λ-Backbone Colorings Along Pairwise Disjoint Stars and Matchings

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Abstract

Given an integer $\lambda \geq 2$, a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a λ-backbone coloring of (G, H) is a proper vertex coloring $V \to \{1, 2, \dots\}$ of G, in which the colors assigned to adjacent vertices in H differ by at least λ. We study the case where the backbone is either a collection of pairwise disjoint stars or a matching. We show that for a star backbone S of G the minimum number ℓ for which a λ-backbone coloring of (G, S) with colors in $\{1, \dots, \ell\}$ exists can roughly differ by a multiplicative factor of at most $2 - \frac{1}{\lambda}$ from the chromatic number $\chi(G)$. For the special case of matching backbones this factor is roughly $2 - \frac{2}{\lambda+1}$. We also show that the computational complexity of the problem "Given a graph G with a star backbone S, and an integer ℓ, is there a λ-backbone coloring of (G, S) with colors in $\{1, \dots, \ell\}$?" jumps from polynomially solvable to NP-complete between $\ell = \lambda + 1$ and $\ell = \lambda + 2$ (the case $\ell = \lambda + 2$ is even NP-complete for matchings). We finish the paper by discussing some open problems regarding planar graphs.

Keywords: λ-backbone coloring, λ-backbone coloring number, star, matching.

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1 Introduction

In [7] backbone colorings are introduced, motivated and put into a general framework of coloring problems related to frequency assignment.

Graphs are used to model the topology and interference between transmitters (receivers, base stations, sensors): the vertices represent the transmitters; two vertices are adjacent if the corresponding transmitters are so close (or so strong) that they are likely to interfere if they broadcast on the same or 'similar' frequency channels. The problem is to assign the frequency channels in an economical way to the transmitters in such a way that interference is kept at an 'acceptable level'. This has led to various different types of coloring problems in graphs, depending on different ways to model the level of interference, the notion of similar frequency channels, and the definition of acceptable level of interference (See, e.g., [16],[20]). Although new technologies have led to different ways of avoiding interference between powerful transmitters, such as base stations for mobile telephones, the above coloring problems still apply to less powerful transmitters, such as those appearing in sensor networks.

We refer to [6] and [7] for an overview of related research, but we repeat the general framework and some of the related research here for convenience and background.

> Given two graphs G_1 and G_2 with the property that G_1 is a spanning subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2.

Many known coloring problems fit into this general framework. We mention some of them here explicitly, without giving details. First of all suppose that $G_2 = G_1^2$, i.e. G_2 is obtained from G_1 by adding edges between all pairs of vertices that are at distance 2 in G_1. If one just asks for a proper vertex coloring of G_2 (and G_1), this is known as the *distance-2 coloring problem.* Much of the research has been concentrated on the case that G_1 is a planar graph. We refer to [1], [4], [5], [18], and [21] for more details. In some versions of this problem one puts the additional restriction on G_1 that the colors should be sufficiently separated, in order to model practical frequency assignment problems in which interference should be kept at an acceptable level. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and G_2 such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. A closely related variant is known as the radio coloring problem and has been studied (under various names) in [2], [9], [10], [11], [12], [13], and [19]. A third variant is known as the radio labeling problem and models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2. We refer to [15] and [17] for more particulars.

In [7], a situation is modeled in which the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means more restrictions are put on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters.

Postponing the relevant definitions, we consider the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that models the backbone) differ by at least $\lambda \geq 2$. This is a continuation of the study in [7]. Throughout the paper we consider two types of backbones: matchings and disjoint unions of stars.

Matching backbones reflect the necessity to assign considerably different frequencies to pairwise very close (or most likely interfering) transmitters. This occurs in real world applications such as military scenarios, where soldiers or military vehicles carry two (or sometimes more) radios for reliable communication. Future applications include the use of sensors or sensor tags in clothes or on bodies.

For star backbones one could think of applications to sensor networks. If sensors have low battery capacities, the tasks of transmitting data are often assigned to specific sensors, called cluster heads, that represent pairwise disjoint clusters of sensors. Within the clusters there should be a considerable difference between the frequencies assigned to the cluster head and to the other sensors within the same cluster, whereas the differences between the frequencies assigned to the other sensors within the cluster, or between different clusters, is of a secondary importance. This situation is well reflected by a backbone consisting of disjoint stars.

We refer to [7] and [6] for a more extensive overview of related research, but we repeat the relevant definitions in the next section.

2 Terminology

For undefined terminology we refer to [3].

Let $G = (V, E)$ be a graph, where $V = V_G$ is a finite set of vertices and $E = E_G$ is a set of unordered pairs of two different vertices, called edges. A function $f : V \to \{1, 2, 3, \ldots\}$ is a *vertex coloring* of V if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A vertex coloring $f : V \to \{1, \ldots, k\}$ is called a *k-coloring*. We say that $f(u)$ is the *color* of u. The *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. A set $V' \subseteq V$ is *independent* if G does not contain edges with both end vertices in V'. By definition, a k-coloring partitions V into k independent sets $V_1, \ldots, V_k$.

Let H be a *spanning subgraph* of G, i.e., $H = (V_G, E_H)$ with $E_H \subseteq E_G$. Given an integer $\lambda \geq 1$, a vertex coloring f is a *λ-backbone coloring* of (G, H), if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$. A λ-backbone coloring $f : V \to \{1, \ldots, \ell\}$ is called a *λ-backbone ℓ-coloring*. The *λ-backbone coloring number* $\mathrm{BBC}_\lambda(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a λ-backbone ℓ-coloring. Since a 1-backbone coloring is equivalent to a vertex coloring, we assume from now on that $\lambda \geq 2$. Throughout the manuscript we will reserve the symbol "ℓ" for λ-backbone ℓ-colorings and the symbol "k" for k-colorings.

A *path* is a graph P whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_P = \{v_1v_2, \ldots, v_{n-1}v_n\}$. A graph G is called *connected* if for every pair of distinct vertices u and v, there exists a path connecting u and v. The *length* of a path is the number of its edges. If a graph G contains a spanning subgraph H that is a path, then H is called a *Hamiltonian path*.

A *cycle* is a graph C whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_C = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$. A *tree* is a connected graph that does not contain any cycles.

A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on n vertices is denoted by K_n. A graph is called *bipartite* if its vertices can be partitioned into two sets A and B such that each edge has one of its endpoints incident with the set A and the other with B. A graph G is *complete p-partite* if its vertices can be partitioned into p nonempty independent sets $V_1, \ldots, V_p$ such that its edge set E is formed by all edges that have one end vertex in V_i and the other one in V_j for some $1 \leq i < j \leq p$.

For $q \geq 1$, a *star* S_q is a complete 2-partite graph with independent sets $V_1 = \{r\}$ and V_2 with $|V_2| = q$; the vertex r is called the *root* and the vertices in V_2 are called the *leaves* of S_q. For the star S_1 we arbitrarily choose one of its two vertices to be the root. In our context a *matching* M is a collection of pairwise vertex-disjoint stars that are all copies of S_1. A matching M of a graph G is called *perfect* if it is a spanning subgraph of G.

We call a spanning subgraph H of a graph G

- a *tree backbone* of G if H is a tree;
- a *path backbone* of G if H is a Hamiltonian path;
- a *star backbone* of G if H is a collection of pairwise vertex-disjoint stars;
- a *matching backbone* of G if H is a perfect matching.

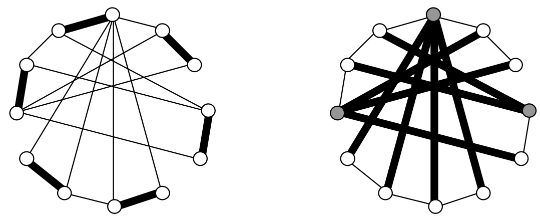

Fig. 1. Matching and star backbones

See Figure 1 for an example of a graph G with a matching backbone M (left) and a star backbone S (right). The thick edges are matching or star edges, respectively. The grey circles indicate root vertices of the stars in S.

Obviously, $\text{BBC}_\lambda(G,H) \geq \chi(G)$ holds for any backbone H of a graph G. In order to analyze the maximum difference between these two numbers the following values can be introduced.

$$\mathcal{T}_\lambda(k) = \max\{\text{BBC}_\lambda(G,T) \mid T \text{ is a tree backbone of } G, \text{ and } \chi(G)=k\}$$
$$\mathcal{P}_\lambda(k) = \max\{\text{BBC}_\lambda(G,P) \mid P \text{ is a path backbone of } G, \text{ and } \chi(G)=k\}$$
$$\mathcal{S}_\lambda(k) = \max\{\text{BBC}_\lambda(G,S) \mid S \text{ is a star backbone of } G, \text{ and } \chi(G)=k\}$$
$$\mathcal{M}_\lambda(k) = \max\{\text{BBC}_\lambda(G,M) \mid M \text{ is a matching backbone of } G, \text{ and } \chi(G)=k\}.$$

3 Results

3.1 Existing results

The behavior of $\mathcal{T}_\lambda(k)$ and $\mathcal{P}_\lambda(k)$ is determined in [7] as summarized in the following two results.

