2-factors with bounded number of components in claw-free graphs

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Abstract

In this paper, we show that every 3-connected claw-free graph G has a 2-factor having at most $\max\left\{\frac{2}{5}(\alpha+1), 1\right\}$ cycles, where α is the independence number of G. As a corollary of this result, we also prove that every 3-connected claw-free graph G has a 2-factor with at most $\left(\frac{4|G|}{5(\delta+2)}+\frac{2}{5}\right)$ cycles, where δ is the minimum degree of G. This is an extension of a known result on the number of cycles of a 2-factor in 3-connected claw-free graphs.

Keywords: Claw-free graphs; 2-factors; Independence number

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1 Introduction

A well-known conjecture by Matthews and Sumner [17] states that every 4-connected claw-free graph is Hamiltonian. Recall that a graph is *claw-free* if it has no claw $K_{1,3}$ as an induced subgraph. Thomassen [20] also posed the following conjecture: every 4-connected line graph is Hamiltonian. Note that Ryjáček [18] showed that these two conjectures are equivalent, using a closure technique. These two conjectures have attracted much attention during the last more than 25 years, but they are still open.

To attack these conjectures, some researchers have considered the Hamiltonicity of claw-free graphs with high connectivity conditions. In fact, Zhan [22], and independently Jackson [12] proved that Thomassen's conjecture is true for 7-connected line graphs. Recently, Kaiser and Vrána [15] improved this result by showing that every 5-connected claw-free graph with minimum degree at least six is Hamiltonian. Like these, several researchers have attacked these conjectures in claw-free graphs with high connectivity. See for example [11, 23].

On the other hand, it is also natural to ask what happens when we consider claw-free graphs with low connectivity. Although it is known that there exist infinitely many 3-connected claw-free graphs (also line graphs) having no Hamiltonian cycles, we would like to find some "good" structures which have some properties close to Hamiltonian cycles in such graphs. The main target of this paper is a 2-*factor* with a bounded number of components. (See the survey [7] for other "good" structures.)

Recall that a 2-factor of a graph is a spanning subgraph in which all vertices have degree two. A Hamiltonian cycle of a graph is actually a 2-factor with exactly one component. In this sense, the fewer components a 2-factor has, the closer to a Hamiltonian cycle it is. Choudum and Paulraj [4], and independently Egawa and Ota [5] proved that if the minimum degree of a claw-free graph G is at least four, then G has a 2-factor (without considering the number of components). Yoshimoto [21] showed that if G is a 2-connected claw-free graph with minimum degree at least three (specifically, if G is 3-connected), then G has a 2-factor.

Now we consider a 2-factor with bounded number of cycles in claw-free graphs. Faudree, Favaron, Flandrin, Li and Liu [6] showed that a claw-free graph with minimum degree $\delta \geq 4$ has a 2-factor with at most $\frac{6|G|}{\delta+2} - 1$ cycles. Gould and Jacobson [10] improved this result for a claw-free graph with large minimum degree; a claw-free graph with minimum degree $\delta \geq \left(4|G|\right)^{\frac{2}{3}}$ has a 2-factor with at most $\left\lceil \frac{|G|}{\delta} \right\rceil$ cycles. Recently, Broersma, Paulusma and Yoshimoto showed the following result.

Theorem 1 (Broersma, Paulusma and Yoshimoto [1]) *Every claw-free graph G with minimum degree $\delta \geq 4$ has a 2-factor with at most*

$$\max\left\{\frac{|G|-3}{\delta-1}, 1\right\} \textit{ cycles.}$$

Also Yoshimoto [21] showed that the coefficient $\frac{1}{\delta-1}$ of $|G|$ is almost best possible.

Now we consider 2-connected or 3-connected claw-free graphs. Jackson and Yoshimoto [13] showed that every 2-connected claw-free graph G with minimum degree at least four has a 2-factor with at most $\frac{|G|+1}{4}$ cycles, and moreover, with at most $\frac{2|G|}{15}$ cycles if G is 3-connected. Čada, Chiba and Yoshimoto [2] proved that every 2-connected claw-free graph G with minimum degree $\delta \geq 4$ has a 2-factor in which every cycle has the length at least δ. This result implies the existence of a 2-factor with at most $\frac{|G|}{\delta}$ cycles in a 2-connected claw-free graph G.

On the other hand, Kužel, Ozeki and Yoshimoto [16] focused on a relationship between a 2-factor and maximum independent sets in a graph, and showed the following:

Theorem 2 (Kužel, Ozeki and Yoshimoto [16]) *For every maximum independent set S in a 2-connected claw-free graph G with minimum degree at least three, G has a 2-factor in which each cycle contains at least one vertex in S, and moreover, at least two vertices in S if G is 3-connected.*

As a direct corollary of Theorem 2, we obtain that every 3-connected claw-free graph G has a 2-factor with at most $\alpha/2$ cycles, where α is the independence number of G. Note that for every claw-free graph G, we have that $\alpha \leq \frac{2|G|}{\delta+2}$, where α is the independence number and δ is the minimum degree of G, respectively. (See for example, Fact 8 in [8].) Therefore, the result of Kužel et al. implies the following corollary.

Theorem 3 (Kužel, Ozeki and Yoshimoto [16]) *Every 3-connected claw-free graph G with minimum degree δ has a 2-factor with at most*

$$\max\left\{\frac{|G|}{\delta+2}, 1\right\} \textit{ cycles.}$$

In this paper, we show the following result, which means that if we do not specify a maximum independent set, for 3-connected claw-free graphs, we can find a 2-factor with fewer cycles than the one obtained by Theorem 2.

Theorem 4 *Every 3-connected claw-free graph with independence number α has a 2-factor with at most*

$$\max\left\{\frac{2}{5}(\alpha+1), 1\right\} \textit{ cycles.}$$

We do not know whether the coefficient $\frac{2}{5}$ of α in Theorem 4 is best possible or not. However, in Section 3, we show two examples to discuss sharpness of the result. By the same argument as above, Theorem 4 implies the following corollary.

Corollary 5 *Every 3-connected claw-free graph G with minimum degree δ has a 2-factor with at most*

$$\left(\frac{4|G|}{5(\delta+2)} + \frac{2}{5}\right) \textit{ cycles.}$$

In Corollary 5, we decrease the coefficient of $|G|$ in Theorem 3. This is the first result that guarantees, in a 3-connected claw-free graph, the existence of a 2-factor having number of cycles with coefficient of $|G|/\delta$ less than 1.

In the next section, we give two statements (Theorems 6 and 7), that are equivalent to Theorem 4. After discussing sharpness of Theorem 4 in Section 3, we show some lemmas in Sections 4 and 5. In Section 6, we prove Theorem 7.

2 Preliminaries

For a graph G and for $S \subset V(G)$, $G[S]$ denotes the subgraph of G induced by the set S. We denote by $N_G(x)$ the neighborhood of a vertex x in a graph G.

For the proof of Theorem 4, we use the *closure* of a claw-free graph which was introduced by Ryjáček [18] as follows. For each vertex x of a claw-free graph G, $N_G(x)$ induces a subgraph $G[N_G(x)]$ with at most two components, and if $G[N_G(x)]$ has two components, both of them must be cliques. In the case where $G[N_G(x)]$ is connected and non-complete, we add edges joining all pairs of nonadjacent vertices in $N_G(x)$. The closure $\mathrm{cl}(G)$ of G is the (unique) graph obtained by recursively repeating this operation, as long as this is possible. Ryjáček, Saito and Schelp [19] proved that a claw-free graph G has a 2-factor with at most c components if and only if $\mathrm{cl}(G)$ has a 2-factor with at most c components. This implies that the following statement is equivalent to Theorem 4.

Theorem 6 *For every 3-connected claw-free graph G with independence number α, $\mathrm{cl}(G)$ has a 2-factor with at most $\max\left\{\frac{2}{5}(\alpha+1), 1\right\}$ cycles.*

Ryjáček [18] proved that for every claw-free graph G, there exists a triangle-free simple (i.e. with no parallel edges) graph H such that $L(H) = \mathrm{cl}(G)$. An *even graph* is a graph in which all vertices have even degree, and a *circuit* is a connected even graph. Let H be a graph. A set $\mathcal{D}$ of circuits and stars with at least three edges in H is called a *D-system of H*, if every edge of H is contained in a member of $\mathcal{D}$ or incident with a vertex in a circuit in $\mathcal{D}$. For a D-system $\mathcal{D}$ of H, let $|\mathcal{D}|$ be the number of circuits and stars in $\mathcal{D}$. Also a D-system $\mathcal{D}$ is called a *strong D-system* of H if $\mathcal{D}$ contains no star and every vertex of degree at least three in H is contained in some circuit in $\mathcal{D}$. Gould and Hynds [9] proved that the line graph $L(H)$ of a graph H has a 2-factor with c components if and only if there is a D-system $\mathcal{D}$ with $|\mathcal{D}| = c$ in H. An edge set E_0 of H is called an *essential edge-cut* if $H - E_0$ contains at least two components having an edge. A graph H is *essentially k-edge connected* if there exists no essential edge-cut with at most $k-1$ edges. Clearly $L(H)$ is k-connected if and only if H is essentially k-edge-connected. Let $\alpha'(H)$ be the number of edges of a maximum matching of H. Note that when $L(H) = G$, then $\alpha(G) = \alpha'(H)$. Then the following is also equivalent to Theorems 4 and 6.

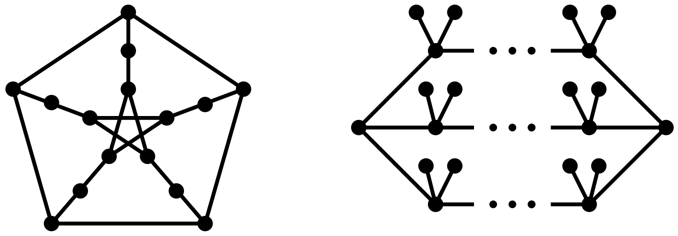


Figure 1: The graphs H and H_i.

Theorem 7 *Let G be an essentially 3-edge connected graph. Then G has a D-system $\mathcal{D}$ with $|\mathcal{D}| \le \max\{\frac{2}{5}(\alpha'(G)+1), 1\}$.*

Note that the statement of Theorem 7 remains equivalent to Theorems 4 and 6 even if restricted to triangle-free simple graphs; however, its present form will be more convenient for our proof.

An edge $e = uv$ of a graph G is said to be *pendant* if the degree of u or v is one in G. For an integer $l \ge 2$, the cycle of length l is denoted by C_l. For an integer $g \ge 2$, $K_{2,2g}$ denotes the complete bipartite graph such that one partite set consists of two vertices and the other consists of $2g$ vertices. For a subgraph H of a graph G and for a set E_1 of edges in $G - E(H)$, we define $H + E_1$ as the graph induced by edges $E(H) \cup E_1$.

