Recent results on closures and forbidden subgraphs for Hamiltonian properties of graphs

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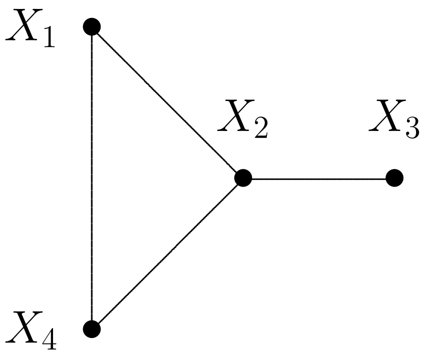
$\mathcal{X} = \{X_1, \ldots, X_k\}$ – a family of sets

$G = G(\mathcal{X})$ – the *intersection graph* of $\mathcal{X}$

$$\mathcal{X}: \quad \begin{array}{c} X_1 = \{1,2,3\} \\ X_2 = \{2,3,4\} \\ X_3 = \{4,5\} \\ X_4 = \{1,2\} \end{array}$$

G :



$V(G)$ – the *vertices* of G ("points")

$$V(G) = \{X_1, \ldots, X_k\}$$

$E(G)$ – the *edges* of G ("lines")

$$X_i X_j \in E(G) \iff X_i \cap X_j \neq \emptyset$$

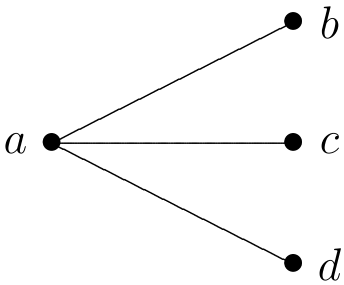
A *graph* is an ordered pair $G = (V(G), E(G))$, where
$V(G)$ is a (finite) set,
$E(G)$ is a subset of $\binom{V(G)}{2}$.

Example:

$V(G) = \{a, b, c, d\}$

$E(G) = \{\{a,b\}, \{a,c\}, \{a,d\}\}$

or simply $E(G) = \{ab, ac, ad\}$



Some classical applications

Connectivity problems, transportation networks

(obvious)

Graph coloring

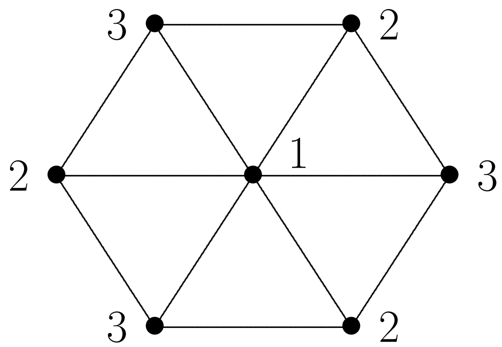
A *k-coloring* of a graph G is a function $t : V(G) \to \{1, \ldots, k\}$ such that, for any $x, y \in V(G)$,

$$xy \in E(G) \Rightarrow t(x) \neq t(y).$$

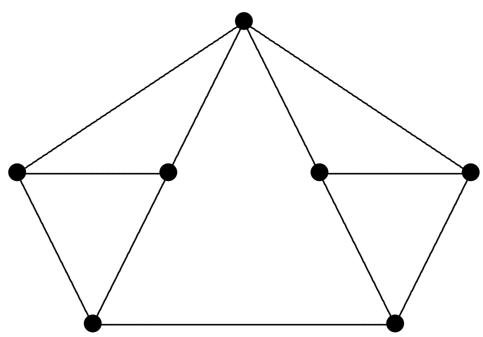
Equivalently: an assignment of "colors" $1, \ldots, k$ to the vertices of G such that adjacent vertices have different colors.

A graph G is *k-colorable* if G has a k-coloring.

Examples



A graph with a 3-coloring



This graph is not 3-colorable

Application: the *Channel assignment problem*

Vertices: transmitters (TV, mobile phone etc.)

Edges: two vertices are adjacent $\iff$ the signal from the corresponding transmitters interferes

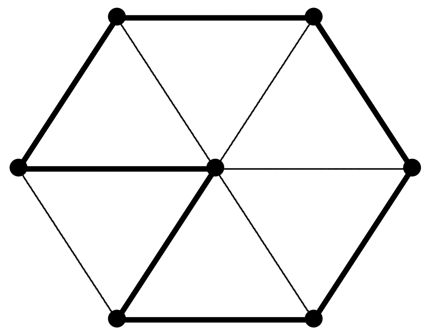
Hamiltonian problems

A *Hamiltonian cycle* in a graph G is a cycle C of length $n = |V(G)|$.

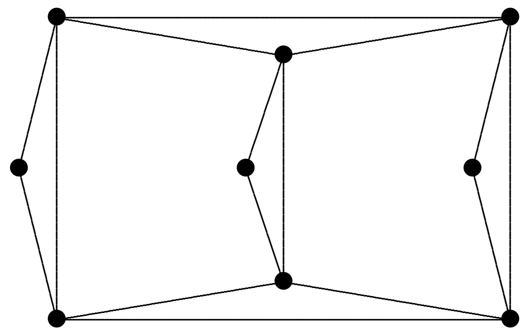
Equivalently: a cycle passing through every vertex exactly once.

A graph G is *Hamiltonian* if G has a Hamiltonian cycle.

Examples



A graph with a Hamiltonian cycle



This graph is not Hamiltonian

Application: the *Travelling Salesman Problem*

Vertices: cities

Edges: connections (roads, railway connections,...) with weights (distances)

Task: find a shortest Hamiltonian cycle

Graph: simple finite undirected, $n = |V(G)|$

G is *hamiltonian*: contains a cycle C_n

HAM
Instance: *A graph G*
Question: *Is G Hamiltonian?*
Output: *Y / N*

Well-known: HAM is **NP-complete**
(no known efficient algorithm)

Thus: sufficient conditions

- degree conditions ("brute force")
- structural conditions

Examples:

- **Theorem [Dirac 1952].** *Let G be a graph of order* $n \geq 3$.
  If $d_G(x) \geq n/2 \quad \forall x \in V(G)$, *then G is hamiltonian.*

  ($d_G(x)$ – the *degree* of x)

- **Theorem [Oberly, Sumner 1979].** *Let G be a graph of order* $n \geq 3$ *such that*
  (*i*) *for every* $x \in V(G)$, *the subgraph induced by the neighbors of* x *is connected,*
  (*ii*) *G does not contain an induced subgraph isomorphic to the graph* $K_{1,3}$.

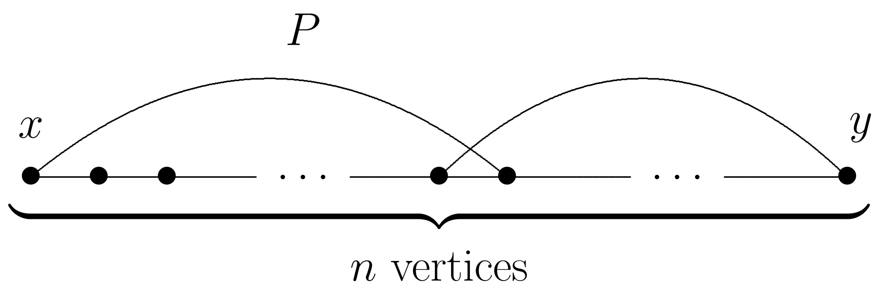
  Then G is hamiltonian.

Theorem [Ore, 1960].

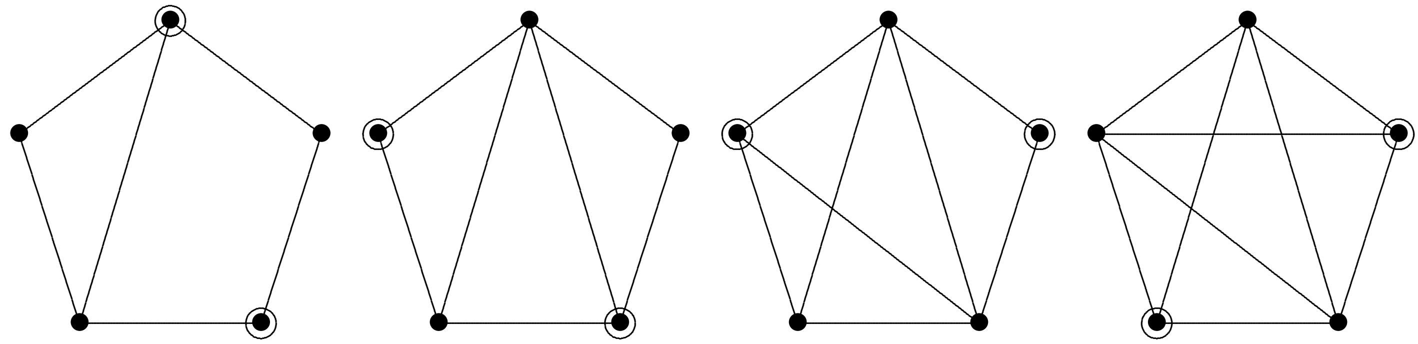
$$d_G(x) + d_G(y) \geq n \quad \forall x, y \in V(G) : xy \notin E(G)$$
$$\Downarrow$$
$$G \text{ is hamiltonian.}$$

Lemma [Ore, 1960].

$$d_G(x) + d_G(y) \geq n$$
$$\Downarrow$$
$$G \text{ is hamiltonian} \iff G + xy \text{ is hamiltonian.}$$



Bondy, Chvátal 1976: the *closure* of G



$\mathrm{cl}_n(G)$: the graph obtained from G by adding edges xy with $d(x) + d(y) \geq n$, as long as this is possible.

Properties:

- unique,
- G is hamiltonian $\iff$ $\mathrm{cl}_n(G)$ is hamiltonian.

Similar:

$$d_G(x) + d_G(y) \geq n - 1 \quad \forall x, y \in V(G) : xy \notin E(G)$$

$$\Downarrow$$

$$G \text{ is traceable (has a hamiltonian path)}$$

$$\downarrow$$

$$G \text{ is traceable} \iff \mathrm{cl}_{n-1}(G) \text{ is traceable}$$

$\mathcal{P}$ - a property:

$\mathcal{P}$ is said to be *k-stable* if $\quad G$ has $\mathcal{P} \iff \mathrm{cl}_k(G)$ has $\mathcal{P}$.

Stability number $s(\mathcal{P})$: smallest k such that $\mathcal{P}$ is *k-stable.*

$\mathcal{P}$ – hamiltonicity: $s(\mathcal{P}) = n$

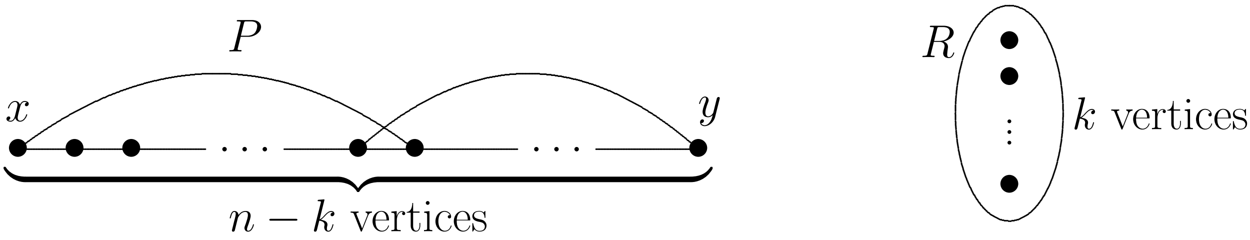
$\mathcal{P}$ – traceability: $s(\mathcal{P}) = n - 1$

$\mathcal{P}$ – Pancyclicity ???

G is *pancyclic*: contains a cycle C_ℓ, $3 \leq \ell \leq n$

Hamiltonicity: having a C_n

$\mathcal{P}_k$ – having a C_{n-k}



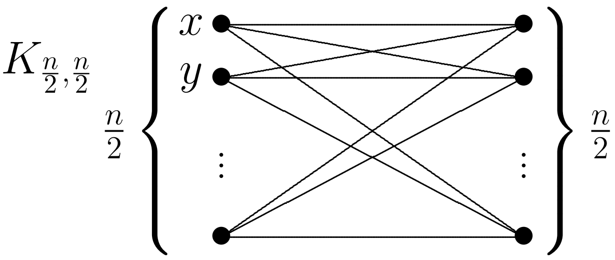
$$??? \qquad d_P(x) + d_P(y) \geq n - k \qquad ???$$

$$d_P(x) + d_P(y) = \underbrace{d(x) + d(y)}_{\geq n+k} - \underbrace{(d_R(x) + d_R(y))}_{\leq 2k} \geq n - k$$

$\mathcal{P}_k$ is $(n + k)$-stable.

$\mathcal{P}_1$: having a C_{n-1}

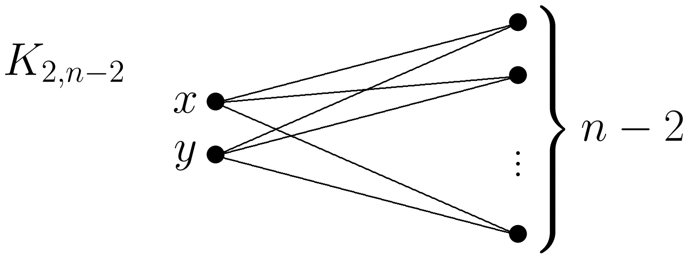
$s(\mathcal{P}_1) = n + 1$



$d(x) + d(y) = n$, no C_{n-1}

$\mathcal{P}_{n-3}$: having a C_3.

$s(\mathcal{P}_{n-3}) = 2n - 3$



$d(x) + d(y) = 2n - 4$, no C_3

Pancyclicity is $(2n-3)$-stable.

However, the graph $K_{2,n-2} + xy$ is not pancyclic.