Theorem 1 $\mathcal{T}_2(k) = 2k-1$ *for all* $k \geq 1$.

Theorem 2 *The function* $\mathcal{P}_2(k)$ *takes the following values:*

(a) for $1 \leq k \leq 4$*:* $\mathcal{P}_2(k) = 2k-1$*;*

(b) $\mathcal{P}_2(5) = 8$ *and* $\mathcal{P}_2(6) = 10$*;*
(c) for $k \geq 7$ *and* $k = 4t$*:* $\mathcal{P}_2(4t) = 6t$*;*
(d) for $k \geq 7$ *and* $k = 4t + 1$*:* $\mathcal{P}_2(4t + 1) = 6t + 1$*;*
(e) for $k \geq 7$ *and* $k = 4t + 2$*:* $\mathcal{P}_2(4t + 2) = 6t + 3$*;*
(f) for $k \geq 7$ *and* $k = 4t + 3$*:* $\mathcal{P}_2(4t + 3) = 6t + 5$*.*

The above theorems show the relation between the 2-backbone coloring number and the classical chromatic number in case the backbone is a tree or a path. We observe that in the worst case the 2-backbone coloring number roughly grows like $2k$ and $3k/2$, respectively, where $\chi = k$.

Problem 3 *What are the values for* $\mathcal{T}_\lambda(k)$ *and* $\mathcal{P}_\lambda(k)$ *for* $\lambda \geq 3$*?*

3.2 Results of this paper

In this paper, we study the functions $\mathcal{S}_\lambda(k)$ and $\mathcal{M}_\lambda(k)$. By definition, $\mathcal{M}_\lambda(k) \leq \mathcal{S}_\lambda(k)$ holds. We completely determine the behavior of these two functions. We first determine all values $\mathcal{S}_\lambda(k)$, and observe that they roughly grow like $(2 - \frac{1}{\lambda})k$. Then we determine all values $\mathcal{M}_\lambda(k)$ and observe that they roughly grow like $(2 - \frac{2}{\lambda+1})k$. Their precise behavior is summarized in our two main theorems.

Theorem 4 *For* $\lambda \geq 2$ *the function* $\mathcal{S}_\lambda(k)$ *takes the following values:*

(a) $\mathcal{S}_\lambda(2) = \lambda + 1$*;*
(b) for $3 \leq k \leq 2\lambda - 3$*:* $\mathcal{S}_\lambda(k) = \lceil \frac{3k}{2} \rceil + \lambda - 2$*;*
(c) for $2\lambda - 1 \leq k \leq 2\lambda$ *with* $\lambda = 2$*:* $\mathcal{S}_\lambda(k) = k + 2\lambda - 2$*;*
(d) for $2\lambda - 2 \leq k \leq 2\lambda - 1$ *with* $\lambda \geq 3$*:* $\mathcal{S}_\lambda(k) = k + 2\lambda - 2$*;*
(e) for $k = 2\lambda$ *with* $\lambda \geq 3$*:* $\mathcal{S}_\lambda(k) = 2k - 1$*;*
(f) for $k \geq 2\lambda + 1$*:* $\mathcal{S}_\lambda(k) = 2k - \lfloor \frac{k}{\lambda} \rfloor$*.*

Theorem 5 *For* $\lambda \geq 2$ *the function* $\mathcal{M}_\lambda(k)$ *takes the following values:*

(a) for $2 \leq k \leq \lambda$*:* $\mathcal{M}_\lambda(k) = k + \lambda - 1$*;*
(b) for $\lambda + 1 \leq k \leq 2\lambda$*:* $\mathcal{M}_\lambda(k) = 2k - 2$*;*
(c) for $k = 2\lambda + 1$*:* $\mathcal{M}_\lambda(k) = 2k - 3$*;*
(d) for $k = t(\lambda + 1)$ *with* $t \geq 2$*:* $\mathcal{M}_\lambda(k) = 2t\lambda$*;*
(e) for $k = t(\lambda + 1) + c$ *with* $t \geq 2$*,* $1 \leq c < \frac{\lambda+3}{2}$*:* $\mathcal{M}_\lambda(k) = 2t\lambda + 2c - 1$*;*
(f) for $k = t(\lambda + 1) + c$ *with* $t \geq 2$*,* $\frac{\lambda+3}{2} \leq c \leq \lambda$*:* $\mathcal{M}_\lambda(k) = 2t\lambda + 2c - 2$*.*

We note that there are many graphs G that have a star backbone S such that $\text{BBC}_\lambda(G, S) < \mathcal{S}_\lambda(\chi(G))$, or that have a matching backbone M such that $\text{BBC}_\lambda(G, M) < \mathcal{M}_\lambda(\chi(G))$. As an example we mention the class of *split graphs*, e.g., graphs whose vertex set can be partitioned into a clique (i.e., a set of pairwise adjacent vertices) and an independent set, with possibly edges in between. In [8] we present (tight) upper bounds on the λ-star and λ-matching backbone coloring number for this graph class. These upper bounds are considerably smaller than the *general* bounds given in Theorem 4 and Theorem 5, respectively.

The rest of the paper is organized as follows. In the next section we consider the computational complexity of computing the λ-backbone coloring number for star and matching backbones. The fifth section gives the proof of Theorem 4, and the sixth section gives the proof of Theorem 5. There are many open problems about backbone colorings. We refer to [7] for details. In the last section of this paper we only focus on some open problems for matching backbone colorings for planar graphs.

4 Complexity Results

The following decision problem can be defined.

λ-Backbone Colorability(ℓ) (λ-BBC(ℓ))

Instance: A graph G with a spanning subgraph H.
Question: Is $\text{BBC}_\lambda(G,H) \le \ell$?

Of course, λ-BBC(ℓ) is NP-complete if ℓ exceeds a certain value. In [7] it has been shown that the complexity of 2-BBC(ℓ) restricted to instance graphs G with a tree backbone H jumps from polynomially solvable to NP-complete between $\ell = 4$ and $\ell = 5$ (difficult even for path backbones). Here we restrict ourselves to instance graphs G with a star backbone S.

Star λ-Backbone Colorability(ℓ) (λ-SBBC(ℓ))

Instance: A graph G with a star backbone S.
Question: Is $\text{BBC}_\lambda(G,S) \le \ell$?

Theorem 6 *λ-SBBC(ℓ) is polynomially solvable if $\ell \le \lambda + 1$, and it is NP-complete if $\ell \ge \lambda + 2$ (even when restricted to matching backbones).*

Proof: Let $G = (V,E)$ be a graph with a star backbone $S = (V, E_S)$. For $\ell \le \lambda$ no λ-backbone coloring exists. Now let $\ell = \lambda + 1$. In any λ-backbone coloring with color set $\{1, 2, \ldots, \lambda + 1\}$, colors $2, 3, \ldots, \lambda$ can not be used at all, since each vertex is incident with an edge of E_S. Hence $\text{BBC}_\lambda(G,S) = \lambda+1$ if and only if G is bipartite.

Let $\ell \geq \lambda + 2$. Obviously the problem λ-SBBC(ℓ) is a member of NP. We prove NP-completeness by reduction from Graph k-Colorability (cf. [14]): Given a graph $G = (V_G, E_G)$, does there exist a k-coloring of G? This problem is known to be NP-complete for any integer $k \geq 3$. We distinguish the following cases.

Case 1 $\lambda + 2 \leq \ell \leq 2\lambda - 1$.
Let $\ell = \lambda + t$ for some $2 \leq t \leq \lambda - 1$, and let $G = (V_G, E_G)$ be an instance of Graph $2t$-Colorability. Let $v_1, v_2, \ldots, v_n$ denote the vertices in V_G. We create n new vertices $u_1, u_2, \ldots, u_n$ and introduce new edges $v_i u_i$ ($i = 1, 2, \ldots, n$). The graph that results from this is denoted by G'. The new edges form a matching backbone M of G'. We claim that $\chi(G) \leq 2t$ if and only if $\text{BBC}_\lambda(G', M) \leq \ell$.

Assume that $\text{BBC}_\lambda(G', M) \leq \ell$, and consider a λ-backbone ℓ-coloring b of G'. Since all vertices in G' are incident with a matching edge, colors $t+1, t+2, \ldots, \lambda$ can not be used at all. Then define a $2t$-coloring c of G by:

- if $b(v) = j$ for $j \in \{1, 2, \ldots, t\}$: $c(v) := j$;
- if $b(v) = \lambda + j$ for $j \in \{1, 2, \ldots, t\}$: $c(v) := t + j$.

