \geq 2k+3$,

$$|N_G(a_0) \cap N_G(x)| \geq 2 \text{ for any } x \in A^*_{k+1}. \tag{5}$$

1. Suppose that there is a component X in $H(A^*_{k+1}, k/9)$ containing a vertex z of C such that $|X| \geq k/9 - 1$. Since X is a clique or $X = H(A^*_{k+1}, k/9)$, obviously for any $2 \leq l \leq k/9$, there is a path $P = x_2 \cdots x_l$ in X where $x_l = z$. For each $y \in N_G(a_0)$ and $y' \in (N_G(a_0) \cap N_G(x_2)) \setminus \{y\}$, there exists a path

$$P_y = ya_0y'x_2y_2 \cdots x_{l-1}y_{l-1}x_l$$

in G such that $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y', y_2, \ldots, y_{i-1}\}$ for all $2 \leq i \leq l-1$. Since $d_G(a_0) \geq k+r$, the set $\{P_y : y \in N_G(a_0)\}$ gives a desired set of $k+1$ paths of order $2l$.

2. Suppose that any component in $H(A^*_{k+1}, k/9)$ contains no vertex in C or the order is less than $k/9 - 1$. Let X be a largest component in $H(A^*_{k+1}, k/9)$. Then $|X| \geq k/9 - 1$.

2.1. Suppose there exist $z \in C$ and $x \in X$ such that $|N_G(z) \cap N_G(x)| \geq 2$. For any $2 \leq l \leq k/9$, let $P = x_1x_2 \cdots x_{l-1}$ be a path in X where $x_{l-1} = x$. For any $y \in N_G(x_1)$, there exist $y' \in (N_G(z) \cap N_G(x)) \setminus \{y\}$ and a path

$$P_y = yx_1y_1x_2 \cdots x_{l-1}y'z$$

where $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y', y_1, y_2, \ldots, y_{i-1}\}$ for $1 \leq i \leq l-2$. Since $d_G(x_1) = k+1$, the set $\{P_y : y \in N_G(x_1)\}$ gives the desired set of $k+1$ paths.

2.2. Suppose

$$\text{for any } z \in C \text{ and } x \in X,\ |N_G(z) \cap N_G(x)| \leq 1. \tag{6}$$

This implies $a_0 \neq c_0$ by (5) and $|N_G(a_0) \cap N_G(c_0)| \geq 1$ from (3). Let $y' \in N_G(a_0) \cap N_G(c_0)$. Then, obviously $\{P_y = ya_0y'c_0 : y \in N_G(a_0) \setminus \{y'\}\}$ contains the desired $k+1$ paths of order 4. Hence in the following we consider when $3 \leq l \leq k/9$.

2.2.a. Suppose $|N_G(a_0) \cap N_G(c_0)| \geq 2$. For any $3 \leq l \leq k/9$, let $P = x_1x_2 \cdots x_{l-2}$ be any path in X. At first, we specify $y \in N_G(x_1)$, and let $y' \in (N_G(a_0) \cap N_G(c_0)) \setminus \{y\}$. If $N_G(a_0) \cap N_G(x_{l-2}) \neq \{y, y'\}$, then we can choose $y'' \in (N_G(a_0) \cap N_G(x_{l-2})) \setminus \{y, y'\}$. If $N_G(a_0) \cap N_G(x_{l-2}) = \{y, y'\}$, then from (6), $y \notin N_G(a_0) \cap N_G(c_0)$. Hence we can choose $y''' \in (N_G(a_0) \cap N_G(c_0)) \setminus \{y, y'\}$. In either case, as in the above, we can construct a path

$$P_y = yx_1y_1x_2y_2 \cdots x_{l-2}y''a_0y'c_0 \text{ or } yx_1y_1x_2y_2 \cdots x_{l-2}y'a_0y'''c_0$$

in G, respectively. Since $d_G(x_1) \geq k+1$, we have the desired $k+1$ paths of order $2l$.

2.2.b. If $|N_G(a_0) \cap N_G(c_0)| \leq 1$, then equality holds and $d_G(c_0) = k - r + 2$ from (3). Let $\{y'\} = N_G(a_0) \cap N_G(c_0)$, and then $N_G(c_0) = (B \setminus N_G(a_0)) \cup \{y'\}$.