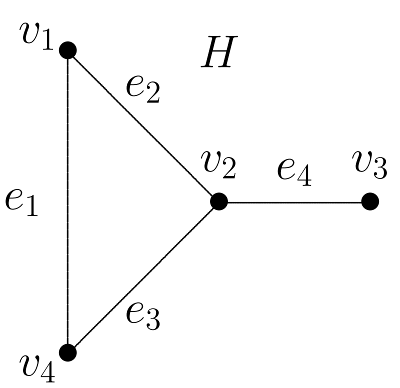
Thus, $s(\mathcal{P}) \le 2n - 3$

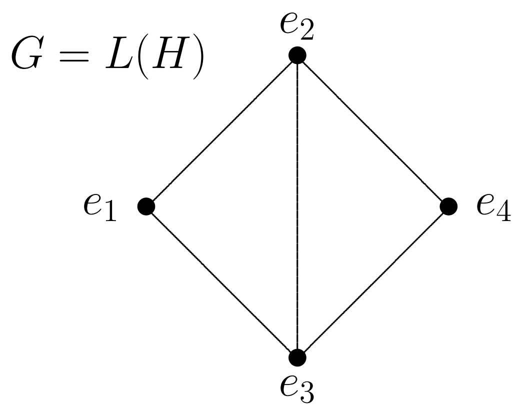
Saito, Schiermeyer (1999): $s(\mathcal{P}) \le \frac{3}{2}(n-3)$.

Recall: the intersection graph of a family of sets

Let H be a graph.

The *line graph* $G = L(H)$ of the graph H: the intersection graph of $E(G)$





$e_1 = \{v_1, v_4\}$

$e_2 = \{v_1, v_2\}$

$e_3 = \{v_2, v_4\}$

$e_4 = \{v_2, v_3\}$

$V(G) = E(H)$

$e_ie_j \in E(G)$ if and only if $e_1 \cap e_j \neq \emptyset$

Correspondences

H	$G = L(H)$
an edge	a vertex
a vertex	a clique (complete subgraph)
the degree of a vertex	the order of a clique
the maximum degree $\Delta(H)$	the clique number $\omega(G)$
an independent set of edges (matching)	an independent set of vertices
the matching number $\nu(G)$	the independence number $\alpha(G)$

Specifically:

the maximum degree $\Delta(H) \longleftrightarrow$ the clique number $\omega(G)$

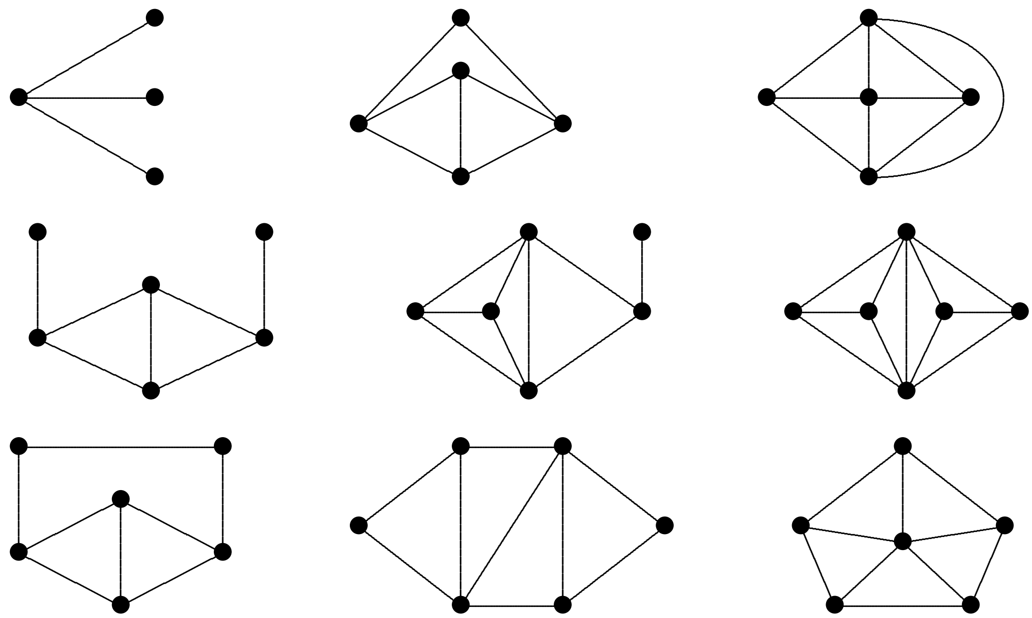
the matching number $\nu(G) \longleftrightarrow$ the independence number $\alpha(G)$

BUT:

- the maximum degree $\Delta(H)$ and the matching number $\nu(G)$ can be found efficiently
- determination of the clique number $\omega(G)$ and of the independence number $\alpha(G)$ are NP-hard problems

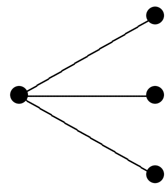
(???)

Theorem [Beineke, 1969]. *A graph G is a line graph (of some graph) if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



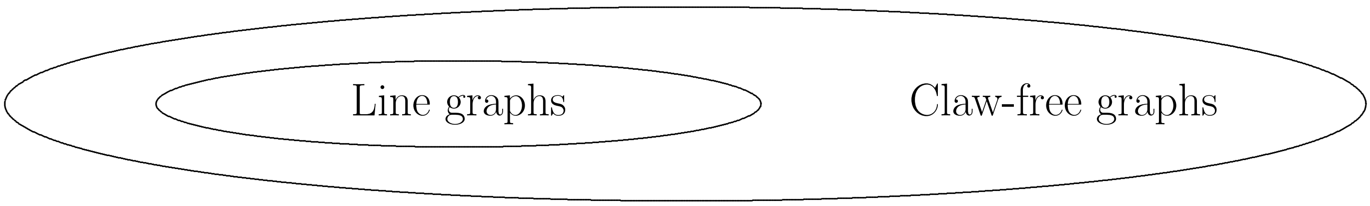
Corollary. *Line graphs are efficiently recognizable.*

The *claw* $K_{1,3}$:



A graph G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph

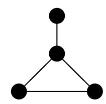
Every line graph is claw-free.



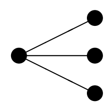
$X_1, \ldots, X_k$ - connected graphs. A graph G is $X_1 \ldots X_k$-*free* if G does not contain a copy of any of the graphs $X_1, \ldots, X_k$ as an induced subgraph.

G - a nonhamiltonian graph:

Every nonhamiltonian graph contains an induced C or Z_1.



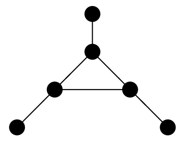
Z_1



$C = K_{1,3}$
(the *claw*)

Theorem [Goodman, Hedetniemi, 1974].
Every 2-connected CZ_1-free graph is hamiltonian.

Theorem [Duffus, Jacobson, Gould, 1980].
Every 2-connected CN-free graph is hamiltonian.

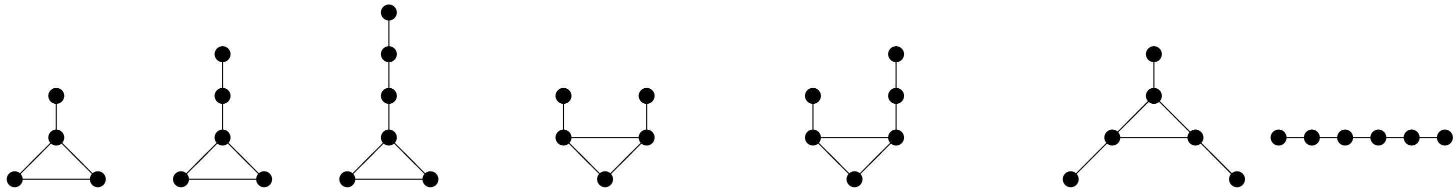

The *net* N

Wanted:

G 2-connected and XY-free $\Rightarrow$ G is hamiltonian

Question:

For which pairs of connected graphs X, Y, G being 2-connected and XY-free implies G is hamiltonian?



Z_1 Z_2 Z_3 The bull B The wounded W The net N P_6

Theorem [Bedrossian 1991].

Let X, Y be connected graphs with $X, Y \not\simeq P_3$ and let G be a 2-connected graph that is not a cycle. Then, G being XY-free implies G is hamiltonian if and only if (up to a symmetry) $X = C$ and Y is an induced subgraph of some of the graphs P_6, Z_2, W or N.

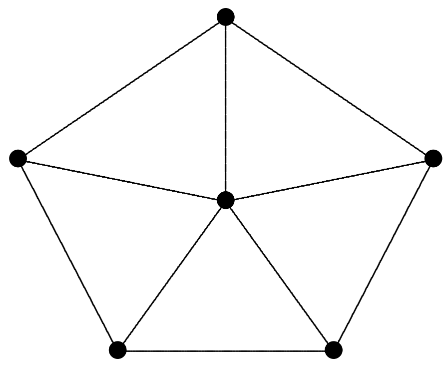
The only two 2-connected nonhamiltonian CZ_3-free graphs

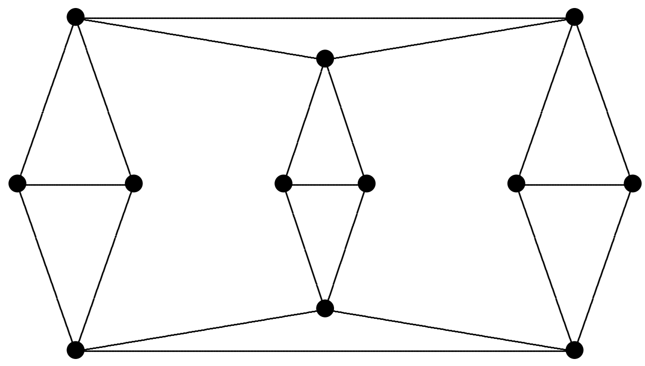
Theorem [Faudree, Gould 1997].

Let X, Y be connected graphs with $X, Y \not\simeq P_3$ and let G be a 2-connected graph of order $n \geq 10$ that is not a cycle. Then, G being XY-free implies G is hamiltonian if and only if (up to a symmetry) $X = C$ and Y is an induced subgraph of some of the graphs P_6, Z_3, W or N.

G claw-free

A vertex $x \in V(G)$ is *locally connected* if the neighborhood $\langle N(x) \rangle$ of x is a connected graph.





Recall:

Theorem [Oberly, Sumner 1979].
Let G be a graph of order $n \geq 3$ such that

(*i*) *for every $x \in V(G)$, the subgraph induced by the neighbors of x is connected,*

(*ii*) *G does not contain an induced subgraph isomorphic to the graph $K_{1,3}$.*

Then G is hamiltonian.

In our terminology:

Theorem [Oberly, Sumner 1979]. *Every locally connected claw-free graph of order $n \geq 3$ is hamiltonian.*

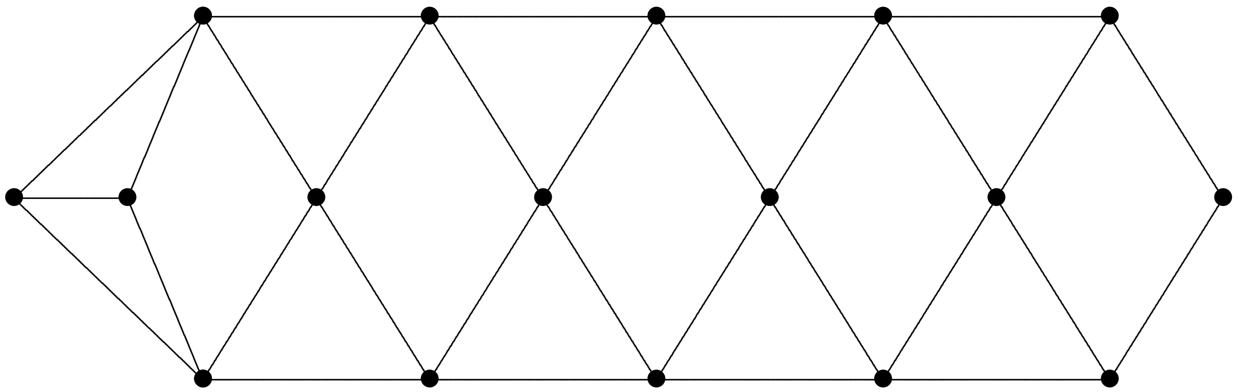
A locally connected vertex with noncomplete neighborhood is called *eligible*.

Let $x \in V(G)$ be an eligible vertex.
The *local completion* of a graph G at x: the graph G'_x with
$V(G'_x) = V(G)$,
$E(G'_x) = E(G) \cup \{xy|\ x, y \in N(x)\}$:
"add to the neighborhood of x all missing edges"

$c(G)$: the *circumference* of G (the length of a longest cycle in G)

Theorem. *Let x be an eligible vertex of a claw-free graph G and let G'_x be the local completion of G at x. Then*
(i) G'_x is claw-free,
(ii) $c(G'_x) = c(G)$.

Corollary. *G'_x is hamiltonian if and only if G is hamiltonian.*



G – a claw-free graph

$\mathrm{cl}(G)$ – the (?) graph obtained from G by recursively performing the local completion operation as long as there is at least one eligible vertex

The graph $\mathrm{cl}(G)$ is called the (claw-free) *closure of* G.

Theorem. *Let* G *be a claw-free graph. Then*

(*i*) $\mathrm{cl}(G)$ *is uniquely determined,*

(*ii*) $c(\mathrm{cl}(G)) = c(G)$,

(*iii*) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Corollary. G *is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

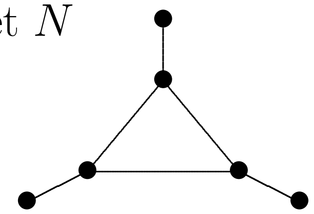
The closure operation $\mathrm{cl}(G)$:

- turns a claw-free graph into a line graph
- preserves hamiltonicity or non-hamiltonicity

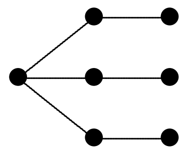
Example:

Theorem [Duffus, Jacobson, Gould, 1981].
Every 2-connected CN-free graph is Hamiltonian.