3 Sharpness of Theorem 4

In this section, we discuss how far Theorem 4 is from being sharp. First, we consider the upper bound on the number of components of a 2-factor. Although we do not know whether the coefficient $\frac{2}{5}$ of α (or α' in Theorem 7) is best possible or not, the following graph shows that it cannot be less than $\frac{2}{7}$. Let H_0 be the Petersen graph, let M_0 be a maximum matching of H_0 and let H be the graph obtained from H_0 by subdividing all edge in M_0 once (see the left side of Figure 1). Suppose that H has a D-system $\mathcal{D}$ with $|\mathcal{D}| = 1$, say, $\{D\} = \mathcal{D}$. Since D has to pass through all vertices of H_0 (because otherwise D cannot dominate a subdivided edge in H incident with a vertex not passed by D), D corresponds to a Hamiltonian cycle of the Petersen graph H_0, a contradiction. Thus, every D-system of H has at least two members. Since $\alpha(L(H)) = \alpha'(H) = 7$, the coefficient of α' in Theorem 7 has to be at least $\frac{2}{7}$. Considering the graph $L(H)$, we can also show that the coefficient of α in Theorem 4 has to be at least $\frac{2}{7}$.

Next we consider the 3-connectedness (or essential 3-edge connectedness) assumption in Theorem 4 (Theorem 7, respectively). Unfortunately, we do not know whether the 2-connectedness (or essential 2-edge connectedness) might be sufficient for the result, but we give examples which show that we cannot decrease the coefficient to less than $\frac{1}{3}$ if we only assume 2-connectedness (or essential 2-edge connectedness, respectively). Let H_i' be obtained from the graph with two vertices and three internally disjoint paths each of which contains i internal vertices. We add two pendant edges to each internal vertex of H_i' and obtain the graph H_i (see the right side of Figure 1). Note that $\alpha'(H_i) = 3i$. Since every circuit of H_i has to miss at least one of the three paths of H_i, each D-system $\mathcal{D}_i$ of H_i has at least i stars, so it has at least $i+1$ members. This implies that

$$\lim_{i\to\infty} \frac{|\mathcal{D}_i|}{\alpha'(H_i)} \geq \lim_{i\to\infty} \frac{i+1}{3i} = \frac{1}{3}.$$

4 Contractions and reconstructions

4.1 Contractions used in this paper

In this paper, in order to make a given graph smaller, we consider the following six types of contractions. Also, we use the reverse operation of those, called *reconstructions*. Let G be a graph. (Possibly, G may have multiple edges.)

A suppressing:
Let x be a vertex of degree two and let e be an edge incident with x. A *suppressing (of x)* is a contraction of the edge e to one vertex and removing the created loop.

A C_2-contraction, a C_3-contraction and a primary $K_{2,2g}$-contraction:
Let C be a cycle of length two in G. A *C_2-contraction (at C)* consists of the following three operations, executed in order:

- contracting C to one vertex,
- removing all created loops,
- adding a new pendant edge to the contracted vertex.

When C is a cycle of length three in G or a subgraph isomorphic to $K_{2,2g}$ with an integer $g \geq 2$, we define similarly a *C_3-contraction (at C)* or *a primary $K_{2,2g}$-contraction*, respectively.

A secondary $K_{2,2g}$-contraction:
Let C be a subgraph of G isomorphic to $K_{2,2g}$ for some $g \geq 2$. Let x_1, x_2 be the two vertices of the smaller partite set of C, and let Y be the other partite set. For $Y_1 \subset Y$ with $Y_1 \neq \emptyset$ and $Y_1 \neq Y$, a *secondary $K_{2,2g}$-contraction at C with respect to Y_1* consists of the following five operations, executed in order:

- identifying all vertices in Y_1 to one vertex, say y_1,

- identifying all vertices in $Y \setminus Y_1$ to one vertex, say y_2,
- replacing multiple edges between x_i and y_j with a single edge for $i, j = 1, 2$,
- removing all loops,
- removing all pendant edges incident with x_1 or x_2.

Note that although the original graph is simple, the graph obtained by a secondary $K_{2,2g}$-contraction might have multiple edges between y_1 (or y_2) and some vertex z with $z \neq x_1, x_2$.

A C_5-contraction:
Let C be a cycle of length five. A C_5*-contraction (at C)* consists of the following two operations, executed in order:

- contracting C to one vertex,
- removing all created loops.

4.2 3-edge connectedness

In this subsection, we consider 3-edge connectedness of a graph obtained by the contractions defined in Section 4.1. By the definition, the following is an easy fact.

Fact 8 *Let G be an essentially 3-edge connected graph, and let G' be a graph obtained from G by a suppressing, a C_2-contraction, a C_3-contraction, a primary $K_{2,2g}$-contraction, or by a C_5-contraction. Then G' is essentially 3-edge connected.*

On the other hand, for a secondary $K_{2,2g}$-contraction, we show the following useful lemma. For a subgraph C of G isomorphic to $K_{2,2g}$ with $g \geq 2$, C is called *good* if all but at most two vertices in Y have degree two in G, where Y is the larger partite set of C, and C is *bad* if C is not good.

Lemma 9 *Let G be an essentially 3-edge connected graph and let C be a subgraph of G isomorphic to $K_{2,2g}$ with $g \geq 2$. Let x_1, x_2 be the two vertices of the smaller partite set of C and let Y be the other partite set. Suppose that C is bad. Then one of the following holds:*

(i) *for some $i = 1, 2$, all edges of C incident with x_i form an essential edge-cut of G,*

(ii) *there exists a subset $Y_1 \subset Y$ with $Y_1 \neq \emptyset$ and $Y_1 \neq Y$ such that the graph obtained by a secondary $K_{2,2g}$-contraction at C with respect to Y_1 is also essentially 3-edge connected.*

Proof of Lemma 9. Let C, x_1, x_2, Y be as in the assumptions of Lemma 9 and suppose that C is bad. We first claim that there exists a path P in $G - E(C)$ connecting some two vertices in Y.

Since C is not good, there exist three vertices $y^1, y^2, y^3 \in Y$ such that $d_G(y^i) \geq 3$ for $i = 1, 2, 3$. Since G is essentially 3-edge connected, there exists a path P_1 in $G - \{x_1y^1, x_2y^1\}$ from y^1 to $V(C) \setminus \{y^1\}$. Note that P_1 is a path in $G - E(C)$. If P_1 reaches y for some $y \in Y \setminus \{y^1\}$, then P_1 is the desired path. Thus we may assume that P_1 reaches x_1 or x_2. Similarly, we can take two paths P_2 and P_3 in $G - E(C)$ from y^2 and y^3, respectively, to x_1 or x_2. Since at least two of the paths P_1, P_2, P_3 have the same end vertex in $\{x_1, x_2\}$, connecting them, we can find a path P in $G - E(C)$ between two vertices in Y.

Let y^1 and y^2 be the end vertices of the path P in $G - E(C)$. Let $Y_1 := \{y^1\}$ and consider the graph G' obtained from G by a secondary $K_{2,2g}$-contraction at C with respect to Y_1. Let y_1, y_2 be the vertices of G' obtained from Y_1 and $Y \setminus Y_1$, respectively.

Suppose that (ii) does not hold, that is, there exists an essential edge-cut $E_1 \subset E(G')$ of G' with $|E_1| \leq 2$. If $\left|E_1 \cap \{x_1y_1, x_2y_1, x_1y_2, x_2y_2\}\right| = 0$, then E_1 is also an essential edge-cut of G, a contradiction. If $\left|E_1 \cap \{x_1y_1, x_2y_1, x_1y_2, x_2y_2\}\right| = 1$, say $x_1y_1 \in E_1$, then x_1 and y_1 are contained in the same component of $G' - E_1$ because $G' - E_1$ has the path $x_1y_2x_2y_1$, a contradiction again. Therefore $|E_1| = 2$ and $E_1 \subset \{x_1y_1, x_2y_1, x_1y_2, x_2y_2\}$.

Since P connects y^1 and y^2 in $G - E(C)$, it also connects y_1 and y_2 in $G' - E_1$. This implies that y_1 and y_2 are contained in the same component of $G' - E_1$, and hence $E_1 = \{x_iy_1, x_iy_2\}$ for some $i = 1, 2$. Since E_1 is an essential edge-cut of G', there exists a component H_1 of $G' - E_1$ with $x_i \in V(H_1)$, $y_1, y_2 \notin V(H_1)$ and $|H_1| \geq 2$. Since we removed all pendant edges incident with x_i, the edges of C incident with x_i correspond to edges in E_1, and hence they form an essential edge-cut of G. Thus, (i) holds. $\square$

4.3 Reconstructions of a C_2- or C_3-contraction

In this subsection, we deal with reconstructions of a C_2- or C_3-contraction. The first statement can be found in several papers, for example [3], and the second one can be easily shown. Hence we omit the proof.

Lemma 10 *Let G be a graph and let C be a cycle of length two or three in G. Let G' be the graph obtained from G by a C_2- or C_3-contraction at C. Then:*

(i) *If G' has a D-system $\mathcal{D}'$, then G also has a D-system $\mathcal{D}$ with $|\mathcal{D}| \leq |\mathcal{D}'|$. In particular, if $\mathcal{D}'$ is strong, then $\mathcal{D}$ is also strong.*

(ii) *For any matching M' in G', there exists a matching M in G with $|M| \geq |M'|$. In particular, $\alpha'(G) \geq \alpha'(G')$.*

4.4 Reconstructions of $K_{2,2g}$-contractions

In this subsection, we deal with reconstructions of a primary or secondary $K_{2,2g}$-contraction. Indeed, we show the following lemma.

Lemma 11 *Let G be a triangle-free graph and let C be a bad $K_{2,2g}$ for some $g \geq 2$ in G. Then all of the following hold:*

(i) *Suppose that C satisfies condition (i) in Lemma 9. Let G' be the graph obtained by a primary $K_{2,2g}$-contraction at C. If G' has a D-system $\mathcal{D}'$, then G also has a D-system $\mathcal{D}$ with $|\mathcal{D}| \leq |\mathcal{D}'|$.*

(ii) *Let G' be the graph obtained by a secondary $K_{2,2g}$-contraction at C. If G' has a D-system $\mathcal{D}'$, then G also has a D-system $\mathcal{D}$ with $|\mathcal{D}| \leq |\mathcal{D}'|$.*

(iii) *Let G' be the graph obtained by a primary or secondary $K_{2,2g}$-contraction at C. Then $\alpha'(G) \geq \alpha'(G')$.*

Proof of Lemma 11.

Since (iii) is obvious, we show only (i) and (ii) at the same time. Let G' be the graph obtained by a primary or secondary $K_{2,2g}$-contraction at C as in the statement (i) or (ii). Suppose that G' has a D-system $\mathcal{D}'$.