Next, assume that $\chi(G) \leq 2t$, and consider a $2t$-coloring $f : V_G \to \{1, \ldots, 2t\}$. We define a λ-backbone ℓ-coloring $g : V_{G'} \to \{1, \ldots, \ell\}$ of (G', M) by:

- if $v \in V_G$ and $f(v) = j$ for $j \in \{1, 2, \ldots, t\}$: $g(v) := j$;
- if $v \in V_G$ and $f(v) = t + j$ for $j \in \{1, 2, \ldots, t\}$: $g(v) := \lambda + j$;
- if $g(v_i) \leq t$: $g(u_i) := \ell$;
- If $g(v_i) \geq \lambda + 1$: $g(u_i) := 1$.

Case 2 $\ell \geq 2\lambda$.
Let $G = (V_G, E_G)$ be an instance of Graph ℓ-Colorability, and denote the vertices in V_G by $v_1, v_2, \ldots, v_n$. We create n new vertices $u_1, u_2, \ldots, u_n$ and introduce new edges $v_i u_i$ ($i = 1, 2, \ldots, n$). The graph that results from this is denoted by G'. The new edges form a matching backbone M of G'. We complete the proof by showing that $\chi(G) \leq \ell$ if and only if $\text{BBC}_\lambda(G', M) \leq \ell$.

Indeed, assume that $\text{BBC}_\lambda(G', M) \leq \ell$ and consider such a λ-backbone ℓ-coloring. Then the restriction to the vertices in V_G yields an ℓ-coloring of G. Next assume that $\chi(G) \leq \ell$, and consider an ℓ-coloring $f : V_G \to \{1, , \ldots, \ell\}$. We extend f to a λ-backbone ℓ-coloring of (G', M): If $f(v_i) \leq \lambda$, then vertex u_i is colored with color ℓ, and otherwise it is assigned color 1. This completes the proof. $\square$

5 Proof of Theorem 4

We prove Theorem 4 in two parts. First we show that $\textsc{bbc}_\lambda(G, S)$ for any graph G with arbitrary star backbone S is at most the value of $\mathcal{S}_\lambda(\chi(G))$ as given in Theorem 4. Next we present a class of graphs G that have a star backbone S such that $\textsc{bbc}_\lambda(G, S)$ is at least the value of $\mathcal{S}_\lambda(\chi(G))$ that is given in Theorem 4. This way we obtain coinciding upper and lower bounds on $\mathcal{S}_\lambda(k)$ that prove the theorem.

5.1 Proof of the Upper Bounds

Let $G = (V, E)$ be a graph with $\chi(G) = k$ and let $V_1, \ldots, V_k$ denote the corresponding independent sets in a k-coloring. Let $S = (V, E_S)$ be a star backbone of G. If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda + 1$. This proves the upper bound for case (a) of the theorem.

Case (b) $3 \leq k \leq 2\lambda - 3$.

Consider the following color sets:

- $C_i = \{i, k + \lambda - 1 - i\}$ for $i = 1, \ldots, \lfloor \frac{k}{2} \rfloor$;
- $C_i = \{i, 2k + \lambda - 1 - i\}$ for $i = \lfloor \frac{k}{2} \rfloor + 1, \ldots, k$.

The union of these k color sets consists of $2k$ colors, namely the colors in $\{1, \ldots, k\}$ together with the colors in $\{k+\lambda-1-\lfloor \frac{k}{2} \rfloor, \ldots, 2k+\lambda-1-(\lfloor \frac{k}{2} \rfloor+1)\}$. The largest color used is $2k + \lambda - 1 - (\lfloor \frac{k}{2} \rfloor + 1) = \lceil \frac{3k}{2} \rceil + \lambda - 2$.

We construct a λ-backbone coloring of (G, S) such that every vertex in V_i ($i = 1, \ldots, k$) is colored with a color in C_i. Since the vertex subsets V_i are independent, we will obtain a vertex coloring this way. To show that we can obtain a λ-backbone $(\lceil \frac{3k}{2} \rceil + \lambda - 2)$-coloring this way, we have to be a bit more careful.

For $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, a root vertex in V_i is colored with the first color of C_i. For $\lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k$, a root vertex in V_i is colored with the second color of C_i.

The leaves in a set V_j of a star with a root in a set V_i for $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ are colored with the second color of C_j. This does not give any conflict, since the smallest gap appears if the root vertex is in $V_{\lfloor \frac{k}{2} \rfloor}$ and one of its leaves is in $V_{\lfloor \frac{k}{2} \rfloor - 1}$, or the other way around. In both cases this gap is $k + \lambda - 1 - \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{2} \rfloor - 1) = k + \lambda - 2\lfloor \frac{k}{2} \rfloor \geq \lambda$.

The leaves in a set V_j of a star with a root in a set V_i for $\lfloor \frac{k}{2} \rfloor + 1 \leq i \leq k$

are colored with the first color of C_j. This is possible, since the smallest gap appears if the root vertex is in V_k and one of its leaves is in V_{k-1}, or the other way around. In both cases this gap is $2k + \lambda - 1 - k - (k-1) = \lambda$. Hence, we indeed have obtained a desired λ-backbone $(\lceil \frac{3k}{2} \rceil + \lambda - 2)$-coloring of (G, S).

Case (c) $\lambda = 2, k = 3$ or $\lambda = 2, k = 4$.

For proving that $\mathcal{S}_2(3) \leq 5$ we use color sets $C_1 = \{1\}$, $C_2 = \{3\}$, $C_3 = \{5\}$. We color the vertices of V_1 by 1, the vertices of V_2 by 3, and the vertices of V_3 by 5. This gives us a 2-backbone 5-coloring of (G, S).

For proving that $\mathcal{S}_2(4) \leq 6$ we use color sets $C_1 = \{1\}, C_2 = \{2, 3\}, C_3 = \{4, 5\}$ and $C_4 = \{6\}$. We use color 1 for all vertices in V_1, and color 6 for all vertices in V_4. We color all roots in V_2 by color 3, and all roots in V_3 by color 4. If $v \in V_2$ is a leaf of a star with root in V_1, we color v by 3. Otherwise we color v by 2. If $w \in V_3$ is a leaf of a star with root in V_4, we color w by 4. Otherwise we color w by 5. This gives us a 2-backbone 6-coloring of (G, S).

For the cases $(d) - (f)$ we need the following lemma.

Lemma 7 *Let $G = (V, E)$ be a graph with $\chi(G) = k$ and let $V_1, \ldots, V_k$ denote the corresponding independent sets in a k-coloring. Let $S = (V, E_S)$ be a star backbone of G. For $i = 1, \ldots, p$, let $C_i = \{a_i\}$ be a set consisting of one color. For $j = 1, \ldots, q$, let $D_j = \{b_j, c_j\}$ be a set consisting of two colors. Let $d = \max\{a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_q\}$. Then (G, S) has a λ-backbone d-coloring if the following conditions are satisfied:*

(i) $p + q = k$.
(ii) $C_i \cap C_j = \emptyset$ *for all* $1 \leq i < j \leq p$.
(iii) $D_i \cap D_j = \emptyset$ *for all* $1 \leq i < j \leq q$.
(iv) $C_i \cap D_j = \emptyset$ *for all* $1 \leq i \leq p$ *and* $1 \leq j \leq q$.
(v) $|a_i - a_j| \geq \lambda$ *for all* $1 \leq i < j \leq p$.
(vi) $|a_i - b_j| \geq \lambda$ *for all* $1 \leq i \leq p$ *and* $1 \leq j \leq q$.
(vii) $|b_j - c_j| \geq 2\lambda - 1$ *for all* $1 \leq j \leq q$.

Proof: Due to (i), we can map each vertex in V_i to a color in C_i for $i = 1, \ldots, p$ and each vertex v in V_j to a color in D_{j-p} for $j = p+1, \ldots, k$. Since the sets V_i are independent, conditions (ii)-(iv) imply that this way we are guaranteed to obtain a vertex coloring of G with colors in $\{1, \ldots, d\}$. Below we explain how we can refine this strategy such that we obtain a λ-backbone d-coloring f of (G, S).

So far only the colors of vertices in V_i for $i = 1, \ldots, p$ have been fixed by a coloring f as above. Due to (v), $|f(u) - f(v)| \geq \lambda$ for all star edges uv with $u \in V_i, v \in V_j$ for some $1 \leq i < j \leq p$.

We let f color a root vertex in V_j for $p+1 \leq j \leq k$ with color b_{j-p}. Due to (vi), we find that $|f(v)-f(u)| \geq \lambda$ holds for all star edges uv with leaf u in V_i for some $1 \leq i \leq p$ and root v in V_j for some $p+1 \leq j \leq k$.