The net N



$L^{-1}(N)$



Tools:

Theorem. *If G is CN-free, then* $\mathrm{cl}(G)$ *is CN-free.*

$L(H)$: the *line graph* of a graph H
$H = L^{-1}(G)$: the *line graph preimage* of G

Observation. *$L(H)$ contains an induced subgraph isomorphic to a graph F if and only if H contains a subgraph isomorphic to the graph $L^{-1}(F)$.*

A *dominating closed trail* (abbreviated DCT) in a graph H is a closed trail T such that every edge of H has at least one vertex in T.
(Trail: vertices can be visited several times.)

Theorem [Harary and Nash-Williams 1965].
Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

G: 2-connected, CN-free, nonhamiltonian

Then cl(G) is also 2-connected, CN-free and nonhamiltonian

But cl(G) is a line graph !!!

Take $H = L^{-1}(G)$:

- essentially 2-edge-connected
- triangle-free
- no subgraph $L^{-1}(N)$
- no DCT

Examples of results which were proved using closure techniques:

Theorem *Every 7-connected claw-free graph is hamiltonian.*

Theorem. *Let G be a 2-connected claw-free graph of order $n \geq 153$ with minimum degree*

$$\delta(G) \geq \frac{n+39}{8}.$$

Then either G is hamiltonian, or $G \in \bigcup_{i=3}^{7} \mathcal{G}_i$.
($\bigcup_{i=3}^{7} \mathcal{G}_i$ – a family of exceptions).

Conjectures

Conjecture 1 (Matthews, Sumner).
Every 4-connected claw-free graph is hamiltonian.

Conjecture 2 (Thomassen).
Every 4-connected line graph is hamiltonian.

Conjecture 3.
Every snark has a dominating cycle.

(Snark: (i) cubic, (ii) cyclically 4-edge connected, (iii) not 3-edge-colorable, (iv) no cycle of length $\ell \leq 4$)

Theorem. *Conjectures 1, 2, 3 are equivalent.*

$\mathcal{C}$ – a class of graphs

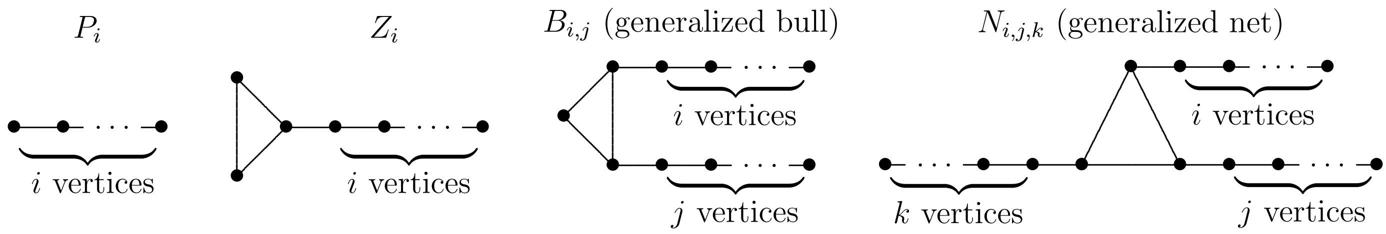
We say that $\mathcal{C}$ *is stable* if $\quad G \in \mathcal{C} \Rightarrow \mathrm{cl}(G) \in \mathcal{C}$.

Examples:

- k-connected claw-free graphs
- chordal claw-free graphs

Question:

For which connected graphs X, the class of CX-free graphs is stable?



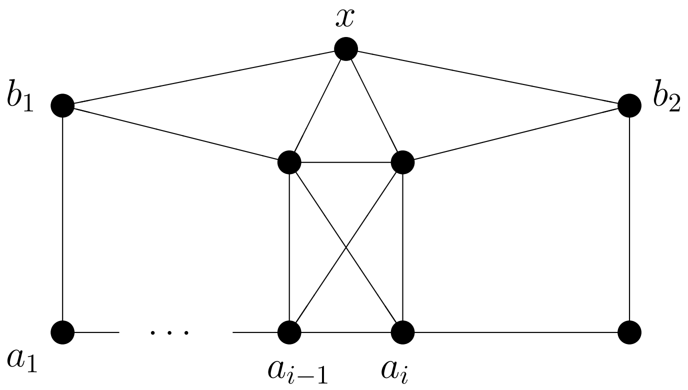
Fact: *G is CP_i-free $\Rightarrow G'_x$ is CP_i-free.*
(Note: no requirement on eligibility of x)

Corollary. *The class of CP_i-free graphs is stable.*

Similarly:

Proposition. *For $i, j, k \geq 1$ G is $CN_{i,j,k}$-free $\Rightarrow G'_x$ is $CN_{i,j,k}$-free.*
(Also no requirement on eligibility of x)

Corollary. *The class of $CN_{i,j,k}$-free graphs is stable for any $i, j, k \geq 1$.*

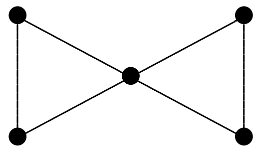


G_i is CZ_i-free, while $\{b_1, b_2, x, a_1, \ldots, a_i\}$ induces a Z_i in G'_x.

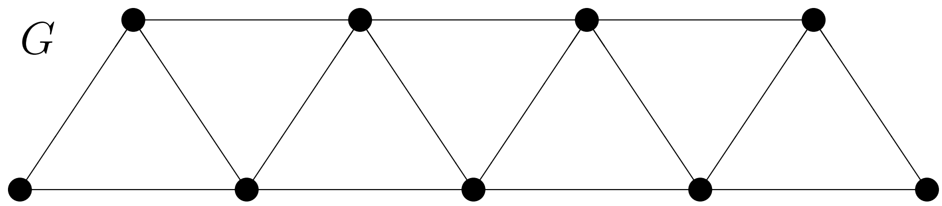
However, the analogue of the corollary is still true:

Theorem. *The class of CZ_i-free graphs is stable for any $i \geq 1$.*
(Note: the proof uses elibility of vertices)

A similar situation: CH-free graphs

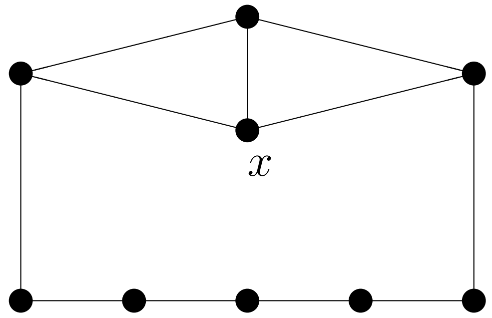


The hourglass H



G_i is CH-free, while G'_x contains an induced H.

Theorem. *The class of CH-free graphs is stable.*
(The proof also uses elibility of vertices)



G is $CB_{i,j}$-free while $G'_x = \mathrm{cl}(G)$ contains an induced $B_{i,j}$.

No hope to get stability of the $CB_{i,j}$-free class.

Characterization:

Theorem [Brousek, Favaron, ZR, Schiermeyer 1999].
Let A be a closed connected claw-free graph. Then the class of CA-free graphs is stable if and only if

$$A \in \{H, T\} \cup \{P_i |\ i \geq 3\} \cup \{Z_i |\ i \geq 1\} \cup \{N_{i,j,k} |\ i, j, k \geq 1\}.$$

Comparison

Bedrossian's characterization of forbidden subgraphs for hamiltonicity:
$X = C,\ \ Y \in \{P_6, Z_3, B_{1,2}, N_{1,1,1}\}$.

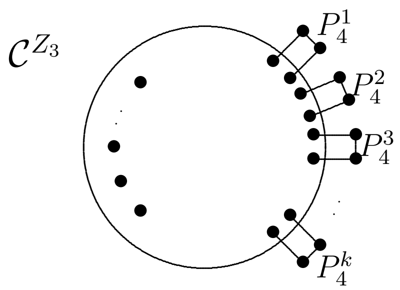
Characterization of stable classes:
$X = C,\ \ Y \in \{H, T\} \cup \{P_i |\ i \geq 3\} \cup \{Z_i |\ i \geq 1\} \cup \{N_{i,j,k} |\ i, j, k \geq 1\}$.

Strange: $B_{i,j}$???

Easy: *If G is a 2-connected CP_6-free graph, then* $\mathrm{cl}(G)$ *is $CN_{1,1,1}$-free.*

G is 2-connected CP_6-free $\Rightarrow$ $\mathrm{cl}(G)$ is 2-connected CP_6-free
If $\mathrm{cl}(G)$ is not $CN_{1,1,1}$-free:

Similar: *If G is a 2-connected CZ_3-free graph, then* $\mathrm{cl}(G)$ *is $CN_{1,1,1}$-free or* $\mathrm{cl}(G) \in \mathcal{C}^{Z_3}$.



Difficulty: the $CB_{1,2}$-free class is not stable

G is 2-connected $CB_{1,2}$-free $\not\Rightarrow$ $\mathrm{cl}(G)$ is 2-connected $CB_{1,2}$-free

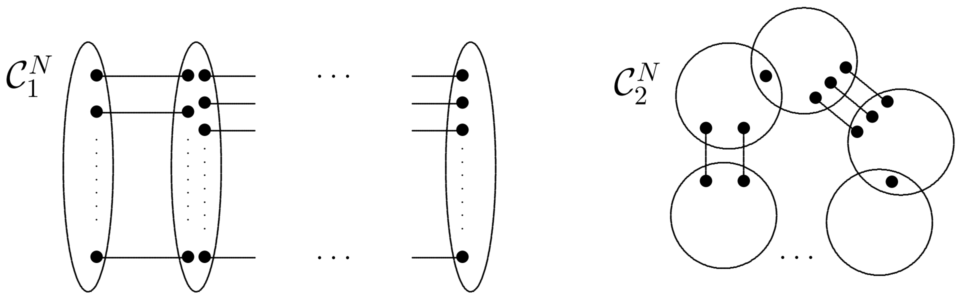
---

Theorem [ZR 2002] *Let G be a 2-connected graph. If G is $CB_{1,2}$-free, then $\mathrm{cl}(G)$ is $CN_{1,1,1}$-free.*

This allows to simplify the Bedrossian's characterization:

Theorem [ZR 2002]. *Let G be a 2-connected graph of order $n \geq 11$. If G is CX-free for $X \in \{P_6, Z_3, B_{1,2}, N_{1,1,1}\}$, then either $\mathrm{cl}(G)$ is CN-free or $\mathrm{cl}(G) \in \mathcal{C}^{Z_3}$.*

Moreover, $\mathrm{cl}(G) \in \mathcal{C}^{Z_3} \cup \mathcal{C}_1^N \cup \mathcal{C}_2^N$.



$\mathcal{P}$ – a property

$\mathcal{C}$ – a stable class

We say that $\mathcal{P}$ *is stable in* $\mathcal{C}$ if, for any $G \in \mathcal{C}$, $\quad G$ has $\mathcal{P} \Leftrightarrow \mathrm{cl}(G)$ has $\mathcal{P}$.

Example:
hamiltonicity is a stable property in the class of k-connected claw-free graphs.

π – a graph invariant

$\mathcal{C}$ – a stable class

We say that π *is stable in* $\mathcal{C}$ if, for any $G \in \mathcal{C}$, $\quad \pi(G) = \pi(\mathrm{cl}(G))$.

Example:
circumference is a stable invariant in the class of k-connected claw-free graphs.

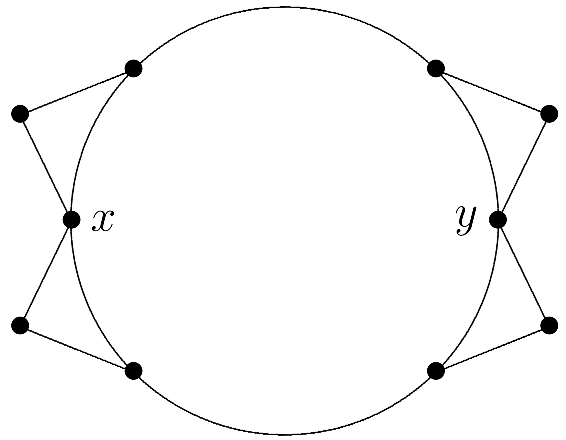
Which properties are stable?

Traceability – stable

(a graph G is *traceable* if G has a hamiltonian path)

G is *Hamilton-connected* if G has a hamiltonian (x, y)-path $\forall x, y \in V(G)$

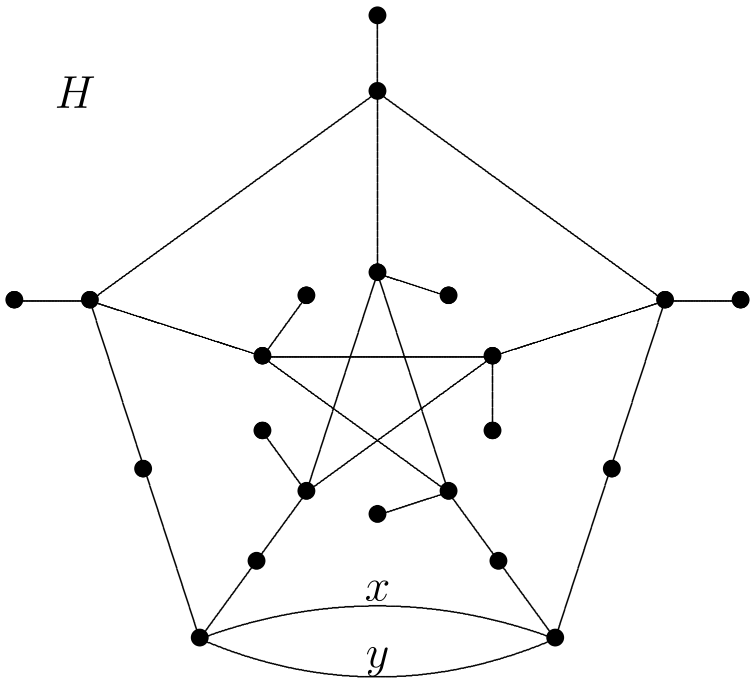
Hamilton-connectedness:



NOT STABLE

Hamilton-connectedness is not stable in 2-connected claw-free graphs.