Let x_1, x_2 be the vertices of the smaller partite set of C and let Y be the other partite set. Since G is triangle-free, Y is an independent set. Let H' be the subgraph of G' such that $V(H')$ is the set of vertices which are contained in some circuit in $\mathcal{D}'$ or are centers of some star in $\mathcal{D}'$, and $E(H')$ is the set of edges in some circuit of $\mathcal{D}'$. Note that H' is an even subgraph of G', and the number of components of H', denoted by $\omega(H')$, is at most $|\mathcal{D}'|$. Notice also that, when (i) occurs, then every edge in H' is also an edge in G, and when (ii) occurs, then every edge in H' except for x_1y_1, x_1y_2, x_2y_1 and x_2y_2 is also an edge in G, where y_1 and y_2 are defined in the operations in a secondary $K_{2,2g}$-contraction. (Recall that we did not replace multiple edges of G' with a single edge, except for the third operation of a secondary $K_{2,2g}$-reduction.) Hence we can regard $E(H')$ for (i) and $E(H') \setminus \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ for (ii) as a subset of edges in G. Let further $\widetilde{H}$ be the graph with $V(\widetilde{H}) = \big(V(H') - \{v_C\}\big) \cup Y \cup \{x_1, x_2\}$ and $E(\widetilde{H}) = E(H')$ for (i), where v_C is the vertex obtained by contracting C; or with $V(\widetilde{H}) = \big(V(H') - \{y_1, y_2\}\big) \cup Y$ and $E(\widetilde{H}) = E(H') \setminus \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ for (ii). Since all edges in $\widetilde{H}$ also appear in G in either case, we can regard $\widetilde{H}$ as a subgraph of G. Note that $V(\widetilde{H})$ dominates all edges in G, and all vertices except for (some of the) vertices in $\{x_1, x_2\} \cup Y$ have even degrees in $\widetilde{H}$ (possibly the degree might be zero).

In the rest of the proof, we will construct an even subgraph H of G by adding some edges of C into $\widetilde{H}$ in such a way that the number of components of H (denoted $\omega(H)$) does not exceed $\omega(H')$, and x_1 and x_2 are contained in the same component of H. Then, since all edges in G, which do not appear in G', are incident with x_1 or x_2, $V(H)$ dominates all edges in G and, since each isolated vertex v of H is also isolated in H', v is a center of some star D_v in $\mathcal{D}'$. Therefore the system

$$\mathcal{D} = \{D : D \text{ is a component of } H\} \cup \{D_v : v \text{ is isolated in } H\}$$

is a D-system of G such that $|\mathcal{D}| = \omega(H) \leq \omega(H') \leq |\mathcal{D}'|$ and we are done.

Let Y_{odd} (or Y_{even}) be the set of vertices y in Y having odd (or even, respectively) degree in $\widetilde{H}$. Note that $|Y_{odd}| + |Y_{even}| = 2g$, that is, $|Y_{odd}| + |Y_{even}|$ is even. Notice also that $d_{\widetilde{H}}(x_1) + d_{\widetilde{H}}(x_2) + |Y_{odd}|$ is even since H' is an even subgraph of G'. We consider the following three cases, depending on the parities of the degrees of x_1 and x_2 in $\widetilde{H}$.

Case 1. Both x_1 and x_2 have even degree in $\widetilde{H}$.

In this case, $|Y_{odd}|$ is even, and hence $|Y_{even}|$ is also even.

Suppose first that $Y_{even} \neq \emptyset$. Then let

$$H := \widetilde{H} + \{x_1 y : y \in Y\} + \{x_2 y : y \in Y_{even}\}.$$

By the choice, every vertex of G has even degree in H. Since $Y_{even} \neq \emptyset$, all vertices in $\{x_1, x_2\} \cup Y$ are contained in the same component in H, and hence we have $\omega(H) \leq \omega(H')$. So, H has the desired properties.

Thus, we may assume that $Y_{even} = \emptyset$, that is, $Y = Y_{odd}$. Since H' is an even subgraph of G', there exists a path P in $\widetilde{H}$ such that either (a) P connects a vertex in Y_1, say y^1, and x_i for some $i = 1, 2$, say $i = 2$, or (b) P connects two vertices in Y_1, say y^1 and y^2. Then we divide Y_{odd} into two sets Y^1_{odd} and Y^2_{odd} so that both Y^1_{odd} and Y^2_{odd} has even number of vertices, $y^1 \in Y^1_{odd}$, and $y^2 \in Y^2_{odd}$ if (b) occurs. Then let

$$H := \widetilde{H} + \{x_1 y : y \in Y^1_{odd}\} + \{x_2 y : y \in Y^2_{odd}\}.$$

Also every vertex of G has an even degree in H. Because of the path P in $\widetilde{H}$, all vertices in $\{x_1, x_2\} \cup Y$ are contained in the same circuit in H. Hence H is a desired even subgraph. $\square$

Case 2. One of x_1 and x_2 has an even degree and the other has an odd degree in $\widetilde{H}$.

By symmetry, we may assume that x_1 has an even degree and x_2 has an odd degree in $\widetilde{H}$. Note that $|Y_{odd}|$ is odd, and hence $|Y_{even}|$ is also odd. Let

$$H := \widetilde{H} + \{x_1 y : y \in Y\} + \{x_2 y : y \in Y_{even}\}.$$

Then every vertex of G has an even degree in H and $\omega(H) \leq \omega(H')$, and hence H is a desired even subgraph. $\square$

Case 3. Both x_1 and x_2 have an odd degree in $\widetilde{H}$.

For (i), we supposed that C satisfies condition (i) in Lemma 9, that is, for some $i = 1, 2$, say $i = 1$, all edges of C incident with x_1 form an essential edge-cut of G. Then by the construction, one component, say R, of $\widetilde{H}$ contains x_1 but does not contain any vertices in $Y \cup \{x_2\}$. Then x_1 is the unique vertex of odd degree in R, a contradiction. Thus, in this case, we need to consider only (ii), and we performed a secondary $K_{2,2g}$-contraction at C.

Case 3.1. $Y_{even} = \emptyset$.

Since y_1 has an even degree in H', as in Case 1, there exists a path in $\widetilde{H}$ connecting two vertices in Y_1, say, y^1 and y^2. Then we divide Y_{odd} into two sets Y_{odd}^1 and Y_{odd}^2 such that both Y_{odd}^1 and Y_{odd}^2 have odd number of vertices and $y^i \in Y_{odd}^i$ for $i = 1, 2$. Then let

$$H := \widetilde{H} + \{x_1 y : y \in Y_{odd}^1\} + \{x_2 y : y \in Y_{odd}^2\}.$$

Then H is a desired even subgraph. $\square$

Case 3.2. $Y_{even} \neq \emptyset$ and $Y_{odd} \neq \emptyset$.

Since $|Y_{odd}|$ is even, we can divide Y_{odd} into two sets Y_{odd}^1 and Y_{odd}^2 such that both Y_{odd}^1 and Y_{odd}^2 have odd number of vertices. Let

$$H := \widetilde{H} + \{x_1 y : y \in Y_{odd}^1 \cup Y_{even}\} + \{x_2 y : y \in Y_{odd}^2 \cup Y_{even}\}.$$

Then H is a desired even subgraph. $\square$

Case 3.3. $Y_{odd} = \emptyset$.

In this case, since x_1 has odd degree in $\widetilde{H}$ and we performed a secondary $K_{2,2g}$-contraction at C, exactly one of the edges x_1y_1 and x_1y_2 is used in H'. By symmetry, we may assume that x_1y_1 is used in H'. Similarly, exactly one of the edges x_2y_1 and x_2y_2 is used in H'. Let D_1 be the circuit in H' using the edge x_1y_1. If y_2 is neither contained in any circuit of $\mathcal{D}'$ nor a center of any star in $\mathcal{D}'$, then every edge of G' incident with y_2 are dominated by some vertex in H', and hence for some $y \in Y \setminus Y_1$,

$$H := \widetilde{H} + \{x_i y' : i = 1, 2,\ y' \in Y \setminus \{y\}\}$$

is the desired even subgraph of G. So we may assume that y_2 is passed by some circuit $D_2 \in \mathcal{D}'$ or is a center of some star $D_2 \in \mathcal{D}'$.

When $D_1 = D_2$, there exists a path P in $\widetilde{H}$ connecting y_2 and a vertex in $\{x_1, x_2, y_1\}$. Let $y \in Y \setminus Y_1$ such that P starts from y in $\widetilde{H}$. When $D_1 \neq D_2$, then we let $y \in Y - Y_1$ be an arbitrary vertex. Let

$$H := \widetilde{H} + \{x_i y' : i = 1, 2,\ y' \in Y \setminus \{y\}\}.$$

Then H is an even subgraph of G. If $D_1 = D_2$, then since H also has a path P, all vertices in $\{x_1, x_2\} \cup Y$ are contained in the same component of H. Thus, we have $\omega(H) \leq \omega(H')$. On the other hand, if $D_1 \neq D_2$, then $\{x_1, x_2\} \cup Y$ are contained in at most two components of H. Since x_1, x_2, y_1, y_2 are contained in the two components D_1 and D_2 of H', we also have that $\omega(H) \leq \omega(H')$. In either case, H is the desired even subgraph. This completes the proof of Lemma 11. $\square$

5 Lemmas

We use the following theorem in the proof of Theorem 7. Recall that a graph is called *even* if all its vertices have even degree.

Theorem 12 (Jackson and Yoshimoto [13]) *Let G be a 3-edge connected graph of order n. Then G has a spanning even subgraph in which every component has at least $\min\{5, n\}$ vertices.*

In the proof of Theorem 7, we will also often use the following observation.

Fact 13 *Let $C \simeq C_5$ be a subgraph of a graph G. Then for any edge uv incident with a vertex of C, say $u \in V(C)$ and $v \notin V(C)$, there exists a matching in $G[V(C) \cup \{v\}]$ with three edges.*

The next lemma concerns the existence of a matching with two or three edges in a circuit. A graph obtained from a star by replacing all edges with multiple edges is called a *flower*.

Lemma 14 *Let D be a circuit of order at least four (D might possibly have multiple edges). Then:*

(i) *D has a matching with two edges unless D is a flower,*

(ii) *If D has at least five vertices and contains no cycle of length two or three, then*

(α) *for all $u \in D$, $D - u$ has a matching with two edges, unless $D \simeq K_{2,2g}$ for some $g \geq 2$,*

(β) *D has a matching with three edges, unless $D \simeq C_5$ or $D \simeq K_{2,2g}$ for some $g \geq 2$.*

Proof. If D contains a cycle of length at least six, then we can easily find a matching with three edges in D, and a matching with two edges in $D - u$ for each $u \in V(D)$. Then we may assume that D contains no cycle of length at least six. If D has a cycle of length five, D has a matching with two edges. Moreover, if D contains no cycle of length two or three, then by Fact 13, D has a matching with at least three edges, or $D \simeq C_5$. So all the statements in (i) and (ii) hold.