What about the other vertices? They are all leaf vertices in sets V_j with $p+1 \leq j \leq k$. Let $v \in V_j$ with $p+1 \leq j \leq k$ be a leaf vertex of a star S with root w. Let x be the color assigned to w. Then colors $x-\lambda+1, \ldots, x+\lambda-1$ are forbidden colors for v. The distance between $x+\lambda-1$ and $x-\lambda+1$ is $2\lambda-2$. Since the two colors in D_{j-p} have pairwise distance at least $2\lambda-1$ due to (vii), at least one of them is feasible for v. This finishes the proof of the lemma. $\square$

Case (d) $2\lambda-2 \leq k \leq 2\lambda-1$ with $\lambda \geq 3$.

We use color sets:

- $C_1 = \{k\}$;
- $D_j = \{j, j+2\lambda-1\}$ for $j = 1, \ldots, k-1$.

We will show that these color sets satisfy the conditions of Lemma 7. First note that these k color sets are pairwise disjoint: the union of these sets consists of all the colors in $\{1, \ldots, k\}$ together with all the colors in $\{2\lambda, \ldots, k+2\lambda-2\}$. We set $b_j := j$ for $1 \leq j \leq \lceil \frac{k}{2} \rceil - 1$. Then $a_1 - b_j = k - j \geq k - \lceil \frac{k}{2} \rceil + 1 \geq \lambda$. for $1 \leq j \leq \lceil \frac{k}{2} \rceil - 1$. We set $b_j := j + 2\lambda - 1$ for $\lceil \frac{k}{2} \rceil \leq j \leq k-1$. Then $b_j - a_1 = j + 2\lambda - 1 - k \geq \lceil \frac{k}{2} \rceil + 2\lambda - 1 - k \geq \lambda$ for $\lceil \frac{k}{2} \rceil \leq j \leq k-1$. We observe that the two colors b_j, c_j in any set D_j have pairwise distance $2\lambda-1$. Hence, all conditions of Lemma 7 are satisfied. This implies that (G,S) has a λ-backbone $(k+2\lambda-2)$-coloring.

Case (e) $k = 2\lambda$ with $\lambda \geq 3$.

We use color sets:

- $C_1 = \{4\lambda-1\}$;
- $D_j = \{j, 2\lambda-1+j\}$ for $j = 1, \ldots, k-1$.

We will show that these color sets satisfy the conditions of Lemma 7. Note that these k color sets are pairwise disjoint: the union of these sets consists of all the colors in $\{1, \ldots, k-1\}$ together with all the colors in $\{2\lambda, \ldots, 4\lambda-1\}$. We set $b_j := j$ for $1 \leq j \leq k-1$. Then $a_1 - b_j = 4\lambda - 1 - j \geq 4\lambda - 1 - (k-1) = 2\lambda \geq \lambda$ for $1 \leq j \leq k-1$. We note that the difference between the two colors b_j and c_j in any set D_j is equal to $2\lambda - 1 + j - j = 2\lambda - 1$. Hence, all conditions of Lemma 7 are satisfied. This implies that (G,S) has a λ-backbone $(4\lambda-1)$-coloring.

Case (f) $k \geq 2\lambda + 1$.

We use color sets:

- $C_i = \{(i-1)\lambda + 1\}$ for $i = 1, \ldots, \lfloor \frac{k}{\lambda} \rfloor$;
- $D_j = \{\lceil \frac{j\lambda}{\lambda-1} \rceil, j+k\}$ for $j = 1, \ldots, \lfloor \frac{k}{\lambda} \rfloor(\lambda-1)$;
- $D_j = \{j - \lfloor \frac{k}{\lambda} \rfloor, j+k\}$ for $j = \lfloor \frac{k}{\lambda} \rfloor(\lambda-1)+1, \ldots, k - \lfloor \frac{k}{\lambda} \rfloor$ and $k > \lfloor \frac{k}{\lambda} \rfloor \lambda$.

We will show that these k color sets satisfy the conditions of Lemma 7. If $j = s(\lambda-1)+t$ for some integers $s \geq 0$ and $0 \leq t \leq \lambda - 2$, then $\lceil \frac{j\lambda}{\lambda-1} \rceil$ is equal to $s\lambda$ in case $t = 0$ and to $s\lambda + t + 1$ in case $t > 0$. Then $C_i \cap D_j$ is empty for all $1 \leq i \leq \lfloor \frac{k}{\lambda} \rfloor$ and $1 \leq j \leq \lfloor \frac{k}{\lambda} \rfloor(\lambda-1)$. Hence the k color sets as defined above are pairwise disjoint, and cover the whole range $1, \ldots, 2k - \lfloor \frac{k}{\lambda} \rfloor$.

We observe that two colors a_i, a_j in two different sets C_i and C_j are at least λ apart from each other. We define $b_j := j+k$ for $1 \leq j \leq k - \lfloor \frac{k}{\lambda} \rfloor$. The smallest gap between a color b_j and a color a_i is $1+k-((\lfloor \frac{k}{\lambda} \rfloor - 1)\lambda + 1) = k - \lfloor \frac{k}{\lambda} \rfloor \lambda + \lambda \geq \lambda$.

For $j = 1, \ldots, \lfloor \frac{k}{\lambda} \rfloor(\lambda-1)$, the distance between two colors in a color set D_j is

$$j+k-\lceil \frac{j\lambda}{\lambda-1} \rceil = j+k-\lceil j + \frac{j}{\lambda-1} \rceil = k - \lceil \frac{j}{\lambda-1} \rceil \geq k - \lceil \frac{\lfloor \frac{k}{\lambda} \rfloor(\lambda-1)}{\lambda-1} \rceil = k - \lfloor \frac{k}{\lambda} \rfloor.$$

Also the distance between two colors in a color set D_j for $j = \lfloor \frac{k}{\lambda} \rfloor(\lambda-1) + 1, \ldots, k - \lfloor \frac{k}{\lambda} \rfloor$ is at least $k - \lfloor \frac{k}{\lambda} \rfloor$. We deduce that

$$k - \lfloor \tfrac{k}{\lambda} \rfloor = \lceil \tfrac{k(\lambda-1)}{\lambda} \rceil \geq \lceil \tfrac{(2\lambda+1)(\lambda-1)}{\lambda} \rceil = \lceil 2\lambda - 1 - \tfrac{1}{\lambda} \rceil = 2\lambda - 1.$$

Hence, all conditions of Lemma 7 are satisfied. This implies that (G, S) has a λ-backbone $(2k - \lfloor \frac{k}{\lambda} \rfloor)$-coloring.

5.2 Proof of the Lower Bounds

Let $\lambda \geq 2$. The case $k = 2$ is trivial. For $k \geq 3$, we consider the *Turán graph* $T(k^2, k)$, i.e., a complete k-partite graph that consists of k independent sets $V_1, \ldots, V_k$ that are all of cardinality k. Let $S = (V, E_S)$ be a star backbone of $T(k^2, k)$ that consists of k stars S_{k-1}. Each V_i contains exactly one root vertex of some star in S and its other $k-1$ vertices are leaves of stars rooted in $k-1$ different sets $V_j \neq V_i$. See Figure 2 for an example of the graph $T(9, 3)$ with star backbone S; the sets V_i are indicated and the thick edges are the star edges. For our case analysis we first prove a number of results for an *arbitrary* λ-backbone ℓ-coloring of $(T(k^2, k), S)$.

Let f be a λ-backbone ℓ-coloring of $(T(k^2, k), S)$. Since $T(k^2, k)$ is complete k-partite, any color that shows up in some set V_i can not show up in any V_j with $j \neq i$. We denote by C_i the set of colors that are used on vertices in

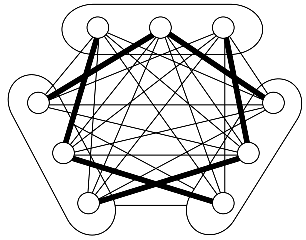

Fig. 2. The graph $T(9,3)$ with star backbone S.

V_i. If $|C_i| = 1$, then V_i is called *monochromatic*, and if $|C_i| \geq 2$, then V_i is called *polychromatic*. We denote by s_1 and s_2 the number of monochromatic and polychromatic sets, respectively. Then we immediately have $s_1 + s_2 = k$ and $s_1 + 2s_2 \leq \ell$ implying the following observation.

Observation 8 *Let f be a λ-backbone ℓ-coloring of $(T(k^2,k),S)$ with s_1 monochromatic sets. Then $s_1 \geq 2k - \ell$ holds.*

Since all stars in S have (exactly) one leaf in any set that does not contain their root vertex, we immediately have the following.

Observation 9 *Let f be a λ-backbone ℓ-coloring of $(T(k^2,k),S)$. Let x be the color for the root in set V_i. Let V_j $(j \neq i)$ be a monochromatic set colored by y. Then the distance between x and y is at least λ.*

We use Observation 9 to prove the following lemma.