$G = L(H)$:



NOT STABLE in 3-connected claw-free graphs

BUT: every 7-connected claw-free graph is Hamilton-connected $\Rightarrow$
Hamilton-connectedness IS STABLE in 7-conected claw-free graphs.

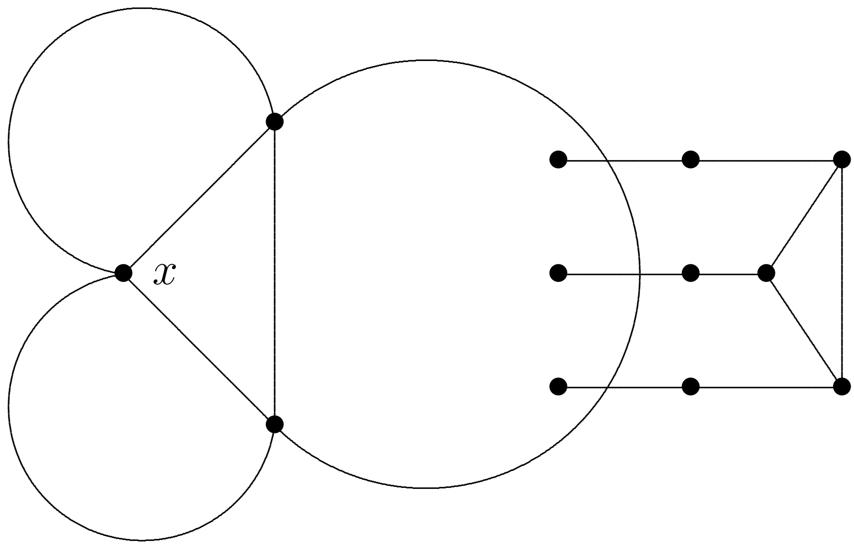
Hamilton-connectedness is:

  not stable in 3-conected claw-free graphs.

  stable in 7-conected claw-free graphs.

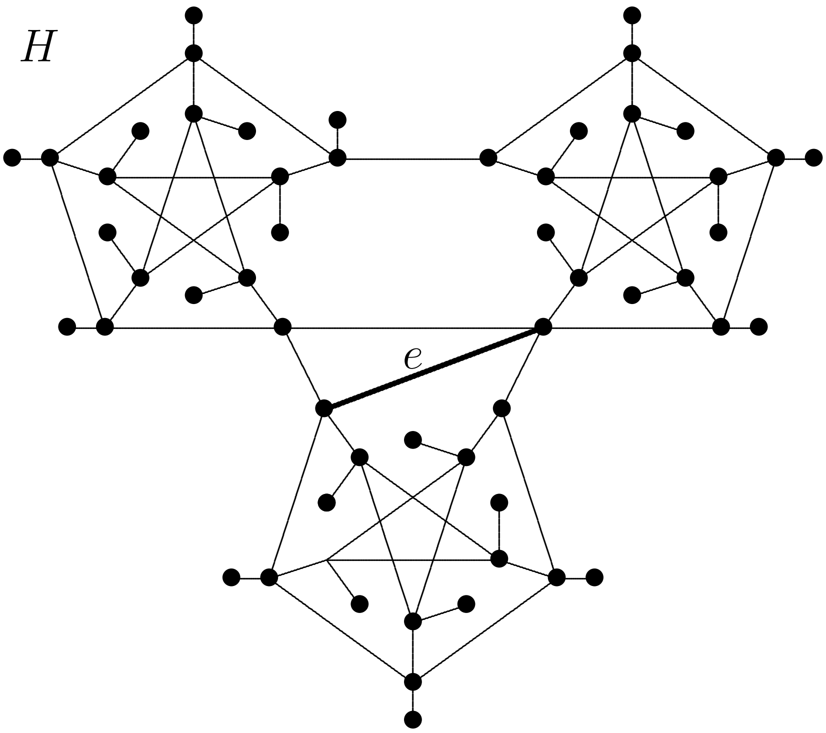
G is *homogeneously traceable* if G has a hamiltonian path with one endvertex at x for any $x \in V(G)$.

Homogeneous traceability:



NOT STABLE

Homogeneous traceability is not stable in 2-conected claw-free graphs.



$G = L(H)$ is 3-connected, not homogeneously traceable

cl(G) is homogeneously traceable

Homogeneous traceability is not stable in 3-conected claw-free graphs.

BUT: every 6-connected claw-free graph is hamiltonian $\Rightarrow$

every 6-connected claw-free graph is homogeneously traceable $\Rightarrow$

homogeneous traceability IS STABLE in 6-conected claw-free graphs.

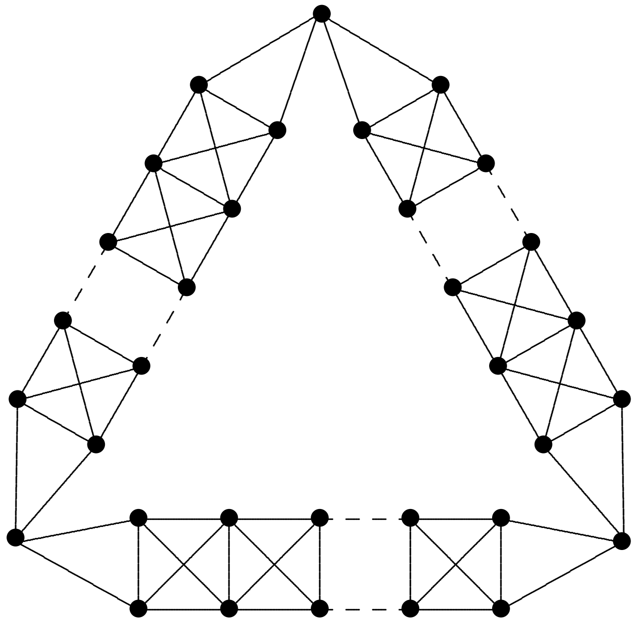
Unstable properties can be made stable by restricting the class of graphs under consideration.

Homogeneous traceability is:

  not stable in 3-conected claw-free graphs.

  stable in 6-conected claw-free graphs.

Pancyclicity (having C_ℓ for all ℓ, $3 \le \ell \le n$).

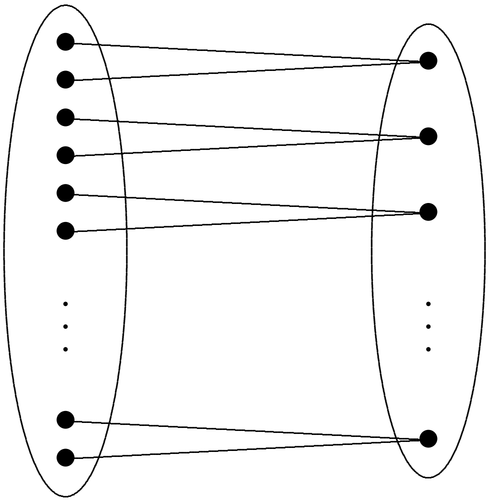


Pancyclicity is NOT STABLE in 2-connected claw-free graphs

Theorem. *For any $k \ge 2$ there is a k-connected nonpancyclic claw-free graph with pancyclic closure.*

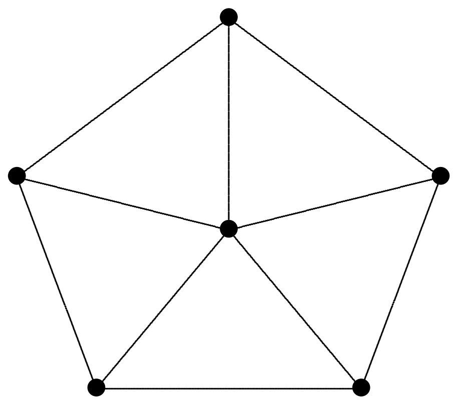
Pancyclicity is not stable in k-connected claw-free graphs for any $k \ge 2$.

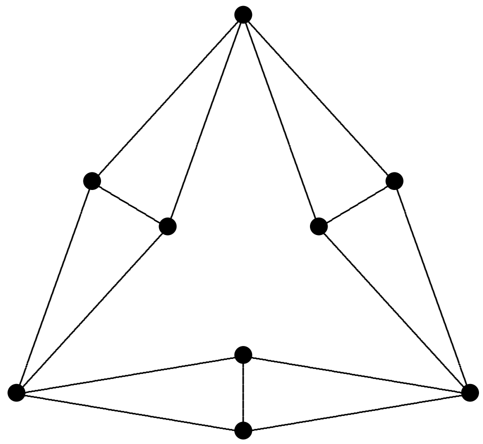
Cycle extendability (for every cycle $C \subset G$ there is a cycle $C' \subset G$ such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$)



Cycle extendability is not stable in k-connected claw-free graphs for any $k \geq 2$.

Having 2-factor with exactly k components:



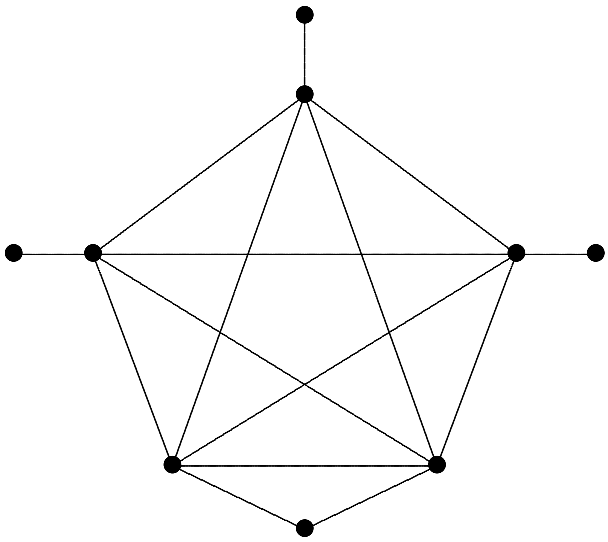


No 2-factor with 2 components

$\mathrm{cl}(G)$ complete

Having a P_3-factor

P_3-factor in G: a factor $F \subset G$ such that each component of F is a P_3



No P_3-factor

"Having a P_3-factor" is not stable in 1-connected claw-free graphs.

Theorem [Kaneko, Kelmans, Nishimura 2001].
Every 2-connected claw-free graph on $3k$ vertices has a P_3-factor.

"Having a P_3-factor" is stable in 2-connected claw-free graphs.

Property / invariant	Stable	Connectivity
Circumference Hamiltonicity	YES YES	1 1
Having a 2-factor with $\leq k$ components Minimum number of components in a 2-factor	YES YES	1 1
Having a cycle cover with $\leq k$ cycles Minimum number of cycles in a cycle cover	YES YES	1 1
(Vertex) pancyclicity (Full) cycle extendability	NO NO	any $\kappa \geq 2$ any $\kappa \geq 2$
Length of a longest path Traceability	YES YES	1 1
Having a path factor with $\leq k$ components Minimum number of components in a path factor	YES YES	1 1 [Ishizuka]
Having a path cover with $\leq k$ paths Minimum number of paths in a path cover	YES YES	1 1 [Ishizuka]
Homogeneous traceability	NO ??? YES	3 $4 \leq \kappa \leq 5$ 6
Hamilton-connectedness	NO ??? YES	3 $4 \leq \kappa \leq 6$ 7
Having a P_3-factor	NO YES	1 2 [K., K., N.]
Flower property	YES	1
Hamiltonian index	YES	1
2-factor index	YES	1 [Xiong, Saito]
Minimum number of components in a 2-factor in k-th iterated line graph	YES	1 [Xiong, Saito]
Supereulerian index	YES	1 [Xiong, Li]
Having hamiltonian prism	YES	1 [Čada]

k-closure

$x \in V(G)$ is *locally k-connected* if $N(x)$ induces a k-connected graph.

$\mathrm{cl}_k(G)$: local completions only at locally k-connected vertices.

Theorem [Bollobás, Riordan, ZR., Saito, Schelp, 1999].

(*i*) $\mathrm{cl}_k(G)$ *is uniquely determined for every* $k \geq 1$.

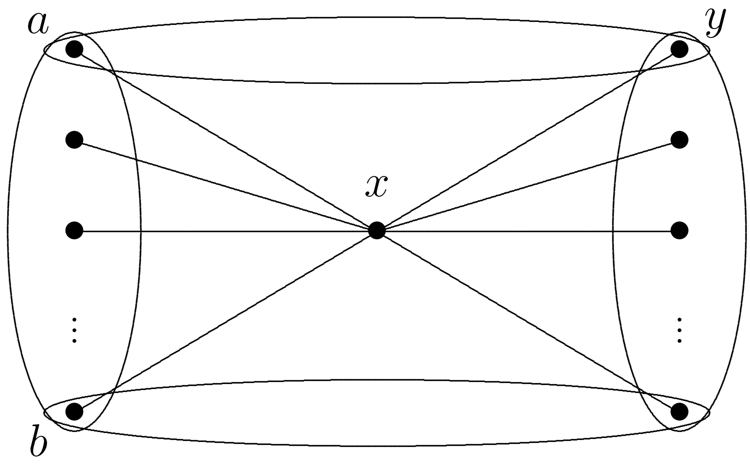
(*ii*) *Homogeneous traceability is stable under* $\mathrm{cl}_2(G)$.

(*iii*) *Hamilton-connectedness is stable under* $\mathrm{cl}_3(G)$.

Conjecture [Bollobás, Riordan, ZR., Saito, Schelp, 1999].
Hamilton-connectedness is stable under $\mathrm{cl}_2(G)$.

Example.

G:



- $\langle N(x) \rangle$ 2-connected
- No hamiltonian (a, b)-path
- There is a hamiltonian (a, b)-path in G'_x.

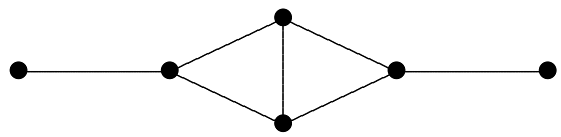
Property "Having a hamiltonian (a, b)-path" is not stable under $\mathrm{cl}_2(G)$.