Thus, we may assume that D has no cycle of length at least five. Suppose next that D has a cycle C of length four, say, $C = x_1x_2x_3x_4$. Clearly, D has a matching with two edges, so the statement (i) holds. Suppose that D contains no cycle of length two or three. If there exists an edge in $D - \{x_1, x_2, x_3, x_4\}$, then we can find a matching with three edges, and hence (ii-α) and (ii-β) hold. So we may assume that the cycle $x_1x_2x_3x_4$ dominates all edges in D. On the other hand, if some vertex y in $D - \{x_1, x_2, x_3, x_4\}$ has consecutive neighbors in C, we can find a cycle of length five, a contradiction. This implies that for any vertex y in $D - \{x_1, x_2, x_3, x_4\}$, $N_D(y) = \{x_1, x_3\}$ or $N_D(y) = \{x_2, x_4\}$. If there exist two vertices y_1, y_2 in $D - \{x_1, x_2, x_3, x_4\}$ with $N_D(y_1) = \{x_1, x_3\}$ and $N_D(y_2) = \{x_2, x_4\}$, then $y_1x_1x_2y_2x_4x_3y_1$ is a cycle of D, a contradiction. Thus, we may assume that $N_D(y) = \{x_1, x_3\}$ for any vertex y in $D - \{x_1, x_2, x_3, x_4\}$, and hence $N_D(x_1) = V(D) \setminus \{x_1, x_3\}$. Since D is a circuit, $|V(D) \setminus \{x_1, x_3\}|$ is even. Thus, $D \simeq K_{2,2g}$, where $2g = |V(D) \setminus \{x_1, x_3\}|$.

Next, we assume that D has no cycle of length at least four. Then D contains a cycle of length two or three, and hence it is enough to show only the statement (i). Now suppose that D has no matching with two edges. If D has a cycle C of length three, say, $C = x_1x_2x_3$, then there exists an edge yx_i ($y \neq x_1, x_2, x_3$) in D for some i, say $i = 1$, since D is connected and D has at least four vertices. Then yx_1 and x_2x_3 form a matching with two edges, a contradiction. So we may assume that D has no cycle of length at least three, that is, D consists only of cycles isomorphic to C_2. If there exist two vertex disjoint cycles isomorphic to C_2 in D, then taking one edge from each cycle, we obtain a matching with two edges. Thus, any two cycles share a vertex. This implies that D is a flower. This completes the proof of Lemma 14. □

6 Proof of Theorem 7

We use induction on $|G|$. When $|G| \leq 5$, we can easily find a desired D-system. Thus we may assume that $|G| \geq 6$ and for all graphs with at most $|G| - 1$ vertices the statement is true.

We divide the proof into five steps. In the first step (Subsection 6.1) we consider some contractions defined in Section 4.1 as a preliminary for C_5-contractions in the second step (Subsection 6.2), where, in the contracted graph, we also construct a strong D-system with "bounded number" of components. In the remaining three steps (Subsections 6.3 to 6.5), we will reconstruct all contracted C_5s one by one. During the reconstruction, in Subsection 6.4 we construct a "sufficiently large" matching, which will be in Subsection 6.5 completed a matching satisfying the statement of Theorem 7.

6.1 C_2- or C_3-contractions and $K_{2,2g}$-contractions

In this subsection, we show the following two claims. Note that the first one is obvious by Lemmas 10 (i) and (ii).

Claim 1 *G has no cycle isomorphic to C_2 or C_3, that is, G is simple and triangle-free.*

Claim 2 *G has no bad $K_{2,2g}$ for any $g \geq 2$.*

Proof. Suppose not and let C be a bad $K_{2,2g}$ for some $g \geq 2$. Let x_1, x_2 be the vertices of the smaller partite set of C and let Y be the other partite set. By Lemma 9, C satisfies one of properties (i) and (ii) in Lemma 9. When (i) holds in Lemma 9, let G' be the graph obtained by a primary $K_{2,2g}$-contraction at C. Then G' is also essentially 3-edge connected by Fact 8. On the other hand, when (ii) holds in Lemma 9, there exists a subset $Y_1 \subset Y$ with $Y_1 \neq \emptyset$ and $Y_1 \neq Y$ such that the graph G' obtained by a secondary $K_{2,2g}$-contraction at C with respect to Y_1 is also essentially 3-edge connected. Note that in either case, $|V(G')| < |V(G)|$, so by the induction hypothesis, G' has a D-system $\mathcal{D}'$ such that $|\mathcal{D}'| \leq \max\{\frac{2}{5}\big(\alpha'(G') + 1\big), 1\}$. By Lemmas 11 (i)–(iii), G also has a D-system $\mathcal{D}$ such that $|\mathcal{D}| \leq |\mathcal{D}'| \leq \max\{\frac{2}{5}\big(\alpha'(G') + 1\big), 1\} \leq \max\{\frac{2}{5}\big(\alpha'(G) + 1\big), 1\}$. Thus, we may assume that G has no bad $K_{2,2g}$ for any $g \geq 2$. $\square$

6.2 C_5-Contractions and a strong D-system

In this subsection, we contract subgraphs isomorphic to C_5 which are bad in the following sense. For a subgraph C of G with $C \simeq C_5$, C is called *normal* if C has a neighbor outside of C that has degree one or two in G; otherwise C is *abnormal*. Now we consider the following contractions.

Let $\mathcal{C}$ be a set of pairwise vertex-disjoint cycles C of G such that C is an abnormal C_5. Take such a set $\mathcal{C}$ so that $|\mathcal{C}|$ is as large as possible. Now we perform C_5-contractions of each $C \in \mathcal{C}$ and let G_1 be the resulting graph. By Fact 8, G_1 is also essentially 3-edge connected (but G_1 might have multiple edges). In addition, we repeat C_2- or C_3-contractions to G_1 until there does not exist a subgraph isomorphic to C_2 or C_3. Let G_1' be the graph obtained by these operations. Again by Fact 8, G_1' is also essentially 3-edge connected.

Let G_1'' be the graph obtained from G_1' by removing all pendant edges, and suppressing all vertices of degree two in G_1'. Since G_1' is essentially 3-edge connected, G_1'' is 3-edge connected. Thus, by Theorem 12, G_1'' has a spanning even subgraph H_1'' in which each component has at least $\min\{5, |G_1''|\}$ vertices.

Let $\mathcal{D}_1'$ be the set of circuits of G_1' corresponding to components of H_1''. In other words, for each $D' \in \mathcal{D}_1'$, there exists a component D'' of H_1'' such that D'' is the

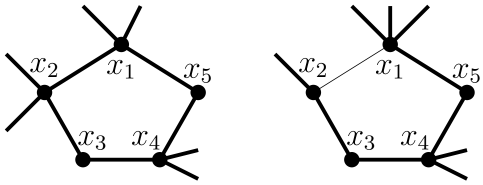


Figure 2: The circuit $\widetilde{D}$.

circuit obtained from D' by suppressing all vertices of degree two in G_1'. Since G_1' is essentially 3-edge connected, $\mathcal{D}_1'$ is a strong D-system in G_1'.

Next we consider reconstructions of C_2's and C_3's. By recursively applying Lemma 10 to $\mathcal{D}_1'$ and G_1', we obtain a strong D-system $\mathcal{D}_1$ of G_1.

6.3 Reconstruction of good C_5s and classification of bad C_5s

Now we consider reconstructions of C_5s. Some vertices obtained by a contraction of a C_5 could be reconstructed without increasing the number of circuits in $\mathcal{D}_1$. We call such a C_5 *good*; otherwise it is a *bad* C_5. More precisely, we define a good C_5 and a bad C_5, respectively, as follows.

Let $C = x_1x_2\dots x_5 \in \mathcal{C}$ and let D be a circuit in $\mathcal{D}_1$ which contains the vertex obtained by contraction of C. Now we regard D as the subgraph in G induced by all edges in D. Although x_i's might have odd degree in D, all other vertices of D have even degree in D. Depending on the parities of degrees of x_i's in D, we consider the following four cases:

a) All x_i's have even degrees in D.

b) Two consecutive x_i's have odd degrees and others have even.

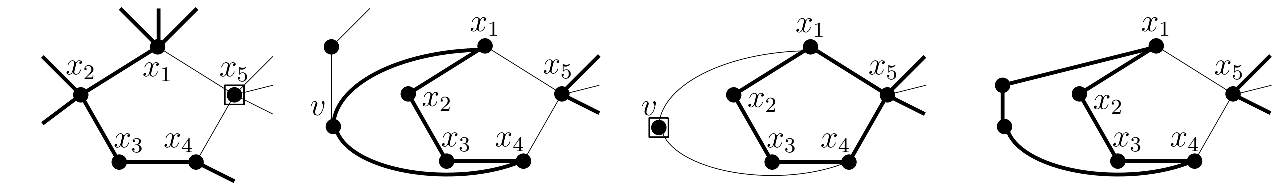
c) Exactly two x_i's but not consecutive have odd degrees.

d) Four x_i's have odd degrees and the fifth one has even degree.

Note that in Cases a) and b), the following $\widetilde{D}$ is also a circuit in the graph obtained from G_1 by reconstruction of C:

$$\widetilde{D} := \begin{cases} D + E(C) & \text{if Case a) occurs,} \\ D + E(C) - \{x_1x_2\} & \text{if Case b) occurs and } x_1 \text{ and } x_2 \text{ have odd degrees.} \end{cases}$$

See Figure 2. Thus, in Case a) and b) we can reconstruct an abnormal $C \simeq C_5$ without changing the number of circuits in $\mathcal{D}_1$. Note that all edges in G are dominated by $\big(\mathcal{D}_1 - \{D\}\big) \cup \{\widetilde{D}\}$. Therefore, such a C_5 is *good*.



A bad C_5 of Type I-i A bad C_5 of Type 0 A bad C_5 of Type I-ii A bad C_5 of Type II-i

Figure 3: The circuit $\widetilde{D}$ in Case c).

Now we consider the remaining two cases. Let $C = x_1 \dots x_5$ be an abnormal $C \simeq C_5$ with Case c), and assume that x_1 and x_4 have odd degrees in D and others have even. In this case, we first consider the even subgraph

$$D^* := D + \{x_1x_2, x_2x_3, x_3x_4\}$$

of the graph obtained from G_1 by reconstruction of C. If the degree of x_5 in G is two, then letting $\widetilde{D} = D^*$, we can reconstruct C without changing the number of circuits in $\mathcal{D}_1$. Therefore, such a C_5 is *good*.

Now we assume that the degree of x_5 in G is at least three. We also consider two cases depending on the degree of x_5 in D^*. If x_5 has degree zero in D, then we let such C be a *bad* C_5 *of Type I-i*. See the left side of Figure 3. In this case, let $\widetilde{D} = D^*$. Note that $\widetilde{D}$ does not pass through the vertex x_5. We call the vertex x_5 *uncovered* and the two edges x_1x_5 and x_4x_5 *D-dominated edges by* x_5. Next, we suppose that x_5 has a degree at least two in D^*. If D^* has only one component, then we let $\widetilde{D} = D^*$ and we can use $\widetilde{D}$ as a circuit of G, so C is *good*; otherwise, C is *bad*.