Lemma 10 *Let f be a λ-backbone ℓ-coloring of $(T(k^2,k),S)$ with s_1 monochromatic sets and s_2 polychromatic sets. Then*

$$\ell \geq \begin{cases} (k-1)\lambda + 1 & \text{if } s_2 = 0; \\ s_1(\lambda - 1) + k & \text{if } s_2 > 0. \end{cases} \tag{1}$$

Proof: Suppose $s_2 = 0$. Then $s_1 = k$, and by Observation 9 there are at least $(k-1)$ gaps of at least $\lambda - 1$ colors that can not be used to color the k roots. Then the total number of colors needed is at least $(k-1)(\lambda-1)+k = (k-1)\lambda + 1$.

If $s_2 > 0$, Observation 9 implies that there are at least s_1 gaps of at least $\lambda - 1$ colors. In this case the total range of colors is at least $s_1(\lambda - 1) + k$. $\square$

A root in a monochromatic set is called *monochromatic* as well. A *root color* is a color that is used for a root. Recall that all stars in S have (exactly) one leaf in any set that does not contain their root vertex. Then we can easily make the following observation.

Observation 11 *Let f be a λ-backbone ℓ-coloring of $(T(k^2,k),S)$. Let x be the color for the root in V_i. Then there are (at least) $k-1$ different colors $y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_k$ that have distance at least λ to x: every V_j $(j\neq i)$ contains a vertex with color y_j.*

Due to Observation 11 we can prove the following lemma.

Lemma 12 *Let f be a λ-backbone ℓ-coloring of $(T(k^2,k),S)$. If $\ell\leq k+2\lambda-3$, then only colors from $A=\{1,\ldots,\ell-k-\lambda+2\}$ and $B=\{k+\lambda-1,\ldots,\ell\}$ can be assigned to root vertices.*

Proof: Suppose a root v is assigned color c with c in $\{\ell-k-\lambda+3,\ldots,k+\lambda-2\}$. By Observation 11 there have to be at least $k-1$ colors with distance at least λ from c. If $\lambda+1\leq c\leq \ell-\lambda$, only colors in $\{1,\ldots,c-\lambda\}$ and in $\{c+\lambda,\ldots,\ell\}$ can be used. These sets together contain $c-\lambda+\ell-(c+\lambda)+1=\ell-2\lambda+1\leq k-2$ colors. Hence either $c\leq\lambda$ or $c\geq \ell-\lambda+1$ holds. If $c\leq\lambda$, then only colors in $\{c+\lambda,\ldots,\ell\}$ are at distance at least λ. The cardinality of this set is $\ell-(c+\lambda)+1\leq \ell-(\ell-k-\lambda+3)-\lambda+1=k-2$. If $c\geq\ell-\lambda+1$, then only colors in $\{1,\ldots,c-\lambda\}$ are at distance at least λ. The cardinality of this set is $c-\lambda\leq k+\lambda-2-\lambda=k-2$. $\square$

We are now ready to make our case analysis.

Case (b) $3\leq k\leq 2\lambda-3$.

Suppose there exists a λ-backbone ℓ-coloring of $(T(k^2,k),S)$ with $\ell=\lceil\frac{3k}{2}\rceil+\lambda-3$ colors. Then $\ell=\lceil\frac{3k}{2}\rceil+\lambda-3\leq k+\lceil\frac{k}{2}\rceil+\lambda-3\leq k+2\lambda-3$ and by Lemma 12 only colors in $A=\{1,\ldots,\lceil\frac{k}{2}\rceil-1\}$ and colors in $B=\{k+\lambda-1,\ldots,\lceil\frac{3k}{2}\rceil+\lambda-3\}$ can be used on roots. Each root is in a different independent set V_i. Therefore the number of different root colors is equal to k. However, the total number of colors in A united with B is $2(\lceil\frac{k}{2}\rceil-1)<k$. This contradiction shows that we must have $\ell\geq\lceil\frac{3k}{2}\rceil+\lambda-2$.

Case (c,d) $2\lambda-1\leq k\leq 2\lambda$ with $\lambda=2$ or $2\lambda-2\leq k\leq 2\lambda-1$ with $\lambda\geq 3$.

Suppose there exists a λ-backbone ℓ-coloring of $(T(k^2,k),S)$ with $\ell=k+2\lambda-3$ colors. By Lemma 12, only colors in $A=\{1,\ldots,\lambda-1\}$ and $B=\{k+\lambda-1,\ldots,k+2\lambda-3\}$ may be used on roots. By Observation 8, $s_1\geq 2k-\ell=k-2\lambda+3\geq 2\lambda-2-2\lambda+3\geq 1$. So there exists at least one monochromatic set. Let y be the (root) color used on this set. Without loss of generality we may assume that y is in A. By Observation 9, all other $k-1$ root colors must be in B. However, B contains $\lambda-1<k-1$ colors. This contradiction shows that we must have $\ell\geq k+2\lambda-2$.

Case (e) $k=2\lambda$ with $\lambda\geq 3$.

Suppose there exists a λ-backbone ℓ-coloring of $(T(k^2, k), S)$ with $2k - 2 = 4\lambda - 2$ colors. If $s_2 = 0$, then by (1) we would have $2k - 2 = \ell \geq (k-1)\lambda + 1 \geq 3(k-1) + 1 = 3k - 2$. Hence $s_2 > 0$.

By Observation 8, $s_1 \geq 2k - \ell = 2$. Together with (1) we then deduce that

$$4\lambda - 2 = \ell \geq s_1(\lambda - 1) + k \geq 2(\lambda - 1) + 2\lambda = 4\lambda - 2.$$

Hence we find that $s_1 = 2$, and $\ell = s_1(\lambda - 1) + k$. Due to Observation 9, there are only three feasible ways to choose k different root colors:

1. monochromatic roots: $1, \lambda + 1$, other roots: $2\lambda + 1, \ldots, 4\lambda - 2$;
2. monochromatic roots: $1, 4\lambda - 2$, other roots: $\lambda + 1, \ldots, 3\lambda - 2$;
3. monochromatic roots: $3\lambda - 2, 4\lambda - 2$, other roots: $1, \ldots, 2\lambda - 2$.

Consider situation **1**. Since color $2\lambda + 1$ is a root color, by Observation 11, in every other color set there must be at least one color that has distance at least λ to color $2\lambda + 1$. This necessary condition is already met for the sets with root color 1, root color $\lambda + 1$ or root colors $3\lambda + 1, \ldots, 4\lambda - 2$. However, the sets with root colors $2\lambda + 2, \ldots, 3\lambda$ need an extra color. Hence, we need $\lambda - 1$ extra colors that have distance at least λ to color $2\lambda + 1$. There are exactly $\lambda - 1$ such colors available, namely colors $2, \ldots, \lambda$. So one of the colors $2, \ldots, \lambda$ must be in the same set with color $2\lambda + 2$.

Simultaneously, since color $2\lambda + 2$ is also a root color, in every other color set there must be at least one color that has distance at least λ to color $2\lambda + 2$. This condition is not met yet for the sets with root color $2\lambda + 1$ or root colors $2\lambda + 3, \ldots, 3\lambda + 1$. To satisfy the condition, we need λ extra colors that have distance at least λ to color $2\lambda + 2$. The only available colors are colors $2, \ldots, \lambda$ and color $\lambda + 2$. This implies that none of the colors $2, \ldots, \lambda$ can be in the same set with color $2\lambda + 2$. This contradiction shows that we must have $\ell \geq 2k - 1$.

Consider situation **2**. Since color $\lambda + 1$ is a root color, by Observation 11, in every other color set there must be at least one color that has distance at least λ to color $\lambda + 1$. This necessary condition is already met for the sets with root color 1, root color $4\lambda - 2$ or root colors $2\lambda + 1, \ldots, 3\lambda - 2$. However, the sets with root colors $\lambda + 2, \ldots, 2\lambda$ need an extra color. Hence, we need $\lambda - 1$ extra colors that have distance at least λ to color $\lambda + 1$. There are exactly $\lambda - 1$ such colors available, namely colors $3\lambda - 1, \ldots, 4\lambda - 3$. So one of the colors $3\lambda - 1, \ldots, 4\lambda - 3$ must be in the same set with color $\lambda + 2$.

Simultaneously, since color $\lambda + 2$ is also a root color, in every other color set there must be at least one color that has distance at least λ to color $\lambda + 2$. This condition is not met yet for the sets with root color $\lambda + 1$ or root colors $\lambda + 3, \ldots, 2\lambda + 1$. To satisfy the condition, we need λ extra colors that have distance at least λ to color $\lambda + 2$. The only available colors are color 2 and the colors $3\lambda-1,\ldots,4\lambda-3$. This implies that none of the colors $3\lambda-1,\ldots,4\lambda-3$ can be in the same set with color $\lambda+2$. This contradiction shows that we must have $\ell \geq 2k-1$.