BUT: G'_x has no hamiltonian (a, y)-path.
Thus: G'_x is not Hamilton-connected.

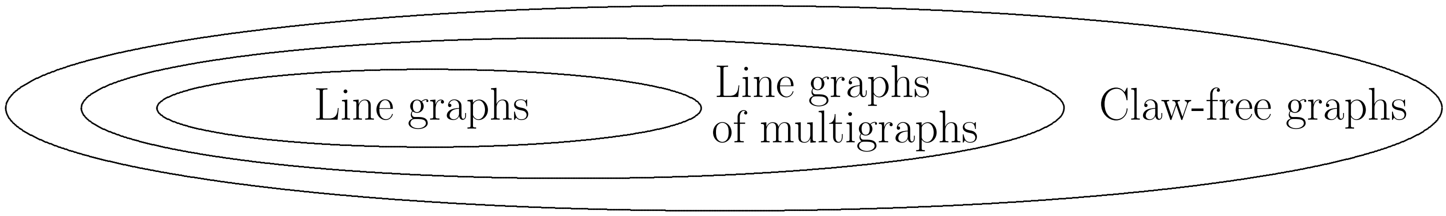
Theorem [ZR., Vrána, 2009].
Hamilton-connectedness is stable under $\mathrm{cl}_2(G)$.

What is the structure of $\mathrm{cl}_2(G)$?

Not a line graph:



Line graph of a multigraph?

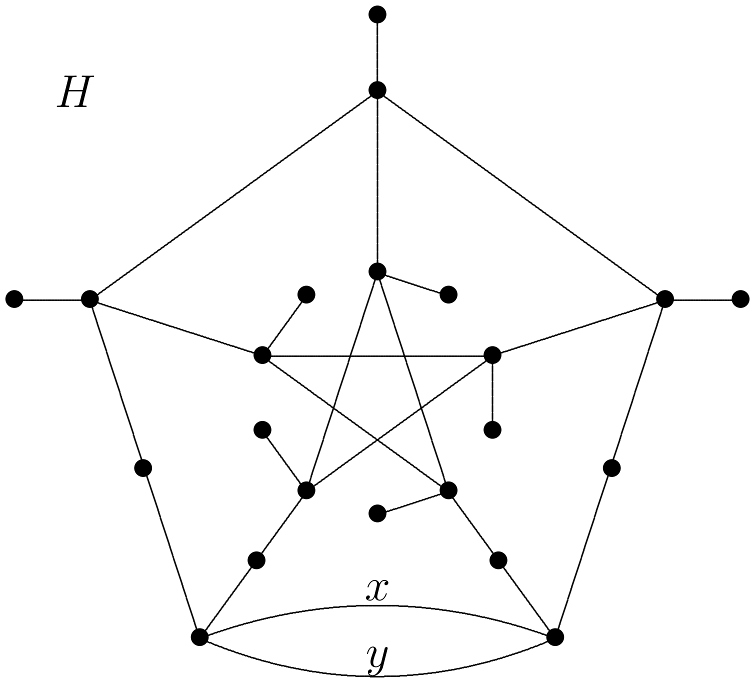


$\mathcal{C}$ – a class of graphs

A *closure* on $\mathcal{C}$ is a mapping $\text{cl} : \mathcal{C} \to \mathcal{C}$ such that

- $V(G) = V(\text{cl}(G))$
- $E(G) \subset E(\text{cl}(G))$

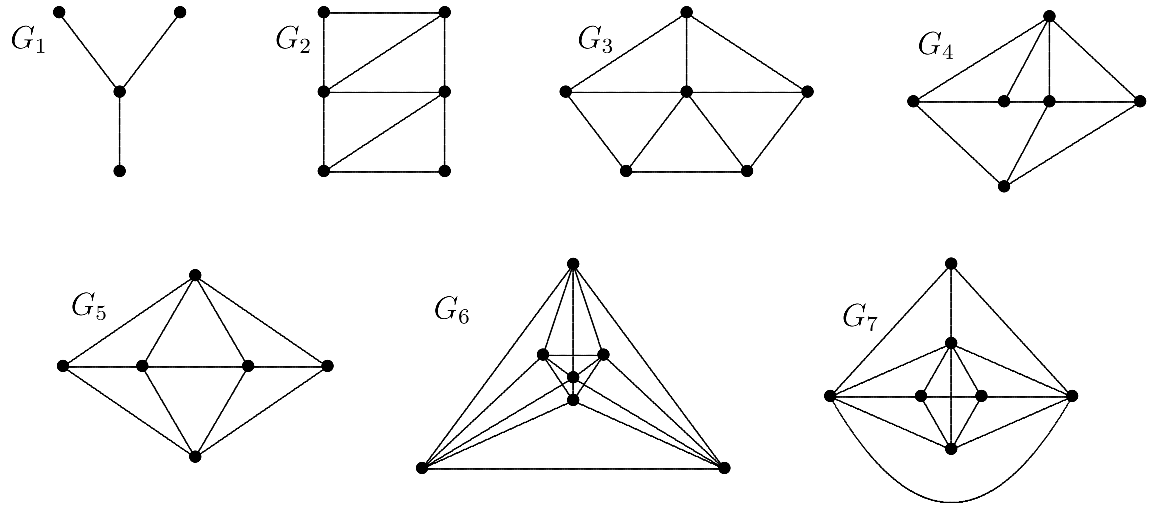
$G = L(H)$:



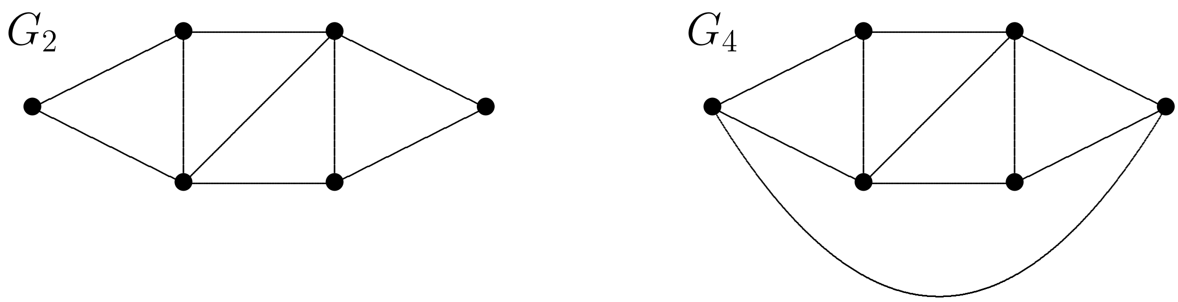
$\mathcal{L}_k$ – the class of k-connected line graphs (of graphs)
$\mathcal{L}_k^M$ – the class of k-connected line graphs of multigraphs

Theorem [Vrána, 2008].
There is no closure cl *on* $\mathcal{L}_3^M$ *such that* $\text{cl} : \mathcal{L}_3^M \to \mathcal{L}_3$ *and Hamilton-connectedness is stable under* cl.

Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



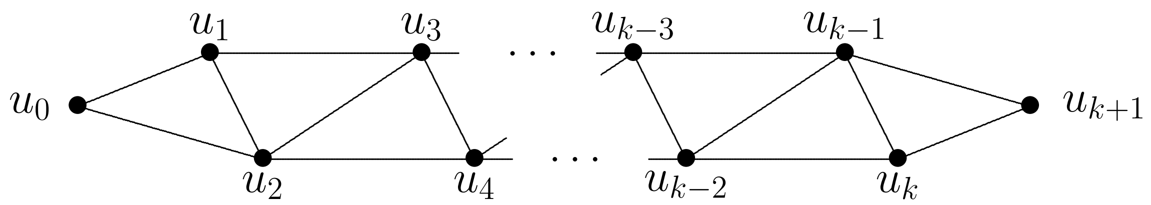
Thus, in $\mathrm{cl}_2(G)$, only G_2 or G_4 can remain.



"Multigraph closure" $\mathrm{cl}^M(G)$ of a graph G:
Recursively perform $\mathrm{cl}_2(G)$ and closing specified vertices in copies of G_2 or G_4, as long as there is something to do.

$J = u_0 u_1 \dots u_{k+1}$ – a walk in G. We say that J is *good in* G, if

- $k \geq 4$,
- $J^2 \subset G$,
- for any i, $0 \leq i \leq k-4$, $\langle \{u_i, u_{i+1}, \dots, u_{i+5}\} \rangle_G \simeq S_1$ or S_2.

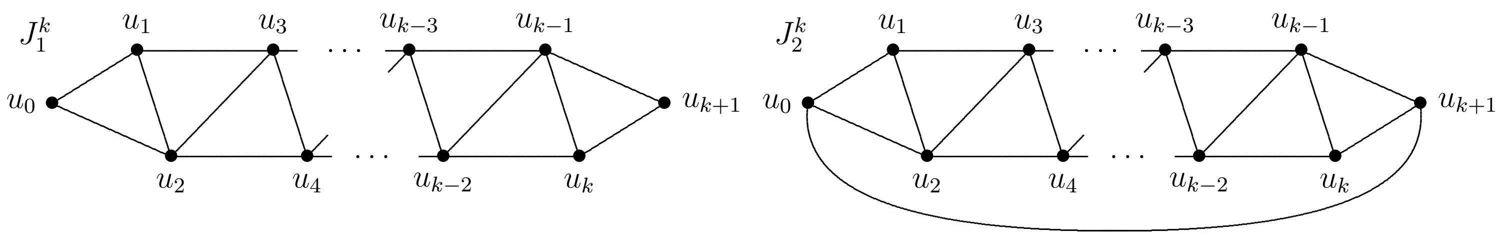


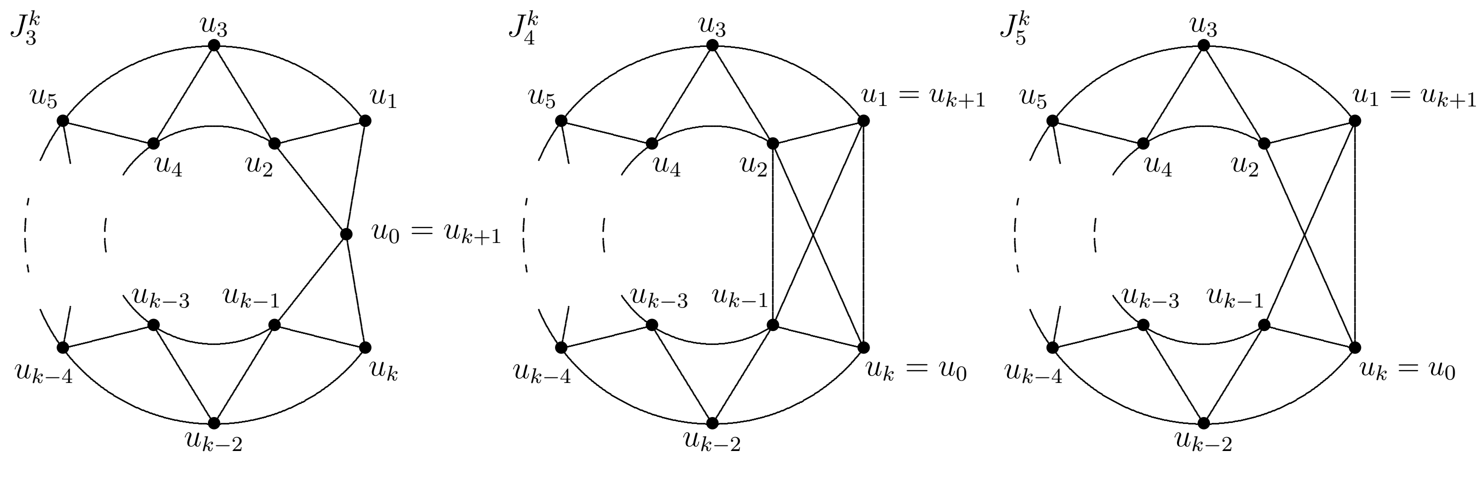
G – claw-free 2-closed, $J = u_0 u_1 \dots u_{k+1}$ – a good walk in G. Then

(i) $J = u_1 \dots u_k$ is a path,

(ii) if $k \geq 5$, then $d_G(u_i) = 4$, $3 \leq i \leq k-2$,

(iii) $\langle N_G[u_1] \setminus \{u_3\} \rangle_G = \langle N_G[u_2] \setminus \{u_3, u_4\} \rangle_G$ is a clique.





Let G be a claw-free graph, and let $\mathrm{cl}^M(G)$ be a graph constructed by the following algorithm.

1. Set $G_1 = \mathrm{cl}_2(G)$, $i := 1$.
2. If G_i contains a good walk, then
   (a) choose a maximal good walk $J = u_0 u_1 \ldots {}_{k+1}$,
   (b) let G_i' be the local completion of G_i at u_1 and G_i'' be the local completion of G_i' at u_k,
   (c) set $G_{i+1} = \mathrm{cl}_2(G_i'')$, $i := i + 1$,
   (d) go to (2).
3. Set $\mathrm{cl}^M(G) = G_i$.

The graph $\mathrm{cl}^M(G)$ will be called (the) M*-closure* of G.

Theorem [ZR., Vrána, 2009].
Let G be a claw-free graph. Then
(*i*) $\mathrm{cl}^M(G)$ *is uniquely determined,*
(*ii*) *there is a multigraph H such that* $\mathrm{cl}^M(G) = L(H)$,
(*iii*) *G is Hamilton-connected if and only if* $\mathrm{cl}^M(G)$ *is Hamilton-connected.*

Hamilton-connectedness is stable under $\mathrm{cl}^M(G)$.

Recall: Every 7-connected line graph of a multigraph is Hamilton-connected [Zhan, 1991].

Corollary [ZR., Vrána, 2009].
Every 7-connected claw-free graph is Hamilton-connected.