Suppose there is a vertex $x \in X$ such that $N_G(x) \setminus N_G(a_0) \neq \emptyset$. Let $y'' \in N_G(x) \setminus N_G(a_0)$. Since $N_G(c_0) = (B \setminus N_G(a_0)) \cup \{y'\}$ and $y' \in N_G(a_0)$, $y'' \in N_G(c_0)$, let P be a path $x_2x_3 \cdots x_{l-1}$ in X where $x_{l-1} = x$ and $3 \leq l \leq k/9$. For any $y \in N_G(a_0)$, we can construct a path

$$P_y = ya_0y_1x_2y_2 \cdots x_{l-1}y''c_0$$

in which $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y'', y_1, y_2, \ldots, y_{i-1}\}$ for $i \leq l-2$. Since $d_G(a_0) = k + r$, there are $k+1$ paths of order $2l$.

Suppose $N_G(X) \subset N_G(a_0)$. Let $x_1x_2 \cdots x_{l-2}$ be a path in X for $3 \leq l \leq k/9$. For any $y \in N_G(x_1) \setminus \{y\}$, there is $y'' \in (N_G(x_{l-2}) \cap N_G(a_0)) \setminus \{y, y'\}$ since $|N_G(x_{l-2}) \cap N_G(a_0)| = d_G(x_{l-2}) \geq k+1$. Thus we can construct a path

$$P_y = yx_1y_1x_2y_2 \cdots x_{l-2}y''a_0y'c_0$$

in which $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y', y'', y_1, y_2, \ldots, y_{i-1}\}$ for $i \leq l-3$. Hence, if $|N_G(x_1) \setminus \{y'\}| \geq k+1$, then there are $k+1$ paths of order $2l$. If $|N_G(x_1) \setminus \{y'\}| = k$, then $y' \in N_G(x_1)$, and so for $y''' \in N_G(a_0) \setminus N_G(x_1)$, we can obtain $(k+1)$th path

$$y'''a_0y''x_{l-2}\cdots y_2x_2y_1x_1y'c_0.$$

Case 3. $|A^*_{k+1}| < 2k/9 - 2$.

From (1), since $2k/9 - 2 > |A^*_{k+1}| = |A_{k+1}| - 1 \geq (k+1)/r - 1$, $r \geq 5$. Let

$$m = d_G(a_{k+1}) \text{ and } p = |\{a_i : d_G(a_i) \leq k \text{ and } 1 \leq i \leq k\}|.$$

Then $p > k - (2k/9 - 2) = 7k/9 + 2$ and

$$\begin{aligned}
&2k^2 + 2k + 1 \leq e(G) \leq \sum_{i=0}^{k-p} d_G(a_i) + \sum_{i=k-p+1}^{k} d_G(a_i) + \sum_{i=k+1}^{2k} d_G(a_i)\\
&\leq (2k+1)(k-p+1) + kp + mk\\
&\Longleftrightarrow km \geq pk + p - k\\
&\Longrightarrow m \geq p > 7k/9 + 2 > 7k/9 + 1/3.
\end{aligned}$$

Hence,

$$\text{the stability of } H(A^*_m, k/9) \text{ is at most two}$$

from (2). Furthermore

$$\begin{aligned}
&2k^2 + 2k + 1 \leq e(G) \leq \sum_{i=0}^{k-p} d_G(a_i) + \sum_{i=k-p+1}^{k} d_G(a_i) + \sum_{i=k+1}^{2k} d_G(a_i)\\
&\leq (k+r)(k-p+1) + kp + mk\\
&\Longleftrightarrow km \geq k^2 + k - kr + pr - r + 1\\
&\Longrightarrow m \geq k + 1 - r + \frac{pr - r + 1}{k}.
\end{aligned}$$

Since $r \geq 5$ and $p > 7k/9 + 2$, the following inequalities hold:

$$\begin{aligned}
&m + (k+r) \geq 2k + 1 + \frac{pr - r + 1}{k} \geq 2k + 3\\
\Longleftrightarrow\quad &r(p-1) \geq 2k - 1.
\end{aligned}$$

Therefore

$$\text{for any } x \in A^*_m \text{ and } a_i \in A_{k+r},\ N_G(x) \cap N_G(a_i) \geq 2. \tag{7}$$

Notice that as $|A_m^*| \geq k+1$, A_m^* contains a vertex z in $C \setminus \{a_0\}$. Let $y' \in N_G(a_0) \cap N_G(z)$. Then, obviously $\{P_y = ya_0y'z : y \in N_G(a_0) \setminus \{y'\}\}$ contains the desired $k+1$ paths of order 4. Hence in the following, we consider the case when $3 \leq l \leq k/9$.