By symmetry, situation **3** yields the same conclusion as situation **1**. Hence, we conclude that any λ-backbone ℓ-coloring of $(T(k^2,k),S)$ has $\ell \geq 2k-1$.

Case (f) $k \geq 2\lambda+1$.

Suppose there exists a λ-backbone ℓ-coloring of $(T(k^2,k),S)$ with $\ell = 2k - \lfloor \frac{k}{\lambda} \rfloor - 1$ colors. Suppose $s_2 = 0$. Then there are only monochromatic sets, i.e., $s_1 = k$. By (1) the total number of colors needed is at least $(k-1)\lambda+1$. However, the difference between this number and ℓ is

$$(k-1)\lambda+1-(2k-\lfloor \tfrac{k}{\lambda} \rfloor-1) = k(\lambda-2)+\lfloor \tfrac{k}{\lambda} \rfloor-\lambda+2 \geq 2\lambda^2-4\lambda+2 > 0.$$

Hence $s_2 > 0$. Write $k = a\lambda+r$ for some integers $a \geq 2$ and $0 \leq r \leq \lambda-1$. By Observation 8, $s_1 \geq 2k-\ell = \lfloor \frac{k}{\lambda} \rfloor+1$ holds. Together with (1) this implies that we need at least $(\lfloor \frac{k}{\lambda} \rfloor+1)(\lambda-1)+k$ colors. However, the difference between this number and ℓ is $(\lfloor \frac{k}{\lambda} \rfloor+1)(\lambda-1)+k-(2k-\lfloor \frac{k}{\lambda} \rfloor-1) = \lfloor \frac{k}{\lambda} \rfloor\lambda+\lambda-k = \lambda-r > 0$. This contradiction shows that we must have $\ell \geq 2k-\lfloor \frac{k}{\lambda} \rfloor$.

This finishes the proof of the lower bounds, and we have completed the proof of Theorem 4.

6 Proof of Theorem 5

We prove Theorem 5 in two parts. First we show that $\textsc{bbc}_\lambda(G,M)$ for any graph G with arbitrary matching backbone M is at most the value of $\mathcal{M}_\lambda(\chi(G))$ as given in Theorem 5. Next we present a class of graphs G that have a matching backbone M such that $\textsc{bbc}_\lambda(G,M)$ is at least the value of $\mathcal{M}_\lambda(\chi(G))$ that is given in Theorem 5. This way we obtain coinciding upper and lower bounds on $\mathcal{M}_\lambda(k)$ proving the theorem.

6.1 Proof of the Upper Bounds

Let $G=(V,E)$ be a graph with $\chi(G)=k$ and let $V_1,\ldots,V_k$ denote the corresponding independent sets in a k-coloring. Let $M=(V,E_M)$ be a matching backbone of G.

Case (a) $2 \leq k \leq \lambda$.

If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda + 1$. Let $k \geq 3$. Let uv be a matching edge with $u \in V_i$ and $v \in V_j$ for some $1 \leq i < j \leq k$. We color u by i and v by $\lambda + j - 1$. Then the difference between the colors of u and v is at least λ. So vertices in V_1 get color 1, vertices in V_i with $2 \leq i \leq k - 1$ get color i or $\lambda + i - 1$, and vertices in V_k get color $\lambda + k - 1$. Hence, we have obtained a λ-backbone $(\lambda + k - 1)$-coloring of (G, M).

Case (b) $\lambda + 1 \leq k \leq 2\lambda$.

Let uv be a matching edge with $u \in V_i$ and $v \in V_j$ for some $1 \leq i < j \leq k$. We color u by i and v by $k + j - 2$. This way we obtain a λ-backbone $(2k - 2)$-coloring of (G, M).

For the cases $(c) - (f)$ we need the following lemma. Observe that this lemma is exactly the same as Lemma 7 except that condition (vi) of Lemma 7 could be dropped. The first two paragraphs of the proof can be copied from the proof of Lemma 7.

Lemma 13 *Let $G = (V, E)$ be a graph with $\chi(G) = k$ and let $V_1, \ldots, V_k$ denote the corresponding independent sets in a k-coloring. Let $M = (V, E_M)$ be a matching backbone of G. For $i = 1, \ldots, p$, let $C_i = \{a_i\}$ be a set consisting of one color. For $j = 1, \ldots, q$, let $D_j = \{b_j, c_j\}$ be a set consisting of two colors. Let $d = \max\{a_1, \ldots, a_p, b_1, \ldots, b_q, c_1, \ldots, c_q\}$. Then (G, M) has a λ-backbone d-coloring if the following conditions are satisfied:*

(i) $p + q = k$.
(ii) $C_i \cap C_j = \emptyset$ *for all* $1 \leq i < j \leq p$.
(iii) $D_i \cap D_j = \emptyset$ *for all* $1 \leq i < j \leq q$.
(iv) $C_i \cap D_j = \emptyset$ *for all* $1 \leq i \leq p$ *and* $1 \leq j \leq q$.
(v) $|a_i - a_j| \geq \lambda$ *for all* $1 \leq i < j \leq p$.
(vi) $|b_j - c_j| \geq 2\lambda - 1$ *for all* $1 \leq j \leq q$.

Proof:

Let uv be a matching edge, where $u \in V_i$ with $1 \leq i \leq p$, and $v \in V_j$ with $p + 1 \leq j \leq k$. Then u has color a_i. Then colors $a_i - \lambda + 1, \ldots, a_i + \lambda - 1$ are forbidden colors for v. The distance between $a_i + \lambda - 1$ and $a_i - \lambda + 1$ is $2\lambda - 2$. Since the two colors in D_{j-p} have pairwise distance at least $2\lambda - 1$ due to (vi), at least one of them is feasible for v.

For all matching edges uv with $u \in V_i$ and $v \in V_j$ for some $p + 1 \leq i < j \leq k$ we choose u to be the root. We color u with b_i. The remaining vertices, whose colors have not yet been fixed, are all leaf vertices in sets V_j with $p+1 \leq j \leq k$. Again due to (vi) we can color them with a feasible color from D_{j-p}. This finishes the proof of the lemma. □

Below we show which color sets we use for each case. To check that these color sets satisfy the conditions of Lemma 13 is a simple exercise and left to the reader.

Case (c) $k = 2\lambda + 1$.

We use color sets:

- $C_i = \{i\lambda + 1\}$ for $i = 0, \ldots, 3$;
- $D_{1,j} = \{j, 2\lambda + j\}$ for $j = 2, \ldots, \lambda$;
- $D_{2,j} = \{\lambda + j, 3\lambda + j\}$ for $j = 2, \ldots, \lambda - 1$ and $\lambda \geq 3$.

Case (d) $k = t(\lambda + 1)$ with $t \geq 2$.

We use color sets:

- $C_i = \{i\lambda + 1\}$ for $i = 0, \ldots, 2t - 1$;
- $D_{i,j} = \{i\lambda + j, (t + i)\lambda + j\}$ for $i = 0, \ldots, t - 1$ and $j = 2, \ldots, \lambda$.

Case (e) $k = t(\lambda + 1) + c$ with $t \geq 2$, $1 \leq c < \frac{\lambda+3}{2}$.

We use color sets:

- $C_i = \{i\lambda + 1\}$ for $i = 0, \ldots, 2t$;
- $D_{0,j} = \{j, 2t\lambda + 2j - 2\}$ for $j = 2, \ldots, c$ and $c \geq 2$;
- $D_{0,j} = \{j, t\lambda + j\}$ for $j = c + 1, \ldots, \lambda$ and $c < \lambda$;
- $D_{i,j} = \{i\lambda + j, (t + i)\lambda + j\}$ for $i = 1, \ldots, t - 1$ and $j = 2, \ldots, \lambda$;
- $D_{t,j} = \{t\lambda + j, 2t\lambda + 2j - 1\}$ for $j = 2, \ldots, c$ and $c \geq 2$.

Case (f) $k = t(\lambda + 1) + c$ with $t \geq 2$, $\frac{\lambda+3}{2} \leq c \leq \lambda$.

We use color sets:

- $C_i = \{i\lambda + 1\}$ for $i = 0, \ldots, 2t$;
- $C_{2t+1} = \{2t\lambda + 2c - 2\}$;
- $D_{0,j} = \{j, 2t\lambda + 2j - 2\}$ for $j = 2, \ldots, c - 1$;
- $D_{0,j} = \{j, t\lambda + j\}$ for $j = c, \ldots, \lambda$;
- $D_{i,j} = \{i\lambda + j, (t + i)\lambda + j\}$ for $i = 1, \ldots, t - 1$ and $j = 2, \ldots, \lambda$;
- $D_{t,j} = \{t\lambda + j, 2t\lambda + 2j - 1\}$ for $j = 2, \ldots, c - 1$.