Uniqueness of the closure

Some notations and cefinitions by Kelmans [2003]

For $X \subset V(G)$, G^*_X denotes the local completion of G at X
(i.e. $V(G^*_X) = V(G)$ and $E(G^*_X) = E(G) \cup \{uv|\ u, v \in X\}$.)

Thus, we simply write G'_x for $G^*_{N_G(x)}$.

We simply write $G^*_{x_1 \ldots x_k}$ for $((G^*_{x_1})^*_{x_2} \ldots)^*_{x_k}$.

$\mathcal{C}$ – a class of graphs
$\mathcal{P}$ – a function on $\mathcal{C}$ such that, for any $G \in \mathcal{C}$, $\mathcal{P}(G) \subset 2^{V(G)}$
(i.e., $\mathcal{P}(G)$ is a set of subsets of $V(G)$).

A graph F is a $\mathcal{P}$-*extension* of G, denoted $G \preceq F$, if there is a sequence of graphs $G_0 = G, G_1, \ldots, G_k = F$ such that $G_i \in \mathcal{C}$, $i = 1, \ldots, k$, and $G_{i+1} = (G_i)^*_{X_i}$ for some $X_i \in \mathcal{P}(G_i)$, $i = 1, \ldots, k-1$.

Clearly, for any graph G exists a $\preceq$-maximal $\mathcal{P}$-extension H, and in this case we say that H is a $\mathcal{P}$-closure of G.
If a $\mathcal{P}$-closure is uniquely determined then it is denoted by $\mathrm{cl}_{\mathcal{P}}(G)$.

A function $\mathcal{P}$ is *non-decreasing* (on a class $\mathcal{C}$), if, for any $H, H' \in \mathcal{C}$, $H \preceq H'$ implies that for any $X \in \mathcal{P}(H)$ there is an $X' \in \mathcal{P}(H')$ such that $X \subset X'$.

Theorem [Kelmans 2003] *If $\mathcal{P}$ is a non-decreasing function on a class $\mathcal{C}$, then, for any $G \in \mathcal{C}$, a $\mathcal{P}$-closure of G is uniquely determined.*

Proof Let $H \neq H'$ be $\mathcal{P}$-closures of G, let $G = G_0, G_1, \ldots, G_k = H'$ be such that $G_{i+1} = (G_i)^*_{X_i}$ for some $X_i \in \mathcal{P}(G_i)$, and let s be a smallest integer such that $G_s \not\subset H$. Since $G_{s-1} \subset H$ and $\mathcal{P}$ is non-decreasing, there is $X \in \mathcal{P}(H)$ such that $X_{s-1} \subset X$. Since H is $\preceq$-maximal, we have $H^*_X = H$, a contradiction.

For a given graph G, let $\mathcal{C}_G$ denote the class of graphs with vertex set $V(G)$.

Easy to observe:

Lemma. *Let G be a graph.*

(i) Let $\mathcal{P}$ be a non-decreasing function on $\mathcal{C}_G$, let $X \subset V(G)$, and for any $H \in \mathcal{C}_G$ set $\mathcal{P}^X(H) = \mathcal{P}(H) \cup \{N_H(x)|\ x \in X\}$. Then $\mathcal{P}^X$ is a non-decreasing function on $\mathcal{C}_G$.

(ii) For any integer $k \geq 1$, the function
$\mathcal{P}_k(H) = \{N_H(x)|\ \langle N_H(x)\rangle_H$ is k-connected $\}$
is a non-decreasing function on $\mathcal{C}_G$. ∎

Thus, for any graph G, integer $k \geq 1$ and a set $X \subset V(G)$, the function $\mathcal{P}^X_k$, defined (for any $H \in \mathcal{C}_G$) by $\mathcal{P}^X_k(H) = (\mathcal{P}_k)^X(H)$, is a non-decreasing function on $\mathcal{C}_G$.

G – connected claw-free (not the square of a cycle),
let $J_1, \ldots, J_t$ be all maximal good walks in $\mathrm{cl}_2(G)$.

For any $J_i = u_0^i u_1^i \ldots u_{k+1}^i$ set
$X_i = \{u_1^i, \ldots, u_{r-1}^i\} \cup \{u_{r+2}^i \ldots u_{2r}^i\}$ if $k = 2r$
or
$X_i = \{u_1^i, \ldots, u_{r-1}^i\} \cup \{u_{r+3}^i \ldots u_{2r+1}^i\}$ if $k = 2r + 1$,
respectively,
and set $X = \cup_{i=1}^t X_i$ (note that the sets X_i are pairwise disjoint)

Then, by the Lemma, $\mathcal{P}^M(H) = \mathcal{P}_2^X(H)$ is a non-decreasing function on $\mathcal{C}_G$. Thus, the corresponding $\mathcal{P}^M$-closure of G is unique.

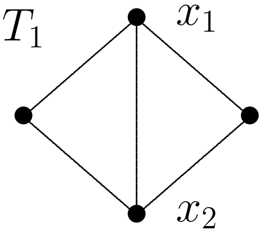
This closure is called the *multigraph closure* (or simply M*-closure*) of G and denoted $\mathrm{cl}^M(G)$.

(Exception - if G is the square of a cycle, we define $\mathrm{cl}^M(G)$ as the complete graph on $V(G)$.)

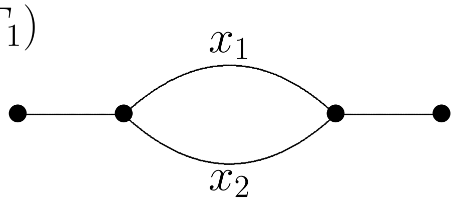
The construction given by the previous algorithm is just one of the possible orderings of vertices during the construction of $\mathrm{cl}^M(G)$.

Drawback: there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$ (i.e., the "preimage" is not uniquely determined).

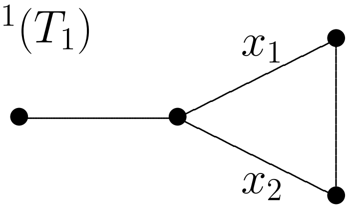
T_1



$L_M^{-1}(T_1)$



$L^{-1}(T_1)$



Theorem. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

If G is a line graph of a graph, then the "multigraph preimage $L_M^{-1}(G)$" and the obvious line graph preimage $L^{-1}(G)$ can be different!

M-closed graphs

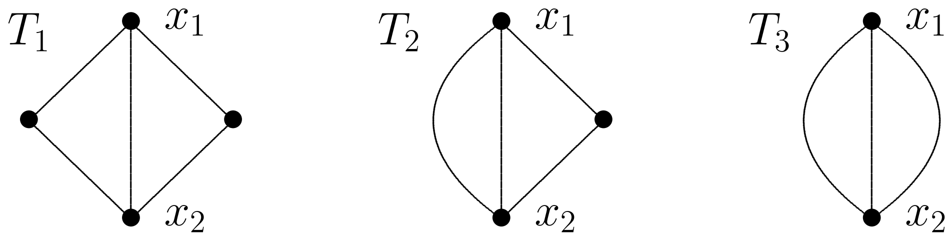
Proposition. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in the Figure. Then G is M-closed if and only if there is a multigraph H such that $G = L(H)$ and H does not contain a subgraph S (not necessarily induced) with any of the following properties:*

(i) $S \simeq T_1$,

(ii) $S \simeq T_2$ and there is a vertex $u \in V(H) \setminus V(S)$ such that $|N_H(u) \cap \{x_1, x_2\}| = 1$,

(iii) $S \simeq T_3$ and there are $u_1, u_2 \in V(H) \setminus V(S)$ such that $u_1 \neq u_2$ and $u_i x_i \in E(H)$, $i = 1, 2$

(where x_1, x_2 are the only vertices in S with $d_S(x_i) = 3$).



Proposition. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in the Figure. Then G is M-closed if and only if G is a line graph of a multigraph and $L_M^{-1}(G)$ does not contain a subgraph (not necessarily induced) isomorphic to any of the graphs T_1, T_2 or T_3.*

Forbidden subgraphs for Hamilton-connectedness

On the positive side:

Theorem.

(*i*) [**Shepherd 1991**] *Every 3-connected $CN_{1,1,1}$-free graph is Hamilton-connected.*

(*ii*) [**Chen,Gould 2000**] *Let G be a 3-connected graph satisfying any of the following:*

(α) *G is CZ_3-free,*

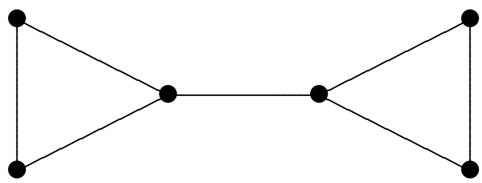
(β) *G is CP_6-free,*

(γ) *G is $CB_{1,2}$-free.*

Then G is Hamilton connected.

(*iii*) [**Broersma, Faudree, Huck, Trommel, Veldman 2002**] *Every 3-connected CH_1-free graph is Hamilton-connected.*

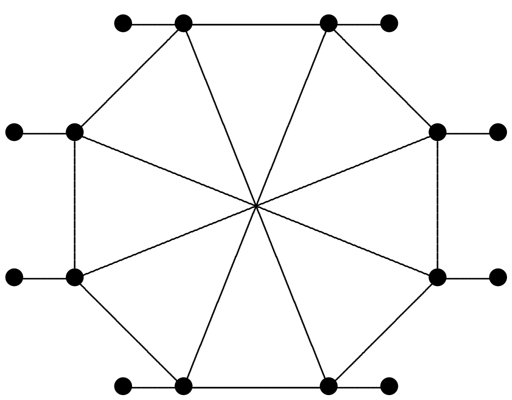
H_1



Opposite direction:

Theorem [Broersma, Faudree, Huck, Trommel, Veldman 2002]; [ZR, Vrána 2009]
If X, Y is a pair of connected graphs such that $X, Y \not\simeq P_3$ and every 3-connected XY-free graph is Hamilton connected, then, up to a symmetry, $X = C$ and Y satisfies each of the following conditions:
(*a*) $\Delta(Y) \leq 3$,
(*b*) *any longest induced path in Y has at most 9 vertices,*
(*c*) *Y contains no cycles of length at least 4,*
(*d*) *the distance between two distinct triangles in Y is either 1 or 3,*
(*e*) *There are at most two triangles in Y,*
(*f*) *Y is claw-free.*

H



$L(H)$ is 3-connected and not Hamilton-connected

Thus, the "longest" P_i, Z_i, $B_{i,j}$ and $N_{i,j,k}$ implying Hamilton-connectedness can be:

P_9

Z_6

$B_{i,j}$ for $i + j = 7$

$N_{i,j,k}$ for $i + j + k = 7$

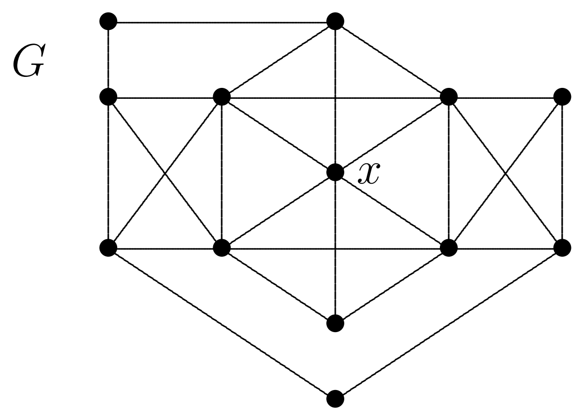
Using $\mathrm{cl}^M(G)$, the following was proved.

Theorem [Faudree, Faudree, ZR, Vrána 2009]
If G is a 3-connected XY-free graph for $X = C$ and $Y = P_8$, $N_{1,1,3}$, or $N_{1,2,2}$, then G is Hamilton-connected.

Recall: the CP_i-free and $CN_{i,j,k}$-free classes are stable without the local connectivity condition.

Problem: the CZ_i-free and $CB_{1,j}$-free classes are not stable:

G is CZ_6-free
x is 2-eligible
G'_x contains an induced Z_6
G'_x contains no 2-eligible vertex



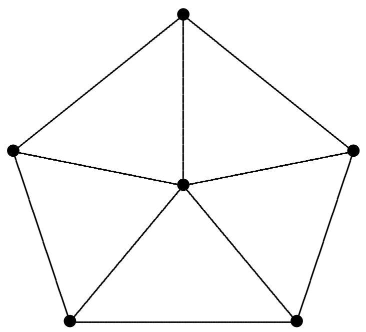
Theorem [ZR, Saburov, Vrána 2010].
Let G be a 3-connected $CB_{1,4}$-free graph. Then $\mathrm{cl}^M(G)$ is either CP_8-free or belongs to one of 9 classes of exceptions.

Corollary.
Every 3-connected $CB_{1,4}$-free graph is Hamilton-connected.

Closure for 2-factors

Theorem [ZR, Saito, Schelp 1999].
Let G be a claw-free graph. Then $\mathrm{cl}(G)$ *has a 2-factor with at most k components if and only if G has a 2-factor with at most k components.*

Corollary. *Let G be a claw-free graph. Then* $\mathrm{cl}(G)$ *has a 2-factor if and only if G has a 2-factor.*



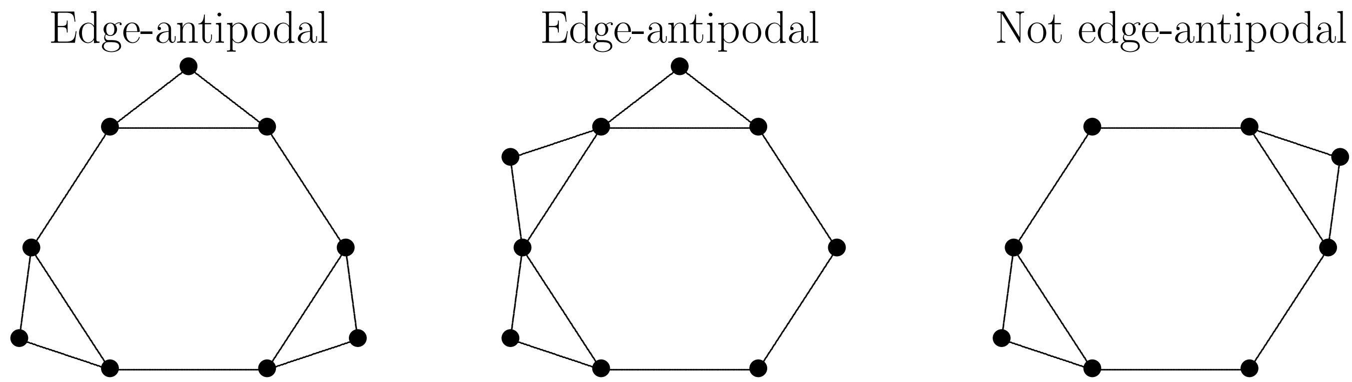
No 2-factor with 2-components
$\mathrm{cl}(G)$ complete

C_k – a cycle of even length $k \geq 4$.