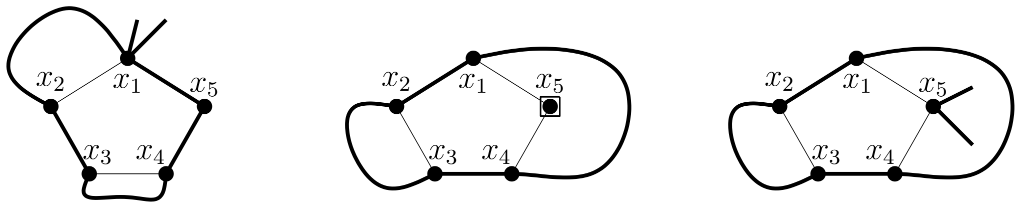
Suppose C is bad in this sense. Then D^* has exactly two components such that one of them contains all vertices in $V(C) \setminus \{x_5\}$ and the other contains x_5. Suppose further that the first one consists of only five vertices, say v, x_1, x_2, x_3 and x_4, and v is not a contracted vertex from a bad C_5. If v is incident with a vertex outside of C and of degree one or two in G, then we call such $C \simeq C_5$ a *bad* C_5 *of Type* 0, say that the circuit $vx_1x_2x_3x_4$ is *generated from* C, and let $\widetilde{D} := D^*$. See the left middle of Figure 3. Otherwise, that is, if v is not incident with a vertex outside of C and of degree one or two in G, then we call such $C \simeq C_5$ a *bad* C_5 *of Type I-ii,* and we let

$$\widetilde{D} := D + E(C).$$

See the right middle of Figure 3. Moreover, we call the vertex v *uncovered*, and the two edges vx_1 and vx_4 are *D-dominated by* v. For other case, that is, if the circuit containing $V(C) - \{x_5\}$ has at least six vertices, or if v is a vertex contracted from a bad C_5, then we say that C is a *bad* C_5 *of Type II-i*, and we let $\widetilde{D} = D^*$. See the right side of Figure 3.

Finally, let C be an abnormal C_5 with Case d), and assume that x_i has odd degree in D for all $1 \le i \le 4$. Then we consider the subgraph

$$D^* := D + \{x_2x_3, x_1x_5, x_4x_5\}$$



A good C_5 A bad C_5 of Type I-iii A bad C_5 of Type II-ii

Figure 4: The circuit $\widetilde{D}$ in Case d).

of the graph obtained from G_1 by reconstruction of C. If D^* has only one component, then we let $\widetilde{D} = D^*$ and we can use $\widetilde{D}$ as a circuit, so C is *good*. See the left side of Figure 4. Suppose now that D^* has two components such that one of them contains x_2 and x_3 and the other contains x_1, x_4 and x_5. Then we consider the following subgraph

$$\widetilde{D} := D + \{x_1x_2, x_3x_4\}$$

of the graph obtained from G_1 by reconstruction of C. Since D^* has two components, there exist a path connecting x_2 and x_3 and a path connecting x_1 and x_4 in D. Therefore, $\widetilde{D}$ has at most two components. If the degree of x_5 in G is two, then we can also use $\widetilde{D}$ as a circuit of G, so C is *good*. Suppose that the degree of x_5 in G is at least three. Similarly to Case c), we consider two cases depending on the degree of x_5 in $\widetilde{D}$. When x_5 has degree zero in $\widetilde{D}$, then we call such C a *bad* C_5 *of Type I-iii*. Also x_5 is *uncovered* and the edges x_1x_5 and x_4x_5 are *D-dominated by* x_5. See the middle of Figure 4. If x_5 has degree at least two in $\widetilde{D}$ and $\widetilde{D}$ consists of two circuits, then such a C_5 is said to be a *bad* C_5 *of Type II-ii*; otherwise C is *good*. Note that when C is a bad C_5 of Type II-ii, $\widetilde{D}$ has exactly two components such that one of them contains $V(C) \setminus \{x_5\}$ and the other contains x_5. Notice also that the first one has at least six vertices. See the right side of Figure 4.

For an abnormal $C \simeq C_5$, we say that C is a bad C_5 *of Type I* if C is of Type I-i or I-ii or I-iii, and C is a bad C_5 of *Type II* if C is of Type II-i or II-ii.

In addition, for an abnormal $C \simeq C_5$, the operation to get $\widetilde{D}$ from $D \in \mathcal{D}_1$ which contains the vertex contracted from C is also called *reconstruction (of C)*. Note that after reconstruction of all bad C_5, we obtain a set of circuits of G that dominates all edges in G except for those connecting two uncovered vertices. By the definition, we can reconstruct all good C_5s without increasing the number of circuits in $\mathcal{D}_1$.

Let G_2 and $\mathcal{D}_2$ be the graph and the D-system of G_2 obtained from G_1 and $\mathcal{D}_1$ by reconstructing all good C_5s and all bad C_5s of Type 0. We call a circuit in $\mathcal{D}_2$ that is not generated from a bad C_5 of Type 0 *original*. Note that the set of original circuits in $\mathcal{D}_2$ has a one-to-one correspondence to $\mathcal{D}_1$, also to $\mathcal{D}'_1$, since any generated circuit from a bad C_5 of Type 0 corresponds to a subcircuit of a circuit in $\mathcal{D}_1$ of length two, so it disappears after C_2-contraction. Notice also that $\mathcal{D}_2$ is a strong D-system of G_2.

It is easy to show the following claim.

Claim 3 *If $\mathcal{D}_2$ has at least two original circuits, then each $D \in \mathcal{D}_2$ with $D \not\simeq C_5$ has at least five vertices even after any sequence of C_2- and C_3-contractions and suppressing.*

Proof. Suppose that some circuit $D \in \mathcal{D}_2$ with $D \not\simeq C_5$ has at most four vertices after a sequence of C_2- and C_3-contractions and suppressing. Since $D \not\simeq C_5$, D is not generated from a bad C_5 of Type 0, and hence there exists a circuit D' in $\mathcal{D}'_1$ that corresponds to D. Since D has at most four vertices after a sequence of C_2- and C_3-contractions and suppressing, D' also has at most four vertices. Since D' corresponds to some component of H''_1, D' has at least $\min\{5, |G''_1|\}$ vertices. This implies that $|\mathcal{D}_1| = |\mathcal{D}'_1| = \omega(H''_1) = 1$, where $\omega(H''_1)$ is the number of components of H''_1. Hence $\mathcal{D}_1$ has only one circuit, which implies that $\mathcal{D}_2$ has exactly one original circuit. $\square$

6.4 Reconstruction of bad C_5s of Type II and construction of a matching

Let G_3 be the graph obtained from G_2 by reconstructing all bad C_5s of Type II. Recursively applying the reconstructions in Section 6.3, we get a strong D-system $\mathcal{D}_3$ of G_3. However, $|\mathcal{D}_3|$ might be larger than $|\mathcal{D}_2|$. In this subsection, we will show the existence of a matching in G_3 having many edges comparing with the number of circuits in $\mathcal{D}_3$. (Actually, we will show Claim 4.)

For a circuit $F \in \mathcal{D}_3$ and a matching M in G_3, we call F *special for M* if F contains a contracted vertex u from a bad C_5 of Type I and (i) no edge incident with u is contained in M, or (ii) $F \simeq K_{2,2g}$ for some $g \geq 2$, u has degree $2g$ in F, and $|E(F) \cap M| = 1$. For a matching M in G_3, we define the function f_M from $\mathcal{D}_3$ to $\left\{\frac{3}{2}, 2, \frac{5}{2}\right\}$ as follows; for every circuit F in $\mathcal{D}_3$,

$$f_M(F) = \begin{cases} \frac{3}{2} & \text{if } F \text{ is special for } M, \\ 2 & \text{if } F \text{ contains a contracted vertex from a bad } C_5 \text{ of Type I} \\ & \quad \text{and } F \text{ is not special for } M, \\ \frac{5}{2} & \text{otherwise.} \end{cases}$$

Claim 4 *There exists a matching M_3 in G_3 such that*

$$\sum_{F \in \mathcal{D}_3} f_{M_3}(F) \leq |M_3| + 1.$$

Proof. Let

$$\begin{aligned} \mathcal{D}_2^1 &:= \{D \in \mathcal{D}_2 : D \text{ contains no contracted vertex from a bad } C_5 \text{ of Type II}\}, \\ \text{and} \quad \mathcal{D}_2^2 &:= \mathcal{D}_2 - \mathcal{D}_2^1. \end{aligned}$$

We define the mapping f from $\mathcal{D}_3$ to $\{2, \frac{5}{2}\}$ as follows; for every circuit F in $\mathcal{D}_3$,

$$f(F) = \begin{cases} 2 & \text{if } F \text{ contains a contracted vertex from a bad } C_5 \text{ of Type I,} \\ \frac{5}{2} & \text{otherwise.} \end{cases}$$

Note that $f_M(F) \leq f(F)$ for each matching M of G_3 and each circuit $F \in \mathcal{D}_3$.

Notice that every circuit D in $\mathcal{D}_2^1$ is also a circuit in $\mathcal{D}_3$. For all $D \in \mathcal{D}_2^1$, let G_D be the subgraph of G_3 induced by $V(D) \cup \{v \in N_{G_3}(D) : d_{G_3}(v) = d_G(v) = 1 \text{ or } 2\}$. We will show the following subclaim.

Subclaim 1 *For every $D \in \mathcal{D}_2^1$, there is a matching M_D in G_D with at least $f(D)-1$ edges such that $\bigcup_{D\in\mathcal{D}_2^1} M_D$ is also a matching in G_3, and*

$$\sum_{D\in\mathcal{D}_2^1} |M_D| \geq \begin{cases} \sum_{D\in\mathcal{D}_2^1} f(D) - 1 & \textit{if } \mathcal{D}_2 \textit{ has only one original circuit and} \\ & \textit{that circuit lies in } \mathcal{D}_2^1, \\ \sum_{D\in\mathcal{D}_2^1} f(D) & \textit{otherwise.} \end{cases}$$

On the other hand, let $D \in \mathcal{D}_2^2$. Note that D is divided into more than one circuit through reconstructions of bad C_5s of Type II. Let $\mathcal{F}_D$ be the set of circuits F in $\mathcal{D}_3$ such that $E(F) \cap E(D) \neq \emptyset$. Note that $\mathcal{D}_3 = \mathcal{D}_2^1 \cup \bigcup_{D\in\mathcal{D}_2^2} \mathcal{F}_D$. We will also show the following.