6.2 Proof of the Lower Bounds

Let $\lambda \geq 2$. For $k \geq 2$, we consider the Turán graph $T(k^2 - k, k - 1)$, i.e., a complete k-partite graph that consists of k independent sets $V_1, \ldots, V_k$ that are all of cardinality $k - 1$. For $1 \leq i \leq k$, let $\{v_{i,j} \mid 1 \leq j \leq k,\ j \neq i\}$ be the vertices of V_i, and let M be a matching backbone of $T(k^2-k,k-1)$ such that $E_M = \{v_{i,j}v_{j,i} \mid 1 \le i < j \le k\}$. See Figure 3 for an example of the graph $T(6,2)$ with matching backbone M. So $V_{T(6,2)} = \{v_{1,2}, v_{1,3}, v_{2,1}, v_{2,3}, v_{3,1}, v_{3,2}\}$ and $E_M = \{v_{1,2}v_{2,1}, v_{1,3}v_{3,1}, v_{2,3}v_{3,2}\}$. For our case analysis we first prove a

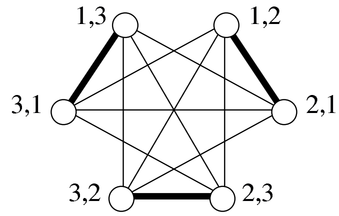

Fig. 3. The graph $T(6,2)$ with matching backbone M.

number of results for an *arbitrary* λ-backbone ℓ-coloring of $(T(k^2-k,k-1),M)$.

Consider some λ-backbone ℓ-coloring f of $(T(k^2-k,k-1),M)$. Since $T(k^2-k,k-1)$ is complete k-partite, any color that shows up in some set V_i can not show up in any V_j with $j \ne i$. As in the star backbone case, we denote by C_i the set of colors that are used on vertices in V_i. Recall that a set V_i is called monochromatic if $|C_i| = 1$, and polychromatic if $|C_i| \ge 2$. Again we denote by s_1 and s_2 the number of monochromatic and polychromatic sets, respectively. Let $m \le \ell$ be the number of different colors used on V. Then we immediately have $s_1 + s_2 = k$ and $s_1 + 2s_2 \le m$ implying the following observation.

Observation 14 *Let f be a λ-backbone ℓ-coloring of $(T(k^2-k,k-1),M)$ using m colors and with s_1 monochromatic sets. Then $s_1 \ge 2k-m$ holds.*

Since there exists a matching edge between any two independent sets V_i and V_j, we obtain the following observation.

Observation 15 *Let f be a λ-backbone ℓ-coloring of $T(k^2-k,k-1),M)$. If color x is assigned to a monochromatic set V_i, and color y is assigned to a monochromatic set V_j, then the distance between x and y is at least λ.*

We use Observation 15 to prove the following lemma.

Lemma 16 *Let f be a λ-backbone ℓ-coloring of $(T(k^2-k,k-1),M)$. Then*

$$\ell \ge \frac{2\lambda k}{\lambda+1} - \frac{\lambda-1}{\lambda+1}. \tag{2}$$

Proof: Let m be the number of different colors that f uses. Observation 15 yields $\ell \ge \lambda(s_1-1)+1$. Together with Observation 14 and $m \le \ell$, we obtain

$\ell \geq \lambda(s_1 - 1) + 1 \geq \lambda(2k - m - 1) + 1 \geq \lambda(2k - \ell - 1) + 1$, which is equivalent to inequality (2). $\square$

Also the following lemma is useful.

Lemma 17 *Let f be a λ-backbone ℓ-coloring of $(T(k^2-k, k-1), M)$. If $\ell \leq k+2\lambda-3$ then only colors from $A = \{1, \ldots, \ell-k-\lambda+2\}$ and $B = \{k+\lambda-1, \ldots, \ell\}$ can be assigned to monochromatic sets.*

Proof: Suppose a vertex v from a monochromatic set is assigned color c with c in $\{\ell - k - \lambda + 3, \ldots, k + \lambda - 2\}$. Recall that there exists a matching edge between any two independent sets V_i and V_j. Then there are at least $k - 1$ colors that have distance at least λ to c. If $\lambda + 1 \leq c \leq \ell - \lambda$, only colors in $\{1, \ldots, c - \lambda\}$ and in $\{c + \lambda, \ldots, \ell\}$ can be used. These sets together contain $c-\lambda+\ell-(c+\lambda)+1 = \ell-2\lambda+1 \leq k-2$ colors. Hence either $c \leq \lambda$ or $c \geq \ell-\lambda+1$ holds. If $c \leq \lambda$, then only colors in $\{c + \lambda, \ldots, \ell\}$ are at distance at least λ. The cardinality of this set is $\ell-(c+\lambda)+1 \leq \ell-(\ell-k-\lambda+3)-\lambda+1 = k-2$. If $c \geq \ell - \lambda + 1$, then only colors in $\{1, \ldots, c - \lambda\}$ are at distance at least λ. The cardinality of this set is $c - \lambda \leq k + \lambda - 2 - \lambda = k - 2$. $\square$

We are now ready to make our case analysis.

Case (a) $2 \leq k \leq \lambda$.

The case $k = 2$ is trivial. Let $k \geq 3$. Suppose $(T(k^2 - k, k - 1), M)$ has a λ-backbone ℓ-coloring with $\ell = k + \lambda - 2$ colors. By Lemma 17, we find that $s_1 = 0$. Colors $k - 1, \ldots, \lambda$ can not be used at all, since there is no color in $\{1, \ldots, \lambda + k - 2\}$ that has distance at least λ to one of them. So we can only use colors in $\{1, \ldots, k - 2\}$ and $\{\lambda + 1, \ldots, \lambda + k - 2\}$. Then the total number m of different colors is at most $2(k-2)$. Hence, by Observation 14 we find that $s_1 \geq 2k - m > 0$. This contradiction shows that we must have $\ell \geq k + \lambda - 1$.

Case (b) $\lambda + 1 \leq k \leq 2\lambda$.

Suppose $(T(k^2 - k, k - 1), M)$ has a λ-backbone ℓ-coloring with $\ell = 2k - 3$ colors. By Observation 14, we find that $s_1 \geq 2k - m \geq 2k - \ell \geq 3$ must hold. By Lemma 17, only monochromatic colors in $A = \{1, \ldots, k - \lambda - 1\}$ and $B = \{k + \lambda - 1, \ldots, 2k - 3\}$ can be used. Both sets have $k - \lambda - 1 \leq \lambda - 1$ elements. Then, by Observation 15, at most one color in A and at most one color in B can be used for monochromatic sets. Hence we find that $s_1 \leq 2$. This contradiction shows that $\ell \geq 2k - 2$.

Case (c) $k = 2\lambda + 1$.

Analogously to the proof of the previous case we can show that $\ell \geq 2k - 3$

must hold for any λ-backbone ℓ-coloring of $(T(k^2-k,k-1),M)$.

Case (d) $k = t(\lambda+1)$ with $t \geq 2$.

Inequality (2) yields $\ell \geq 2t\lambda - \frac{\lambda-1}{\lambda+1} = 2t\lambda - 1 + \frac{2}{\lambda+1}$ for any λ-backbone ℓ-coloring of $(T(k^2-k,k-1),M)$. Since ℓ is an integer, this implies that $\ell \geq 2t\lambda$.

Case (e) $k = t(\lambda+1)+c$ with $t \geq 2$ and $1 \leq c < \frac{\lambda+3}{2}$.

Inequality (2) yields $\ell \geq 2t\lambda + \frac{2\lambda c}{\lambda+1} - \frac{\lambda-1}{\lambda+1} = 2t\lambda + 2c - 1 + \frac{2-2c}{\lambda+1} > 2t\lambda + 2c - 1 + \frac{2-\lambda-3}{\lambda+1} = 2t\lambda + 2c - 2$ for any λ-backbone ℓ-coloring of $(T(k^2-k,k-1),M)$. Since ℓ is an integer, we have found that $\ell \geq 2t\lambda + 2c - 1$.

Case (f) $k = t(\lambda+1)+c$ with $t \geq 2$ and $\frac{\lambda+3}{2} \leq c \leq \lambda$.

Inequality (2) yields $\ell \geq 2t\lambda + \frac{2\lambda c}{\lambda+1} - \frac{\lambda-1}{\lambda+1} = 2t\lambda + 2c - 1 + \frac{2-2c}{\lambda+1} \geq 2t\lambda + 2c - 1 + \frac{2-2\lambda}{\lambda+1} = 2t\lambda + 2c - 3 + \frac{2}{\lambda+1}$ for any λ-backbone ℓ-coloring of $(T(k^2-k,k-1),M)$. Since ℓ is an integer, this implies that $\ell \geq 2t\lambda + 2c - 2$.