1. Recall that the stability of $H(A_m^*, k/9)$ is two. Therefore if $H(A_m^*, k/9)$ has a component X which contains a vertex z of C with $|X| \geq k/9 - 1$, then we can construct the desired $k+1$ paths of order $2l$ for any $3 \leq l \leq k/9$ as done in part 1 in Case 2.

2. Suppose that each component X in $H(A_m^*, k/9)$ has no vertex of C or $|X| < k/9 - 1$. Since $A_m^* \cap (C \setminus \{a_0\}) \neq \emptyset$, $H(A_m^*, k/9)$ is the union of two cliques X and X' such that $|X| > 8k/9 + 1$, $X \cap C = \emptyset$, $|X'| < k/9 - 1$ and $X' \cap C \neq \emptyset$.

2.1. Suppose that there are $x \in X$ and $z \in C \setminus \{a_0\}$ such that $N_G(x) \cap N_G(z) \neq \emptyset$. Let $y' \in N_G(x) \cap N_G(z)$ and $P = x_2x_3 \cdots x_{l-1}$ be a path in X where $x_{l-1} = x$ for any $3 \leq l \leq k/9$. Let $y'' \in (N_G(a_0) \cap N_G(x_2)) \setminus \{y'\}$. Then for any $y \in N_G(a_0) \setminus \{y', y''\}$, we can construct a path

$$P_y = ya_0y''x_2y_2 \cdots x_{l-1}y'z$$

in which $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y', y'', y_2, \ldots, y_{i-1}\}$ for $2 \leq i \leq l-2$. Since $d_G(a_0) \geq k + r \geq k + 5$, we obtain the desired $k+1$ paths of order $2l$.

2.2. Assume that $N_G(X) \cap N_G(C \setminus \{a_0\}) = \emptyset$, i.e., $N_G(C \setminus \{a_0\}) \subset B \setminus N_G(X)$.

2.2.a. Suppose $d_G(X) \geq k+1$. If $a_0 \in C$, then for any $y \in N_G(X)$ and $3 \leq l \leq k/9$, there is a path $P = x_1x_2 \cdots x_{l-1}$ in X such that $y \in N_G(x_1)$, and $y'' \in (N_G(x_{l-1}) \cap N_G(a_0)) \setminus \{y\}$, and so there is a path

$$P_y = yx_1y_1x_2y_2 \cdots x_{l-1}y''a_0$$

in which $y_i \in (N_G(x_i) \cap N_G(x_{i+1})) \setminus \{y, y', y_1, y_2, \ldots, y_{i-1}\}$ for $1 \leq i \leq l-2$. Since $d_G(X) \geq k+1$, we obtain desired $k+1$ paths of length $2l$.

If $a_0 \notin C$, then from (3), $N_G(a_0) \cap N_G(c_0) \neq \emptyset$. Let $y' \in N_G(a_0) \cap N_G(c_0)$. Notice that $y' \notin N_G(X)$. For any $y \in N_G(X)$ and $3 \leq l \leq k/9$, there is a path $P = x_1x_2 \cdots x_{l-2}$ in X such that $y \in N_G(x_1)$. For $y'' \in (N_G(x_{l-2}) \cap N_G(a_0)) \setminus \{y\}$, we can construct a path

$$P_y = yx_1y_1x_2y_2\cdots x_{l-2}y''a_0y'c_0$$

in which $y_i \in (N_G(x_i)\cap N_G(x_{i+1}))\setminus\{y,y',y'',y_1,y_2,\ldots,y_{i-1}\}$ for $1\le i\le l-3$. Since $d_G(X)\ge k+1$, we obtain the desired $k+1$ paths of order $2l$.

2.2.b. Assume $d_G(X)\le k$. Let $U=\{c_0,c_1,\ldots,c_{\lceil 2k/9-2\rceil}\}\setminus\{a_0\}$.

Suppose

$$\delta_G(U)\ge \frac{d_G(U)}{3}+\frac{k}{9},$$

and then from Claim 1 the stability of $H(U,k/9)$ is at most 2. Let X_C be a largest component in $H(U,k/9)$.