Subclaim 2 *For every $D \in \mathcal{D}_2^2$, there is a matching M_D in $G_3\big[\bigcup_{F\in\mathcal{F}_D} V(F)\big]$ such that*

$$|M_D| \geq \begin{cases} \sum_{F\,\in\,\mathcal{F}_D} f_{M_D}(F) - 1 & \textit{if } D \textit{ is the only original circuit in } \mathcal{D}_2, \\ \sum_{F\,\in\,\mathcal{F}_D} f_{M_D}(F) & \textit{otherwise.} \end{cases}$$

Suppose that both Subclaims 1 and 2 hold. Then $M_3 := \bigcup_{D\in\mathcal{D}_2} M_D$ is a matching in G_3. Moreover, since the first case in the inequality in Subclaim 1 and the first case in the inequality in Subclaim 2 do not occur at the same time, we have

$$\begin{aligned} |M_3| &\geq \sum_{D\in\mathcal{D}_2^1} |M_D| + \sum_{D\in\mathcal{D}_2^2} |M_D| \\ &\geq \sum_{D\in\mathcal{D}_2^1} f(D) + \sum_{D\in\mathcal{D}_2^2} \sum_{F\in\mathcal{F}_D} f_{M_D}(F) - 1 \\ &\geq \sum_{F\in\mathcal{D}_3} f_{M_3}(F) - 1, \end{aligned}$$

which completes the proof of Claim 4. Therefore, it suffices to prove Subclaims 1 and 2.

Proof of Subclaim 1. Recall that for $D \in \mathcal{D}_2^1$, D is also a circuit in $\mathcal{D}_3$ since D has no contracted vertex from a bad C_5 of Type II. Recall also that G_D is the subgraph of G_3 induced by $V(D) \cup \{v \in N_{G_3}(D) : d_{G_3}(v) = d_G(v) = 1 \text{ or } 2\}$.

We will first show that for all $D \in \mathcal{D}_2^1$ with $D \not\simeq C_5$, there is a matching M_D in G_D with $|M_D| \geq f(D) - 1$ if D is the only original circuit in $\mathcal{D}_2$, and $|M_D| \geq f(D)$ otherwise. Let $D \in \mathcal{D}_2^1$ with $D \not\simeq C_5$.

Suppose first that D is the only original circuit in $\mathcal{D}_2$. If D contains a contracted vertex from a bad C_5 of Type I, then $D \subset G_D$ contains a matching M_D with $|M_D| = 1 = f(D) - 1$. On the other hand, if D contains no contracted vertex from a bad C_5 of Type I, then D is also a circuit in G, and hence D is not a flower by Claim 1. Then we can find a matching M_D in D with $|M_D| = 2 > f(D) - 1$ by Lemma 14 (i). So we may assume that $\mathcal{D}_2$ has at least two original circuits.

By Claim 3, D has at least five vertices and we can find a matching with two edges in D by Lemma 14 (ii). Hence if D has a contracted vertex from a bad C_5 of Type I, then we can find a matching M_D with at least $f(D)$ edges in D, and we are done. So we may assume that D has no contracted vertex from a bad C_5 of Type I, and hence D is also a circuit in G. Hence D contains no cycle of length two or three by Claim 1. By Lemma 14 (iii), if $D \not\simeq K_{2,2g}$ with $g \geq 2$, then we can find a matching with three edges, and we are done. (Recall that $D \not\simeq C_5$.) So we may also assume that $D \simeq K_{2,2g}$ with $g \geq 2$. By Claim 2, D is good. Thus, D has only at most four vertices of degree at least three in G. This implies that after suppressing all vertices of degree two, D has only at most four vertices, but this contradicts Claim 3. Thus, for all $D \in \mathcal{D}_2^1$ with $D \not\simeq C_5$, there is a matching M_D in G_D with $|M_D| \geq f(D) - 1$ if D is the only original circuit in $\mathcal{D}_2$, and $|M_D| \geq f(D)$ otherwise.

We next consider all $D \in \mathcal{D}_2^1$ that are isomorphic to C_5. Note that D has no contracted vertex from a bad C_5 of Type II. If D is generated from a bad C_5 of Type 0, then by the definition, D has a neighbor of degree one or two in G. Otherwise D is vertex-disjoint from any $C \in \mathcal{C}$. Therefore, if D is abnormal, this contradicts the maximality of $|\mathcal{C}|$. Thus, D is normal. In either cases, D has a neighbor of degree one or two in G.

If D has a neighbor of degree one, then by Fact 13, there exist three edges forming a matching in $G_D - \bigcup_{D' \in \mathcal{D}_2^1 \setminus \{D\}} V(G_{D'})$ and we are done. Therefore, it suffices to consider only the set, say $\mathcal{C}_2$, of circuits D in $\mathcal{D}_2^1$ such that $D \simeq C_5$, D has no vertex contracted from a bad C_5 and D is adjacent with a vertex of degree two.

Let R be the bipartite graph such that one vertex set of the bipartition of R is $\mathcal{C}_2$, the other one is the set of vertices of degree two in G_3, and $D \in \mathcal{C}_2$ is adjacent with v in R if and only if v is adjacent with a vertex of D in G_3. By the definition, each $D \in \mathcal{C}_2$ has a degree at least one in R. Let R' be a component of R containing at least one vertex in $\mathcal{C}_2$. If R' has only one vertex in $\mathcal{C}_2$, say $D \in \mathcal{C}_2$, then $D \cup \{\varphi(D)\}$ has a matching in G_3 with three edges, where $\varphi(D)$ is a vertex of degree two in G_3

which is a neighbor of D in R. (Note that the matching is also in G_D.) So we may assume that $|\mathcal{C}_2 \cap V(R')| \geq 2$, and let $\overrightarrow{T}$ be a rooted spanning tree of R' with root D^* for some $D^* \in \mathcal{C}_2$. Since each $D \in \mathcal{C}_2$ has a vertex incident with a vertex of degree two in G_3, each $D \in \mathcal{C}_2 \cap V(R')$ has a parent $\varphi(D)$ in $\overrightarrow{T}$, except for $D = D^*$. Let $\varphi(D^*) = \emptyset$. By Fact 13, $D \cup \{\varphi(D)\}$ has a matching M_D in G_3 with three edges for each $D \in \mathcal{C}_2 \cap V(R')$ with $D \neq D^*$, and with two edges for $D = D^*$. Then

$$\begin{aligned}
\sum_{D \in \mathcal{C}_2 \cap V(R')} |M_D| &\geq 3\big(|\mathcal{C}_2 \cap V(R')| - 1\big) + 2 \\
&= 3|\mathcal{C}_2 \cap V(R')| - 1 \\
&= \frac{5}{2}|\mathcal{C}_2 \cap V(R')| + \frac{1}{2}|\mathcal{C}_2 \cap V(R')| - 1 \\
&\geq \sum_{D \in \mathcal{C}_2 \cap V(R')} f(D).
\end{aligned}$$

Considering all components of R, this completes the proof of Subclaim 1. $\square$

Proof of Subclaim 2. Let $D \in \mathcal{D}_2^2$. Recall that $\mathcal{F}_D$ is the set of circuits F in $\mathcal{D}_3$ such that $E(F) \cap E(D) \neq \emptyset$. For a circuit $F \in \mathcal{F}_D$, let D_F be the subcircuit of D such that $E(D_F) = E(F) \cap E(G_2)$.

By the definition, each contracted vertex from a bad C_5 of Type II is a cut vertex of D (otherwise we can reconstruct such a vertex without increasing the number of circuits, so it is good). Therefore D has a tree-like structure. More precisely, let T be the graph such that the vertex set of T is $\mathcal{F}_D$ and two vertices F and F' are joined by an edge in T if and only if D_F and $D_{F'}$ share a contracted vertex from a bad C_5 of Type II. Note that T is a tree.

Let $F \in \mathcal{F}_D$ be a leaf of T. Note that D_F has exactly one contracted vertex from a bad C_5 of Type II, say u. Let $C = x_1x_2 \dots x_5$ be the bad C_5 in G_3 corresponding to u. Suppose that D_F has only two vertices, and let v be the (only) vertex in $V(D_F) \setminus \{u\}$. If $|F| = 2$, then v is a contracted vertex from a bad C_5 of Type I since G is simple. Otherwise, that is, if $|F| > 2$, then C is a bad C_5 of Type II-i by the definition, and we may assume that F consists of five vertices x_1, x_2, x_3, x_4 and v. Then, by the definition of Type II-i, v has to be a contracted vertex from a bad C_5 of Type I. This implies that if $|D_F| = 2$, then F contains a contracted vertex from a bad C_5 of Type I.

Let L be the set of leaves of T, and let $L' \subset L$ be the set of circuits $F \in \mathcal{F}_D$ such that D_F contains at least three vertices. By the above fact, each component in $L \setminus L'$ has a contracted vertex from a bad C_5 of Type I, and hence at least $|L \setminus L'|$ circuits in $\mathcal{F}_D$ contain a contracted vertex from a bad C_5 of Type I. Thus, for every matching M in G_3,

$$\begin{aligned}
\sum_{F\,\in\,\mathcal{F}_D} f_M(F) &\le \sum_{F\,\in\,\mathcal{F}_D} f(F)\\
&\le \frac{5}{2}\big(|T|-|L\setminus L'|\big)+2|L-L'|\\
&= \frac{5}{2}|T|-\frac{1}{2}|L|+\frac{1}{2}|L'|. \qquad (1)
\end{aligned}$$

Let $T' = T - L'$ and let I be a maximum independent set of T'. Note that $|I| \ge \frac{1}{2}|T'|$, since T' is bipartite. Taking one edge from D_F for each $F \in I$, we can find a matching M' in D of order at least $|I|$. Moreover, for each $F \in L'$, D_F has an edge which is not incident with the vertex u, where u is the unique contracted vertex in D_F from a bad C_5 of Type II. Therefore, in D, we can find a matching $\widetilde{M}$ with at least $|I| + |L'|$ edges.

Let u be a vertex in D that contracted from a bad C_5 of Type II, and let C be the bad C_5 in G_3 corresponding to u. By Fact 13, for each edge e in D incident with u, we can find two edges in C which together with e form a matching in G_3. This implies that for each contracted vertex in D from C that is a bad C_5 of Type II, we can add two edges into the matching $\widetilde{M}$ through the reconstruction of C. Since D has $|T|-1$ contracted vertices from bad C_5s of Type II, there exists a matching M_D in $G_3\big[\bigcup_{F\in\mathcal{F}_D} V(F)\big]$ such that

$$\begin{aligned}
|M_D| &\ge |I|+|L'|+2\big(|T|-1\big)\\
&\ge \frac{1}{2}|T'|+|L'|+2|T|-2\\
&= \frac{5}{2}|T|+\frac{1}{2}|L'|-2.
\end{aligned}$$

If D is the only original circuit in $\mathcal{D}_2$, then by the inequality (1),

$$\begin{aligned}
|M_D| &\ge \frac{5}{2}|T|+\frac{1}{2}|L'|-2\\
&\ge \frac{5}{2}|T|+\frac{1}{2}|L'|-\frac{1}{2}|L|-1\\
&\ge \sum_{F\,\in\,\mathcal{F}_D} f_{M_D}(F)-1,
\end{aligned}$$

and hence M_D is a desired matching. On the other hand, if $|L| \ge 4$, then

$$\begin{aligned}
|M_D| &\ge \frac{5}{2}|T|+\frac{1}{2}|L'|-2\\
&\ge \frac{5}{2}|T|+\frac{1}{2}|L'|-\frac{1}{2}|L|\\
&\ge \sum_{F\,\in\,\mathcal{F}_D} f_{M_D}(F),
\end{aligned}$$

and we are also done. Thus, we may assume that $\mathcal{D}_2$ has at least two original circuits and $|L| \leq 3$.