$e_1, e_2 \in E(G)$ are *antipodal in* C_k, if they are at maximum distance in C_k (i.e., $\text{dist}_{C_k}(e_1, e_2) = k/2 - 1$),

C_k is *edge-antipodal in* G, abbreviated EA, if $\min\{\omega_G(e_1), \omega_G(e_2)\} = 2$ for any two antipodal edges $e_1, e_2 \in E(C)$.

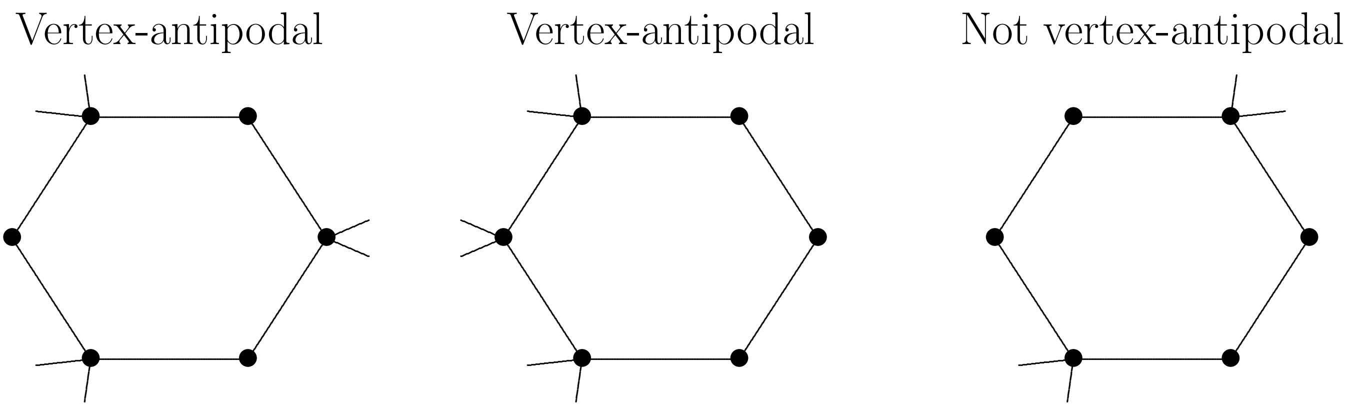
($\omega_G(e)$ – the largest order of a clique containing e)



Analogously:

$x_1, x_2 \in V(C_k)$ are *antipodal in* C_k if they are at maximum distance in C_k (i.e. $\text{dist}_{C_k}(x_1, x_2) = k/2$),

C_k is *vertex-antipodal in* G, abbreviated VA, if $\min\{d_G(x_1), d_G(x_2)\} = 2$ for any two antipodal vertices $x_1, x_2 \in V(C_k)$.



A vertex $x \in V(G)$ is *2f-eligible*, if x satisfies one of the following:

(*i*) $x \in \mathrm{EL}(G)$,

(*ii*) $x \notin \mathrm{EL}(G)$ and x is in an induced cycle of length 4 or 5 or in an induced EA-cycle of length 6.

The set of all 2f-eligible vertices of G will be denoted $\mathrm{EL}^{2f}(G)$.

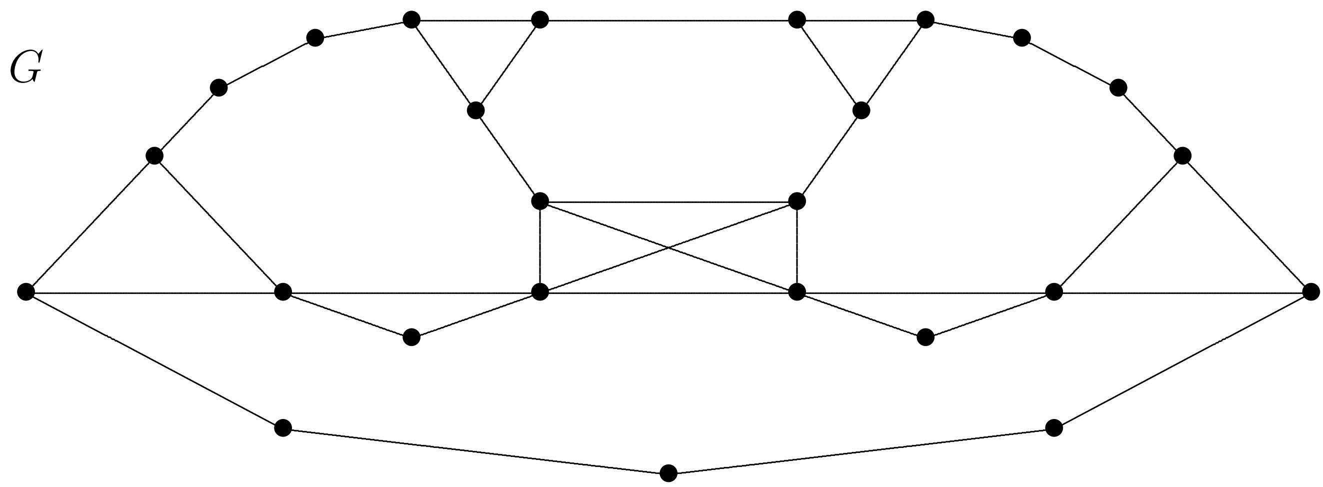
We say that a graph $\mathrm{cl}^{2f}(G)$ is a *2-factor-closure* (abbreviated 2f-closure) of a claw-free graph G, if there is a sequence of graphs $G_1, \ldots, G_k$ such that

(*i*) $G_1 = G$,

(*ii*) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in \mathrm{EL}^{2f}(G_i)$, $i = 1, \ldots, k-1$,

(*iii*) $G_k = \mathrm{cl}^{2f}(G)$ and $\mathrm{EL}^{2f}(G_k) = \emptyset$.

Thus, the 2f-closure of is obtained by recursively repeating the local completion operation at 2f-eligible vertices, as long as this is possible.



Theorem [ZR, Xiong, Yoshimoto 2009].

Let G be a claw-free graph. Then

(i) *the closure* $\mathrm{cl}^{2f}(G)$ *is uniquely determined,*

(ii) *there is a graph H such that*

(α) $L(H) = \mathrm{cl}^{2f}(G)$,

(β) $g(H) \geq 6$,

(γ) *H does not contain any vertex-antipodal cycle of length 6,*

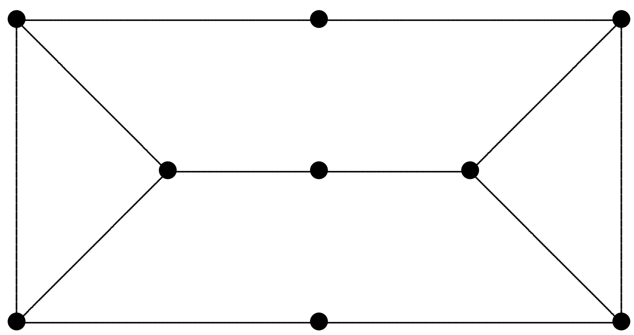
(iii) *G has a 2-factor if and only if* $\mathrm{cl}^{2f}(G)$ *has a 2-factor.*

Corollary. *Let G be a claw-free graph in which every locally disconnected vertex is in an induced cycle of length 4 or 5, or in an induced EA-C_6. Then G has a 2-factor.*

Note: $\mathrm{cl}^{2f}(G)$ does not preserve

– (non)-hamiltonicity

– minimum number of components of a 2-factor



No 2-factor

Every vertex in a cycle of length at most 6

Thus, the antipodality condition cannot be omitted

Forbidden subgraphs for 2-factors

Theorem [Faudree, Faudree, ZR 2008].
Let X and Y be connected graphs with $X, Y \not\cong P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then, G being XY-free implies that G has a 2-factor if and only if, up to the order of the pairs, either
(i) $\{X, Y\} = \{K_{1,4}, P_4\}$, *or*
(ii) $X = K_{1,3}$ *and* Y *is an induced subgraph of at least one of the graphs* P_7, $B_{1,4}$ *or* $N_{1,1,3}$.

Theorem [Aldred, Fujisawa, Saito 2009]. *Let H_1 and H_2 be connected graphs of order at least three. Suppose $\mathcal{G}(\{H_1, H_2\})$ is an infinite class but $\mathcal{G}(\{H_1, H_2\}) \setminus \mathcal{F}_2$ is a finite class. Then either H_1 or H_2 is $K_{1,2}$ or $K_{1,3}$.*

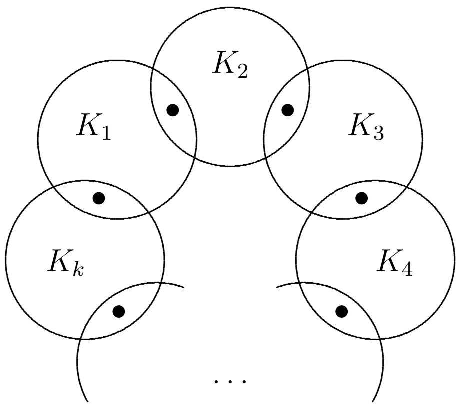
$\mathcal{G}(\{H_1, H_2\})$: the class of H_1H_2-free graphs
$\mathcal{F}_2$: the class of graphs that have a 2-factor

Theorem [ZR, Saburov 2009].
If G is 2-connected and CP_7-free or $CB_{1,4}$-free, then $\mathrm{cl}^{2f}(G)$ is $CN_{1,1,3}$-free.

Corollary. *Let G be a 2-connected XY-free graph of order $n \geq 10$, where X, Y is a pair of connected graphs such that G being XY-free implies G has a 2-factor. Then either*
(i) $\{X, Y\} = \{K_{1,4}, P_4\}$, *or*
(ii) $X = K_{1,3}$ and $\mathrm{cl}^{2f}(G)$ *is* $N_{1,1,3}$*-free.*

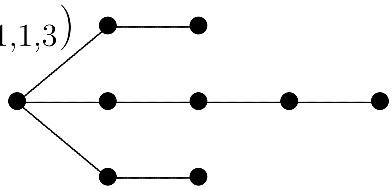
Moreover, in the last case $\mathrm{cl}^{2f}(G) \in \mathcal{C}^1 \cup \mathcal{C}^{\geq 6}$.

$\mathcal{C}^k$

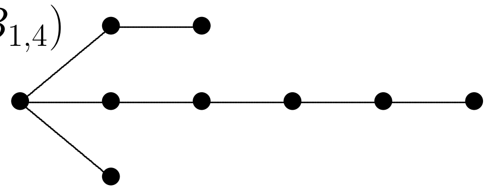


Proposition. *Let G be a claw-free graph, G'_x the local completion of G at a vertex $x \in V(G)$, and let F be a connected triangle-free graph with $\Delta(F) \leq 3$ such that $H = L(F) \overset{\text{IND}}{\subset} G'_x$. Then either $H \overset{\text{IND}}{\subset} G$, or there is a graph F' such that $H' = L(F') \overset{\text{IND}}{\subset} G$ and F' is obtained from F by subdivision or rotation of an edge.*

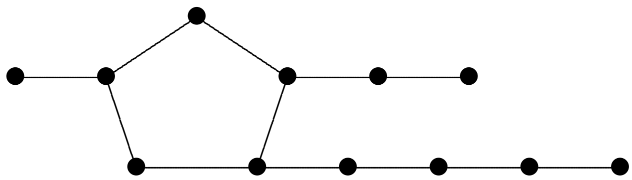
$L^{-1}(N_{1,1,3})$



$L^{-1}(B_{1,4})$



A sun



Good sun: contains $L^{-1}(B_{1,4})$
Good suns: $\mathcal{S}_G^A \cup \mathcal{S}_G^B$.

Proposition. *Let G be a $K_{1,3}B_{1,4}$-free graph. Then $\mathrm{cl}^{2f}(G)$ contains no induced subgraph F such that $L^{-1}(F) \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$.*

Proposition. *Let G be a 2-connected 2f-closed claw-free graph. If G is not $K_{1,3}N_{1,1,3}$-free, then G contains an induced subgraph F such that $L^{-1}(F) \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$.*

Thomassen's conjecture

Recall:

Conjecture 1 (Matthews, Sumner).
Every 4-connected claw-free graph is hamiltonian.

Conjecture 2 (Thomassen).
Every 4-connected line graph is hamiltonian.

Conjecture 3.
Every snark has a dominating cycle.

(Snark: (i) cubic, (ii) cyclically 4-edge connected, (iii) not 3-edge-colorable, (iv) no cycle of length $\ell \leq 4$)

Theorem. *Conjectures 1, 2, 3 are equivalent.*

A graph G is *k-Hamilton-connected* if, for any $X \subset V(G)$ with $|X| = k$, the graph $G - X$ is Hamilton-connected.

Easy: *k-Hamilton-connected* $\Rightarrow$ $(k+3)$*-connected*

$E^+(G) = \{xy|\ x, y \in V(G)\}$

for $X \subset E^+(G)$ set $G + X = (V(G), E(G) \cup X)$

X is a set of "new" edges that are "added" to G; if $e_1 = \{x, y\} \in E(G)$ and $e_2 = \{x, y\} \in X$, we consider e_1 and e_2 as parallel edges of $G + X$

G is *k-edge-Hamilton-connected* if, for any $X \subset E^+(G)$ such that $|X| = k$ and the the edges of X determine a path system, the graph $G + X$ has a hamiltonian cycle containing all edges in X.