If there is $z\in X_C$ such that $N_G(z)\cap N_G(a_0)\ne\emptyset$, then there is a path $P=z_2z_3\cdots z_l$ in X_C for any $3\le l\le k/9$ where $z_2=z$. Let $y'\in N_G(z_2)\cap N_G(a_0)$. For any $y\in N_G(a_0)\setminus\{y'\}$, we can construct

$$P_y = ya_0y'z_2y_2\cdots z_l$$

in which $y_i\in(N_G(z_i)\cap N_G(z_{i+1}))\setminus\{y,y',y_2,\ldots,y_{i-1}\}$ for $2\le i\le l-1$. Since $d_G(a_0)\ge k+r$, we obtain the desired $k+1$ paths of order $2l$.

If $N_G(X_C)\cap N_G(a_0)=\emptyset$, then $\Delta_G(X_C)\le k-r+1$. This implies that $d_G(c_{\lceil 2k/9-2\rceil+1})\le k-r+1$. Let $L=\{c_i : \lceil 2k/9-2\rceil+1\le i\le k\}$. Since $|X|>8k/9+1$, $|X_C|\ge k/9-1$,

$$|L|=\lfloor 7k/9+2\rfloor>7k/9+1,\ |A\setminus(X\cup X_C\cup L)|<2k/9,$$

and $k+r>\max\{k,k-r+1\}$,

$$\begin{aligned}
&2k^2+2k+1\le e(G)\\
\le\ & \sum_{x\in X}d_G(x)+\sum_{z\in X_C}d_G(z)+\sum_{c_i\in L}d_G(c_i)+\sum_{a_i\in A\setminus(X\cup X_C\cup L)}d_G(a_i)\\
\le\ & k|X|+(k-r+1)|X_C|+(k-r+1)|L|+(k+r)|A\setminus(X\cup X_C\cup L)|\\
<\ & k(\frac{8}{9}k+1)+(k-r+1)(\frac{k}{9}-1)+(k-r+1)(\frac{7}{9}k+1)+(k+r)\frac{2}{9}k\\
=\ & 2k^2+\frac{17}{9}k-\frac{2r}{3}k<2k^2+2k+1.
\end{aligned}$$

This is a contradiction.

Therefore
$$\delta_G(U) < \frac{d_G(U)}{3} + \frac{k}{9} \leq \frac{2k+1-d_G(X)}{3} + \frac{k}{9}.$$

Since
$$d_G(X) \geq \delta_G(X) \geq m \geq k+1-r+\frac{pr-r+1}{k},$$

$2k+1-d_G(X) < k+r$, and so
$$k+r > \max\{d_G(X), 2k+1-d_G(X), \frac{2k+1-d_G(X))}{3} + \frac{k}{9}\}.$$

Since $|X| > 8k/9+1$, $|U| \geq 2k/9-2$, $|L| > 7k/9+1$ and $|A \setminus (X \cup U \cup L)| < k/9+1$,

$$\begin{aligned}
& 2k^2+2k+1 \leq e(G) \\
\leq \quad & \sum_{x \in X} d_G(x) + \sum_{z \in U} d_G(z) + \sum_{c_i \in L} d_G(c_i) + \sum_{a_i \in A \setminus (X \cup U \cup L)} d_G(a_i) \\
\leq \quad & d_G(X)(\frac{8}{9}k+1) + (2k+1-d_G(X))(\frac{2}{9}k-2) \\
& +(\frac{2k+1-d_G(X)}{3} + \frac{k}{9})(\frac{7}{9}k+1) + (k+r)(\frac{k}{9}+1) \\
= \quad & \frac{94}{81}k^2 + \frac{r}{9}k + \frac{11d_G(X)}{27}k + \frac{8d_G(X)}{3} - \frac{47}{27}k - \frac{5}{3} + r \\
\Longleftrightarrow \quad & \frac{r}{9}k + \frac{11d_G(X)}{27}k + \frac{8d_G(X)}{3} + r \geq \frac{68}{81}k^2 + \frac{101}{27}k + \frac{8}{3}
\end{aligned}$$

Since $d_G(X) \leq k$ and $r \leq k+1$

$$\begin{aligned}
& \frac{r}{9}k + \frac{11d_G(X)}{27}k + \frac{8d_G(X)}{3} + r \leq \frac{k+1}{9}k + \frac{11k}{27}k + \frac{8k}{3} + k + 1 \\
= \quad & \frac{11}{27}k^2 + \frac{34}{9}k + \frac{10}{9} < \frac{68}{81}k^2 + \frac{101}{27}k + \frac{8}{3},
\end{aligned}$$

a contradiction. This completes the proof of this case and the proof of the theorem.

□

References

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