Moreover, we may also assume that

(T0) D has no matching with at least $\frac{1}{2}|T'| + |L'| + 1$ edges, and if $|L| = 3$, then D has no matching with at least $\frac{1}{2}|T'| + |L'| + \frac{1}{2}$ edges.

This also implies the following facts.

(T1) T' has no independent set of order at least $\frac{1}{2}|T'| + 1$.

(T2) If $|L| = 3$, then T' has no independent set of order at least $\frac{1}{2}\big(|T'| + 1\big)$, that is, T' is a balanced bipartite graph.

(T3) For each $F \in L'$, $D_F - \{u\}$ has no matching with at least two edges, where u is the unique contracted vertex from a bad C_5 of Type II.

Suppose first that $|L| = 3$. Let $F^* \in \mathcal{F}_D$ such that the degree of F^* in T is exactly 3. Note that D_{F^*} has at least three vertices. Let T^1, T^2 and T^3 be the three paths in $T' - F^*$ (possibly $T^i = \emptyset$ for some i, which could happen when F^* is adjacent with a member of L' in T). By (T2), T' is a balanced bipartite graph, and hence at least one of the paths T^1, T^2, T^3, say, T^1, has odd number of vertices. Since $T' - T^1$ is a path of odd order, $T' - T^1$ has an independent set I_0 with $|I_0| \geq \frac{1}{2}|T' - T^1| + \frac{1}{2}$. Similarly, T^1 has an independent set I_1 with $|I_1| \geq \frac{1}{2}|T^1| + \frac{1}{2}$. Since D_{F^*} has at least three vertices, even if $F^* \in I_0$, we can take an edge from D_F for each $F \in I_0 \cup I_1$ so that such edges form a matching M' in D. Adding one edge from D_F for each $F \in L'$, we can obtain a matching in D with

$$|M'| + |L'| = |I_0| + |I_1| + |L'| \geq \frac{1}{2}|T'| + |L'| + 1$$

edges, contradicting (T0). Thus, we may assume that $|L| = 2$, that is, T is a path. Let $T = F^1 F^2 \dots F^l$.

A circuit is called *redundant* if it is reduced to one vertex by a sequence of C_2- or C_3-contractions. If for all circuits $F \in \mathcal{F}_D$, D_F is redundant, then D is also redundant, contradicting Claim 3 and the fact that $|\mathcal{D}_2| \geq 2$. Hence there exists a circuit $F^* \in \mathcal{F}_D$ such that D_{F^*} is not redundant.

Suppose also that there exists a circuit $F^{**} \in \mathcal{F}_D$ such that $F^{**} \neq F^*$ and $D_{F^{**}}$ is not redundant. We may assume that $F^* = F^i$ and $F^{**} = F^j$ for some $i < j$. Let $T^1 = F^1 \cdots F^i$, $T^2 = F^i \cdots F^j$ and $T^3 = F^j \cdots F^l$. If $|T^1|$ is odd, then we can find a matching in $\bigcup_{F \in V(T^1)} D_F$ with at least $\frac{1}{2}\big(|T^1| + 1\big)$ edges. On the other hand, if $|T^1|$ is even, then we can find a matching in $\bigcup_{F \in V(T^1)} D_F$ with at least $\frac{1}{2}|T^1|$ edges if $F^1 \notin L'$, and with at least $\frac{1}{2}|T^1| + 1$ edges if $F^1 \in L'$. In either case, $\bigcup_{F \in V(T^1)} D_F$ has a matching M^1 with at least $\frac{1}{2}\big(|T^1| + |L' \cap T^1|\big)$ edges. Similarly, $\bigcup_{F \in V(T^2)} D_F$

and $\bigcup_{F\in V(T^3)} D_F$ have matchings M^2 and M^3 with at least $\frac{1}{2}|T^2|$ edges and with at least $\frac{1}{2}\big(|T^3|+|L'\cap T^3|\big)$ edges, respectively. Since both F^i and F^j are not redundant, both have a cycle of length at least 4. Therefore, we can take such matchings M^1, M^2 and M^3 so that $M^1\cup M^2\cup M^3$ is also a matching with $|M^1|+|M^2|+|M^3|$ edges. Therefore, $G_3\big[\bigcup_{F\in\mathcal{F}_D} V(F)\big]$ has a matching with at least

$$\begin{aligned}
&\frac{1}{2}\big(|T^1|+|L'\cap T^1|\big)+\frac{1}{2}|T^2|+\frac{1}{2}\big(|T^3|+|L'\cap T^3|\big)+2\big(|T|-1\big)\\
&\quad= \frac{1}{2}\big(|T|+2+|L'|\big)+2|T|-2\\
&\quad= \frac{5}{2}|T|+\frac{1}{2}|L'|-1\\
&\quad= \frac{5}{2}|T|+\frac{1}{2}|L'|-\frac{1}{2}|L|
\end{aligned}$$

edges, and we are done by the inequality (1). Thus, we may also assume that F^* is the only circuit in $\mathcal{F}_D$ such that D_{F^*} is not redundant. Note that after C_2- or C_3-contractions and suppressing all vertices of degree 2, D_{F^*} has at least five vertices by Claim 3.

Suppose that $D_{F^*}\not\simeq K_{2,2g}$ for any $g\geq 2$. Then by Lemma 14 (ii) and by (T3), $D_{F^*}\notin L'$. Recall that I is a maximum independent set of T'. Then $|I|\geq\frac{1}{2}|T'|$. We can take two edges from D_{F^*} when $F^*\in I$ and one edge from D_{F^*} when $F^*\notin I$, such that they, together with an edge in D_F for each $F\in I$, form a matching. This implies that there exists a matching in D with at least $\frac{1}{2}|T'|+|L'|+1$ edges, contradicting (T0). Thus, we obtain that $D_{F^*}\simeq K_{2,2g}$ for some $g\geq 2$.

Let F,F' be circuits in $\mathcal{F}_D$ such that D_F and $D_{F'}$ share a contracted vertex, say u, from a bad C_5 of Type II. By the definition of the reconstruction of a bad C_5 of Type II, exactly one of D_F and $D_{F'}$ does not change through the reconstruction of C_u, where C_u is the bad C_5 of Type II corresponding to u. Since D has $|T|-1$ contracted vertices from bad C_5s of Type II, at least one circuit in $\mathcal{F}_D$, say F, is also a circuit in D. So, $F=D_F$. Since F^* is the unique circuit in $\mathcal{F}_D$ such that D_{F^*} is not redundant, we have that $F=D_F$ is redundant or $F=F^*\simeq K_{2,2g}$ for some $g\geq 2$.

If $F=F^*$ and F contains no contracted vertex from a bad C_5 of Type I, then F is also a circuit in G. Then by Claim 2, F is a good $K_{2,2g}$. However, after C_2- and C_3-contractions and suppressing all vertices of degree 2, D has only at most four vertices in G_1'', contradicting Claim 3 and the fact $|\mathcal{D}|\geq 2$. Thus, if $F=F^*$, then F contains a contracted vertex from a bad C_5 of Type I. On the other hand, even when F is redundant, F contains a contracted vertex from a bad C_5 of Type I, since G is simple and triangle-free by Claim 1. In either case, F contains a contracted vertex from a bad C_5 of Type I, say u.

We will show that

$$\begin{array}{ll} \text{there exists a matching } \widetilde{M} \text{ in } D \text{ with at least } \dfrac{1}{2}|T'|+|L'|+\dfrac{1}{2} \text{ edges} & \\ \text{such that (i) } \widetilde{M} \text{ does not contain any edge incident with } u \text{ or} & (2) \\ \text{(ii) } F=F^* \simeq K_{2,2g} \text{ and } u \text{ has degree } 2g \text{ in } F \text{ and } |E(\widetilde{M})\cap E(F)|=1. & \end{array}$$

This implies that after reconstructing all contracted vertices in D from bad C_5s of Type II, adding some edges into $\widetilde{M}$, we can find a matching M_D in $G_3[\bigcup_{F'\in\mathcal{F}_D} V(F')]$ with at least

$$\begin{aligned} \frac{1}{2}|T'|+|L'|+\frac{1}{2}+2\big(|T|-1\big) &= \frac{5}{2}|T|+\frac{1}{2}|L'|-\frac{3}{2} \\ &= \frac{5}{2}\big(|T|-|L\setminus L'|\big)+2\big(|L\setminus L'|-1\big)+\frac{3}{2} \end{aligned}$$

edges. (Recall that $|L|=2$.) By the choice (2) and the definition, F is special for M_D, and hence $\sum_{F'\in\mathcal{F}_D} f_{M_D}(F') \le \frac{5}{2}\big(|T|-|L\setminus L'|\big)+2\big(|L\setminus L'|-1\big)+\frac{3}{2}$. This completes the proof of Subclaim 2.

In the rest of the proof of Subclaim 2, we will show (2). Recall that $T = F^1F^2\cdots F^l$. We may assume that $F=F^l$ if $F\in L'$. Let $F^i=F^*$. Let $T^1=F^1\cdots F^i$ if $F^1\notin L'$; otherwise let $T^1=F^2\cdots F^i$. Similarly, let $T^2=F^i\cdots F^l$ if $F^l\notin L'$; otherwise $T^2=F^i\cdots F^{l-1}$. Note that if $F^*=F^i=F^l$, then $T^2=\emptyset$.

Since both T^1 and T^2 is a path, for $j=1,2$, we can find a matching M^j in $\bigcup_{F'\in V(T^j)} D_{F'}$ with at least $\frac{1}{2}|T^j|$ edges. Since $D_{F^i}=D_{F^*}$ is not redundant, D_{F^i} has a cycle of length at least 4, and hence we can choose M^1 and M^2 such that $M^1\cup M^2$ is also a matching with $|M^1|+|M^2|$ edges. Since we can take an edge from each circuit F' in L' which is not incident with the contracted vertex from bad C_5 from Type II, there exists a matching in D with

$$\begin{aligned} |M^1|+|M^2|+|L'| &\ge \frac{1}{2}|T^1|+\frac{1}{2}|T^2|+|L'| \\ &= \frac{1}{2}|T'|+|L'|+\frac{1}{2} \end{aligned}$$

edges. Moreover, if $|M^1|\ge\frac{1}{2}|T^1|+\frac{1}{2}$, then together with M^2, it forms a matching of D with at least $\frac{1}{2}|T^1|+\frac{1}{2}+\frac{1}{2}|T^2|\ge\frac{1}{2}|T'|+1$ edges, contradicting (T1). Thus, we have that $|M^1|=\frac{1}{2}|T^1|$, that is, T^1 has even number of vertices. Similarly, T^2 also has even number of vertices.