Easy to observe:

(i) A graph G is 1-edge-Hamilton-connected if and only if G is Hamilton-connected.

(ii) A graph G is 2-edge-Hamilton-connected if and only if

- (i) G is 1-Hamilton-connected (i.e., $G-x$ is Hamilton-connected for any vertex $x \in V(G)$), and
- (ii) for any four distinct vertices $x_1, x_2, x_3, x_4 \in V(G)$, G has a path factor consisting of 2 paths P_1, P_2 such that both P_1 and P_2 have one endvertex in $\{x_1, x_2\}$ and one endvertex in $\{x_3, x_4\}$.

(iii) If G is 2-edge-Hamilton-connected, then G is 4-connected.

Theorem. *The following statements are equivalent:*

(i) *Every snark has a dominating cycle*

(ii) *Every 4-connected claw-free graph is hamiltonian*

(iii) *Every 4-connected line graph is hamiltonian*

(iv) *Every 4-connected line graph is Hamilton-connected*

(v) *Every 4-connected line graph is 1-Hamilton-connected*

(vi) *Every 4-connected line graph is 2-edge-Hamilton-connected*

(iv): [Kužel, Xiong 2004]

(v), (vi): [Kužel, ZR, Vrána 2009]

Consider the following two decision problems.

k-HC

Instance: *A graph G.*

Question: *Is G k-Hamilton-connected?*

k-HCL

Instance: *A line graph G.*

Question: *Is G k-Hamilton-connected?*

(i.e., k-HCL is k-HC restricted to line graphs).

---

k-E-HC

Instance: *A graph G.*

Question: *Is G k-edge-Hamilton-connected?*

k-E-HCL

Instance: *A line graph G.*

Question: *Is G k-edge-Hamilton-connected?*

(i.e., k-E-HCL is k-E-HC restricted to line graphs).

Question 1: *Determine the complexity of 1-HCL and/or of 2-E-HCL.*

Known:

HAM

Instance: *A graph* G.

Question: *Does* G *contain a hamiltonian cycle?*

HAM $\in$ NPC, even if restricted to line graphs.

H-PATH

Instance: *A graph* G *and distinct vertices* $u, v \in V(G)$.

Question: *Does* G *contain a hamiltonian* (u, v)*-path?*

H-PATH $\in$ NPC, even if restricted to line graphs [Bertossi 1981]

H-CONN

Instance: *A graph* G.

Question: *Is* G *Hamilton-connected?*

H-CONN $\in$ NPC [Dean 1993]

1-H-CONN

Instance: *A graph* G.

Question: *Is* G *1-Hamilton-connected?*

1-H-CONN $\in$ NPC [Vrána, personal communication]

Thus, a common guess would be that probably 2-E-HCL $\in$ NPC.

Question 2: *Why is Question 1 interesting?*

Recall:

Theorem. *The following statements are equivalent:*

(i) *Every 4-connected line graph is hamiltonian*

(ii) *Every 4-connected line graph is 1-Hamilton-connected*

(iii) *Every 4-connected line graph is 2-edge-Hamilton-connected*

If the Thomassen's conjecture is true, then:

- A line graph G is 1-Hamilton-connected $\Longleftrightarrow$ G is 4-connected
- both 1-HCL and 2-E-HCL is polynomial

Proving the "common guess" 2-E-HCL $\in$ NPC would mean

- disproving the Thomassen's conjecture,
- proving the existence of a snark with no dominating cycle,

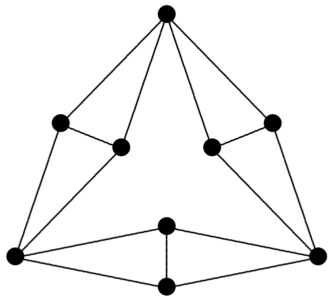
unless P=NP.

Claw-free graphs with complete closure

Theorem. *G claw-free with complete closure $\Rightarrow$ G has a C_{n-1}.*

Easy: C_3, C_4

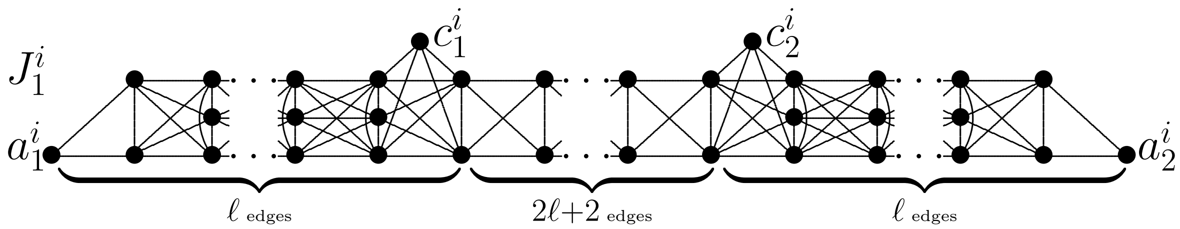
C_5 (?)

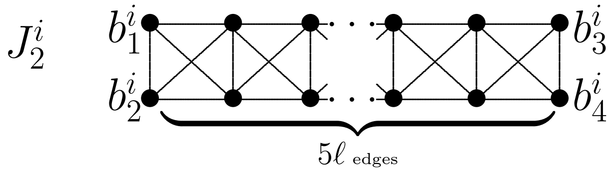


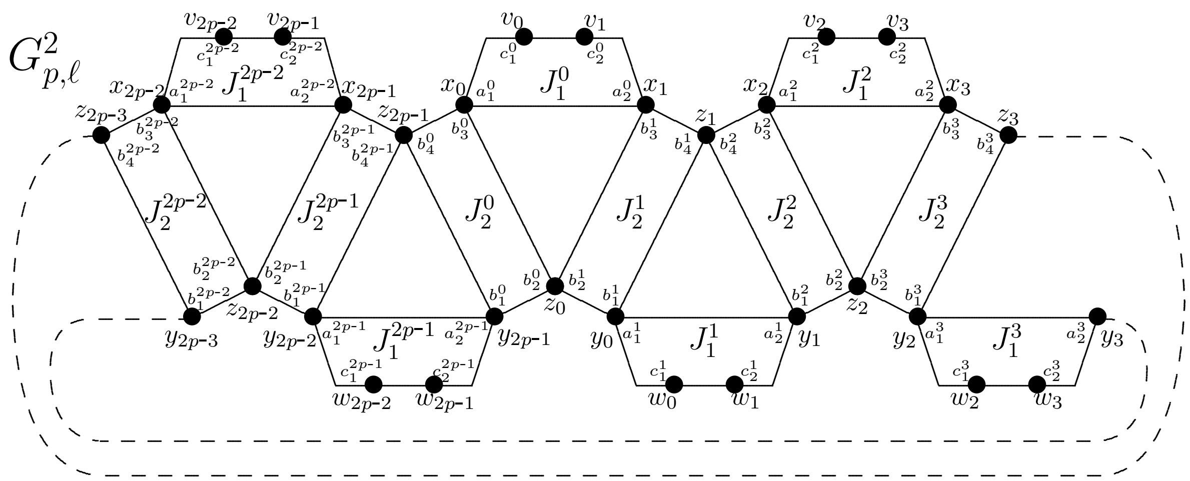
Conjecture. *Let c_1, c_2 be fixed constants. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, where $3 \leq i \leq c_1$ and $n - c_2 \leq i \leq n$.*

Theorem. *Let κ, λ be integers, $2 \leq \kappa \leq 5$, $\lambda \geq 33$ if $\kappa \in \{2, 3, 4\}$ and $\lambda \geq 52$ if $\kappa = 5$. Then there is an infinite family of claw-free graphs of connectivity κ with complete closure and not containing a cycle of length λ.*

$\kappa = 4$







Construction of G:

(i) take a cubic hamiltonian graph H with "sufficiently large" girth,

(ii) denote C a hamiltonian cycle in H,

(iii) label the vertices of H "along" C,

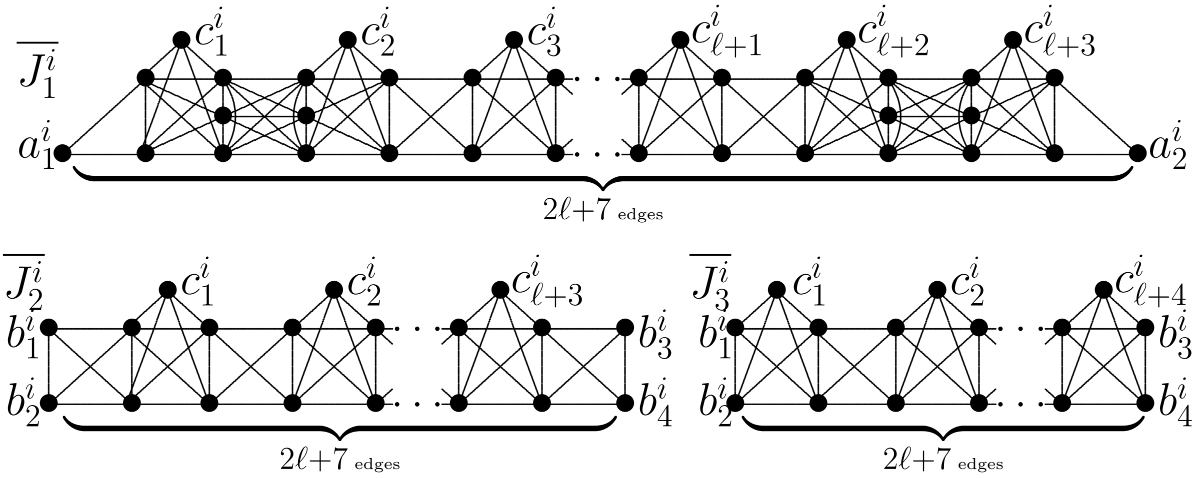
(iv) set $H' = H - C$ (i.e., H' is a matching)

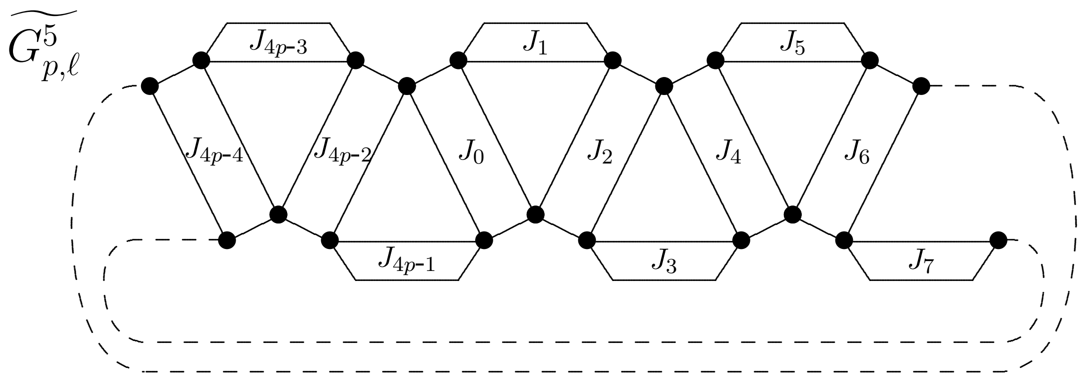
(v) Take 2 copies H_1, H_2 of H' and identify the vertices of H_1 and H_2 with the vertices $v_0, v_1, \ldots, v_{2p-1}$ and $w_0, w_1, \ldots, w_{2p-1}$, respectively.

Sufficient:

$H = 272_{16}(45, 59, -119, -89, 101, -109, -72, 72, 109, -101, 89, 119, -59, -45, 72, -72)$ (the Hoare's 13-cage on 272 vertices), cyclically repeated along C

$\kappa = 5$





We need a 3-connected cubic hamiltonian graph of "sufficiently large" girth.

$\mathcal{G}_{n,d}$ – the uniform probability space of d-regular graphs on n vertices, dn even

Fact 1 [Bollobás, Wormald].

For fixed $d \geq 3$, any $G \in \mathcal{G}_{n,d}$ is asymptotically almost surely d-connected.

Fact 2 [Robinson, Wormald].

For fixed $d \geq 3$, any $G \in \mathcal{G}_{n,d}$ is asymptotically almost surely hamiltonian.

Fact 3 [Wormald].

For fixed d, let $X_i = X_{i,n}$ $(i \geq 3)$ be the number of cycles of length i in a graph in $\mathcal{G}_{n,d}$. For fixed $k \geq 3$, $X_3, \ldots, X_k$ are asymptotically independent Poisson random variables with means $\lambda_i = \frac{(d-1)^i}{2i}$.

From Facts 1-3 we easily conclude the following consequence.

Fact 4. *There is an infinite family of cubic hamiltonian 3-connected graphs with arbitrarily large fixed girth.*

Conjecture 1. *Let c be a fixed constant. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, $n - c \leq i \leq n$.*

Conjecture 2. *Every 6-connected claw-free graph with complete closure is pancyclic.*

Conjecture 2 $\Rightarrow$ pancyclicity is stable in the class of 6-connected claw-free graphs with complete closure.

Oberly, Sumner 1980:

$$\left.\begin{array}{l}\text{locally connected}\\ \text{claw-free}\end{array}\right\} \Longrightarrow \text{vertex pancyclic}$$

G is *weakly pancyclic* if G has a C_ℓ for all ℓ, $3 \leq \ell \leq c(G)$.

Conjecture. *Every locally connected graph is weakly pancyclic.*

Known for:

- claw-free graphs
- chordal graphs (easy observation)
- squares (follows from a result by Fleischner)
- lexicographic product $G[H]$ if G is connected and H has at least 1 edge [Kaiser, Kriesell]
- graphs with maximum degree ≤ 4 [Kriesell]
- planar triangulations [Balister]