This finishes the proof of the lower bounds, and we have completed the proof of Theorem 5.

7 Matching Backbones for Planar Graphs

7.1 *Implications of the Four Color Theorem*

In the last section of this paper we focus on some open problems for matching backbone colorings on planar graphs. For simplicity we assume $\lambda = 2$. The Four Color Theorem together with Theorem 5 implies that $\mathrm{BBC}_2(G,M) \leq 6$ holds for any planar graph G with a matching backbone M. It seems likely that this bound 6 is not best possible. However, the planar graph G_1 with indicated matching backbone M consisting of edges ab', bc', cd', da' as in Figure 4 shows that one can not improve this bound to 4.

We prove here that we can not find a backbone coloring of (G_1, M) with color set $\{1,2,3,4\}$. First of all observe that G_1 can be obtained from a plane embedding of the K_4 induced by the vertices a,b,c,d, by putting a new vertex in each face and adding edges from this new vertex to the three vertices on the boundary of the face, and assigning the label x' to the new vertex in the triangular face bounded by the cycle $uvwu$, where $\{u,v,w,x\} = \{a,b,c,d\}$. Suppose we only use colors $1,2,3,4$. Then it is clear from this construction that a, b, c and d get different colors, and that the colors of a vertex and its primed counterpart are the same. Without loss of generality assume that a and a' get color 2. Then both b' and d must get color 4, a contradiction. It is routine to check that $\text{BBC}_2(G_1, M) = 5$.

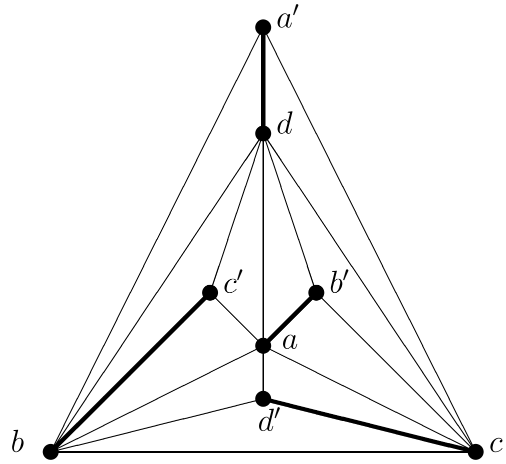

Fig. 4. A graph G_1 with a matching backbone M such that $\text{BBC}_2(G_1, M) = 5$.

The following problems are still open.

Problem 18 *Is* $\text{BBC}_2(G, M) \leq 5$ *for any planar graph* G *with a matching backbone* M*?*

Problem 19 *Is there a proof of* $\text{BBC}_2(G, M) \leq 6$ *that does not require the Four Color Theorem?*

7.2 Cyclic Backbone Colorings

In the last part of this section we introduce a special kind of 2-backbone coloring with a cyclic property as defined below. Our motivation for doing this is to get a better understanding of the structure of the original (acyclic) 2-matching backbone colorings of planar graphs. We prove a sharp result with respect to the upper bound on the number of colors needed to color planar graphs in the way explained below.

Let $H = (V, E_H)$ be a backbone of the graph $G = (V, E_G)$. A 2-backbone coloring $f : V \to \{1, \ldots, \ell\}$ of (G, H) is called an *ℓ-cyclic 2-backbone coloring* of (G, H), if no edge of E_H joins two vertices with color 1 and color ℓ in V. In a 2-backbone coloring we say that two colors x and y are *adjacent* if $|x - y| \leq 1$. In an ℓ-cyclic 2-backbone coloring we also say that color 1 and color ℓ are adjacent.

The study of cyclic colorings in the context of frequency assignment is well-motivated in [17].

For the proof of Theorem 21 below we first construct the following useful gadget.

Lemma 20 *Let H be the graph with a matching M consisting of edges ab, cd, eu, fg and hi as in Figure 5(a). Let G be a graph with a matching backbone M'. If $H \subseteq G$ and $M \subseteq M'$, then vertex u and vertex v can not be colored with two adjacent colors in a 5-cyclic 2-backbone coloring of (G, M').*

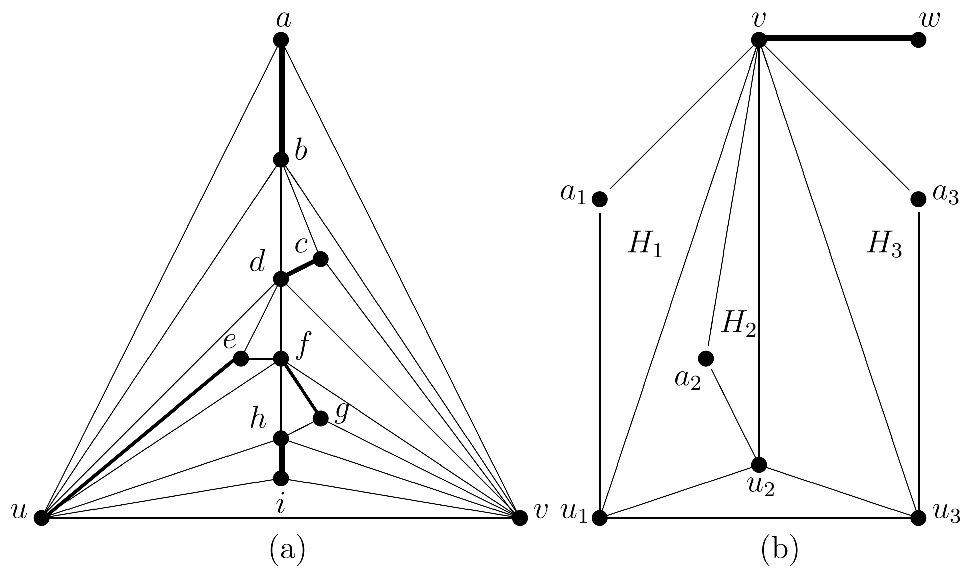

Fig. 5. (a) Graph H with matching M (b) Planar graph G_2

Proof: Suppose vertex u and vertex v can be colored with two adjacent colors in a 5-cyclic 2-backbone coloring of (G, M'). Since we use a 5-cyclic 2-backbone coloring, we can without loss of generality assume that vertex u is colored with color 1 and vertex v is colored with color 2. This leaves us with three possible colors for vertex d: color 3, color 4 or color 5.

- If vertex d is colored with color 3, then vertex e must get color 4. Continuing this way, vertex f gets color 5, vertex g gets color 3 and vertex h gets color 4. Since there is no feasible color for vertex i, this implies a contradiction.
- If vertex d is colored with color 4, then vertex e gets color 3, vertex f gets color 5, vertex g gets color 3 and vertex h gets color 4. Again, we find a contradiction, since there is no feasible color for vertex i.
- If vertex d is colored with color 5, then vertex c must get color 3 and the only feasible color for vertex b is color 4. We get a contradiction, since there is no feasible color for vertex a.

This completes the proof of Lemma 20. □

Theorem 21

(a) Let G be a planar graph with a matching backbone M. Then (G, M) has

a 6-cyclic 2-backbone coloring.

(b) There exist planar graphs that do not have a 5-cyclic 2-backbone coloring along a matching.

Proof: (a) By the Four Color Theorem, we know that the chromatic number of a planar graph G is at most 4. For a vertex v in G, we denote by $n(v)$ the only neighbor of v in M. We can construct a 6-cyclic 2-backbone coloring b of (G, M) by replacing the colors of a 4-coloring c of G as follows:

- if $c(v) = 1$: $b(v) := 1$;
- if $c(v) = 2$: $b(v) := 3$;
- if $c(v) = 3$: $b(v) := 5$;
- if $c(v) = 4$ and $c(n(v)) = 1$: $b(v) := 4$;
- if $c(v) = 4$ and $c(n(v)) = 2$: $b(v) := 6$;
- if $c(v) = 4$ and $c(n(v)) = 3$: $b(v) := 2$.

(b) We construct a planar graph G_2 as follows. First we make three copies $(H_1, M_1), (H_2, M_2), (H_3, M_3)$ of the pair (H, M) from Figure 5(a), and glue them together at vertex v. Then we add one new vertex w and four new edges: the edge vw and the edges u_1u_2, u_2u_3, u_3u_1 (see Figure 5(b)). The vertex w and the edge vw are only added to guarantee that G_2 has a perfect matching. Let M' be a matching backbone of G_2 that contains all matchings M_i $(i = 1, 2, 3)$ and the edge vw.

Suppose there exists a 5-cyclic 2-backbone coloring of (G_2, M'). Without loss of generality we may assume that the vertex v is colored with color 1. Then, by Lemma 20, the vertices u_1, u_2 and u_3 can not be colored with colors $1, 2$ and 5. On the other hand, u_1, u_2 and u_3 must all get different colors, since they induce a K_3. This contradiction completes the proof of Theorem 21. $\square$

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