Therefore, if $F\in\mathcal{F}_D\setminus L'$, then we can choose M^1 and M^2 such that every edge in F is not used in $M^1\cup M^2$. This together with appropriate edges in D_{F^1} (if $F^1\in L'$) and in D_{F^l} (if $F^l\in L'$) forms a matching $\widetilde{M}$ in D, which is a desired one in (2)-(i).

So we may assume that $F\in L'$. Suppose first that $F=F^1$. Note that $F\neq F^*$ by the choice of F^1. Since $|T^1|$ is even, we can choose a matching M^1 such that any edge in D_{F^2} is not used in M^1. Then there exists an edge e in D_{F^1} which is not incident with the vertex u, since D_{F^1} has at least three vertices. Therefore, $M^1 \cup \{e\}$, together with M^2 (and an edge in a circuit in D_{F^l} if $F^l \in L'$), forms a matching $\widetilde{M}$ in D, which is a desired one for (2)-(i). When $F = F^l$ but $F \neq F^*$, or when $F = F^l = F^*$ and the degree of u in D_{F^*} is 2, then similarly we can find a desired matching $\widetilde{M}$ in D for (2)-(i). So we may assume that $F = F^l = F^*$ and the degree of u in D_{F^*} is $2g$. In this case, $M^1 \cup M^2$ together with an appropriate edge in D_{F^1} (if $F^1 \in L'$) and an edge incident with u in F^l forms a matching $\widetilde{M}$ in D, which is a desired one for (2)-(ii). This completes the proofs of Subclaim 2 and Claim 4. □

6.5 Reconstruction of bad C_5s of Type I

In this subsection, we reconstruct all bad C_5s of Type I in G_3. After reconstructing all such bad C_5s, we get the original graph G, and we also get a set of circuits of G from $\mathcal{D}_3$, say $\widetilde{\mathcal{D}}$. Note that $\widetilde{\mathcal{D}}$ might not be a D-system of G, because some edges incident with uncovered vertices might be not dominated by any circuit in $\widetilde{\mathcal{D}}$. In order to dominate all such edges, we shall add some circuits and some stars with centers at uncovered vertices. In this process, the number of members in the D-system increases, but we will show that we do not need to add too many circuits and stars.

Let K be the subgraph of G induced by the set of uncovered vertices. Note that any edge of K is not dominated by any circuit in $\widetilde{\mathcal{D}}$. For an uncovered vertex v contained in a bad C_5 of Type I, say C, we call v *special* if F is special for M_3, where F is the circuit in $\mathcal{D}_3$ passing the vertex corresponding to C. An uncovered vertex v is *non-special* if v is not special for M_3.

Let $C^1, C^2, \ldots, C^l$ be vertex disjoint cycles in K. Taking as many such cycles as possible, we can assume that K' has no cycle, where $K' = K - \bigcup_{i=1}^{l} V(C^i)$. Let V_0^S and V_0^N be the set of special vertices and the set of non-special vertices in $\bigcup_{i=1}^{l} V(C^i)$, respectively. Since G is simple and triangle-free, for all $1 \leq i \leq l$, C^i has at least four vertices, and hence

$$l \leq \frac{1}{4}|\bigcup_{i=1}^{l} V(C^i)| = \frac{1}{4}\Big(|V_0^S| + |V_0^N|\Big). \tag{3}$$

Taking a smaller partite set of each component of K', we obtain an independent set I of K' which dominates all edges in K'. Thus, there exists a mapping ψ from $E(K')$ to I such that for all $e \in E(K')$, e is incident with $\psi(e) \in I$. Note that I does not contain an isolated vertex in K', and hence $|\psi^{-1}(v)| \geq 1$ for each $v \in I$.

Let $v \in I$. Since v is uncovered with respect to some $F \in \mathcal{D}_3$, there exist two D-dominated edges by v. Let S_v be the star which is formed by a center v together with the edges in $\psi^{-1}(v)$ and two D-dominated edges by v. In particular, S_v is a star with at least three edges for all $v \in I$, and $E(K') \subset \bigcup_{v\in I} E(S_v)$. So, $\mathcal{D}$ is a D-system of G, where

$$\mathcal{D} := \widetilde{\mathcal{D}} \cup \{C^1, \dots, C^l\} \cup \{S_v : v \in I\}.$$

Let $\mathcal{S}_1$ be the set of stars S'_v in $\{S_v : v \in I\}$ such that S'_v contains an edge both of whose end vertices are special. Let $\mathcal{S}_2 := \{S_v : v \in I\} - \mathcal{S}_1$. For $i = 1, 2$, let V_i^S and V_i^N be the set of special vertices and the set of non-special vertices in $\mathcal{S}_i$, respectively. Since for all $v \in I$, S_v contains at least two vertices in $V(K')$, we have that

$$|\mathcal{S}_1| \le \frac{1}{2}\big(|V_1^S| + |V_1^N|\big) \quad \text{and} \quad |\mathcal{S}_2| \le \frac{1}{2}\big(|V_2^S| + |V_2^N|\big). \tag{4}$$

On the other hand, since each star in $\mathcal{S}_2$ has to contain a non-special vertex, we obtain

$$|\mathcal{S}_2| \le |V_2^N|. \tag{5}$$

Let $\mathcal{D}_3^S \subset \mathcal{D}_3$ be the set of special circuits for M_3, and let $\mathcal{D}_3^N \subset \mathcal{D}_3$ be the set of circuits F in $\mathcal{D}_3$ such that F contains a contracted vertex from a bad C_5 of Type I and F is not special for M_3. Since every circuit in $\mathcal{D}_3^S$ (and in $\mathcal{D}_3^N$) corresponds to at least one special uncovered vertex for M_3, (one non-special uncovered vertex, respectively,) we obtain that

$$\begin{aligned} |\mathcal{D}_3^S| &\le |V_0^S| + |V_1^S| + |V_2^S|, && (6)\\ \text{and} \qquad |\mathcal{D}_3^N| &\le |V_0^N| + |V_1^N| + |V_2^N|. && (7) \end{aligned}$$

On the other hand, when we reconstruct each bad C_5, by Fact 13, we can find two edges which can be added into the matching M_3 of G_3.

Moreover, let $S_v \in \mathcal{S}_1$ and let vv' be an edge of S_v both of whose end vertices are special. Let $C_v = x_1x_2\dots x_5$ and $C_{v'} = x'_1x'_2\dots x'_5$ be the bad C_5s corresponding to v and v', respectively. Let F_v be the circuit in $\mathcal{D}_3$ containing v.

Suppose first that C_v is a bad C_5s of Type I-i or I-iii. In this case, by symmetry, we may assume that $v = x_5$. If all edges of F_v incident with x_i for $1 \le i \le 5$ are not used in M_3, then let $e_1 = x_1x_2$ and $e_2 = x_3x_4$. Otherwise, $F_v \simeq K_{2,2g}$ for some $g \ge 2$ since v is special. In this case, we may also assume that x_1 is incident with an edge in M_3. Since C_v is a bad C_5 of Type I-i or I-iii, there exists an edge of F_v incident with x_4 in G. Then let e_1 be such an edge and let $e_2 = x_2x_3$. Suppose next that C_v is a bad C_5 of Type I-ii. In this case, we may assume that for all $1 \le i \le 4$, x_i is not incident with an edge in M_3, and we let $e_1 = x_1x_2$ and $e_2 = x_3x_4$. In either case, note that e_1 and e_2 can be added into M_3 as a matching.

Similarly, we can find two edges e'_1 and e'_2 from $C_{v'}$ such that e'_1 and e'_2 can be added into M_3 as a matching. Moreover, when we reconstruct C_v and $C_{v'}$, we can add

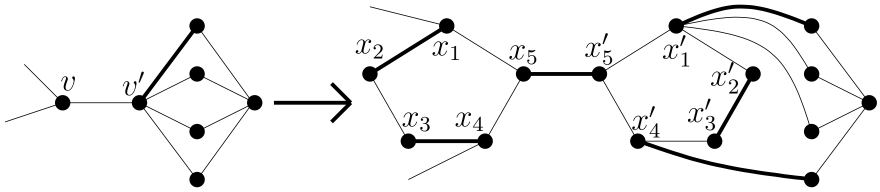


Figure 5: The matching in G.

the five edges e_1, e_2, e_1', e_2' and vv' into M_3 and obtain a matching in G. See Figure 5. This implies that for each $S_v \in \mathcal{S}_1$, we can add $2|V(S_v) \cap (V_1^S \cup V_1^N)| + 1$ edges into M_3 through reconstructions at C_u for all uncovered vertices u in S_v, where C_u is a bad C_5 corresponding to u. Thus, G has a matching with at least $|M_3| + 2|V(K)| + |\mathcal{S}_1|$ edges, that is,

$$\alpha'(G) \geq |M_3| + 2|V(K)| + |\mathcal{S}_1|. \tag{8}$$

Hence by Claim 4 and by the inequalities (3) – (8),

$$\begin{aligned}
&\alpha'(G) \\
&\quad\geq |M_3| + 2|V(K)| + |\mathcal{S}_1| \\
&\quad\geq \sum_{F \in \mathcal{D}_3} f(F) - 1 + 2|V(K)| + |\mathcal{S}_1| \\
&\quad\geq \frac{3}{2}|\mathcal{D}_3^S| + 2|\mathcal{D}_3^N| + \frac{5}{2}\Big(|\widetilde{\mathcal{D}}| - |\mathcal{D}_3^S| - |\mathcal{D}_3^N|\Big) + 2|V(K)| + |\mathcal{S}_1| - 1 \\
&\quad= \frac{5}{2}|\widetilde{\mathcal{D}}| - |\mathcal{D}_3^S| - \frac{1}{2}|\mathcal{D}_3^N| + 2\big(|V_0^S| + |V_1^S| + |V_2^S| + |V_0^N| + |V_1^N| + |V_2^N|\big) + |\mathcal{S}_1| - 1 \\
&\quad\geq \frac{5}{2}|\widetilde{\mathcal{D}}| + |V_0^S| + |V_1^S| + |V_2^S| + \frac{3}{2}\Big(|V_0^N| + |V_1^N| + |V_2^N|\Big) + |\mathcal{S}_1| - 1 \\
&\quad\geq \frac{5}{2}|\widetilde{\mathcal{D}}| + \frac{5}{8}\Big(|V_0^S| + |V_0^N|\Big) + \frac{3}{4}\Big(|V_1^S| + |V_1^N|\Big) + |\mathcal{S}_1| + \Big(|V_2^S| + |V_2^N|\Big) + \frac{1}{2}|V_2^N| - 1 \\
&\quad\geq \frac{5}{2}|\widetilde{\mathcal{D}}| + \frac{5}{2}l + \frac{3}{2}|\mathcal{S}_1| + |\mathcal{S}_1| + 2|\mathcal{S}_2| + \frac{1}{2}|\mathcal{S}_2| - 1 \\
&\quad= \frac{5}{2}|\mathcal{D}| - 1,
\end{aligned}$$

or

$$|\mathcal{D}| \leq \frac{2}{5}\big(\alpha'(G) + 1\big).$$

This completes the proof of Theorem 7. $\square$

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