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# Cycles through specified vertices in triangle-free graphs

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## Abstract

Let $G$ be a triangle-free graph with $\delta(G) \geq 2$ and $\sigma_4(G) \geq |V(G)|+2$. Let $S \subset V(G)$ consist of less than $\sigma_4/4+1$ vertices. We prove the following. If all vertices of $S$ have degree at least three, then there exists a cycle $C$ containing $S$. Both the upper bound on $|S|$ and the lower bound on $\sigma_4$ are best possible.

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## 1 Introduction

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is a finite set of vertices and $E(G)$ is a set of unordered pairs of two different vertices, called edges. All notation and terminology not explained is given in [6]. For simplicity, the order of a graph is denoted by $n$ and $G - V(H)$ by $G - H$. Let

$$\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(x_i) \mid x_1, x_2, \ldots, x_k \text{ are independent}\},$$

where $d_G(x_i)$ is the degree of a vertex $x_i$. If the independence number of $G$ is less than $k$, then we define $\sigma_k(G) = \infty$.

Ore [11] showed that a graph $G$ with $\sigma_2 \geq n$ is hamiltonian and Bondy [3] proved that if $G$ is a 2-connected graph with $\sigma_3 \geq n+2$, then for any longest cycle $C$, $E(G - C) = \emptyset$. Enomoto et al. [9] generalized this theorem as follows: if $G$ is a

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2-connected graph with $\sigma_3 \geq n+2$, then $p(G) - c(G) \leq 1$, where $p(G)$ and $c(G)$ are the order of longest paths and the circumference, respectively.

In this paper we study triangle-free graphs. For triangle-free graphs with $\sigma_2 \geq (n+1)/2$, all longest cycles are dominating [16]. This lower bound is almost best possible by the examples due to Ash and Jackson [1]. Corresponding to the theorem by Enomoto et al., the following result has been proven.

**Theorem 1** ([13]). *Let $G$ be a triangle-free graph with $\delta \geq 2$. If $\sigma_4 \geq n+2$, then for any path $P$, there exists a cycle $C$ such that $|V(P-C)| \leq 1$ or $G$ is isomorphic to the graph in Figure 1.*

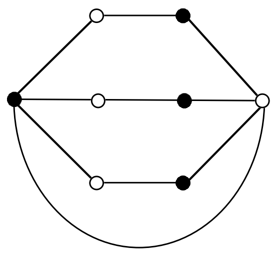

Figure 1:

In the literature the question has been studied whether for a given graph $G$ any subset $S$ of vertices of restricted size has some cycle passing through it. Many results on general graphs and graph classes are known (see, e.g., [2], [4], [5] [7], [8], [10], [12], [14], [15], [17]). For triangle-free graphs the following result has been proven.

**Theorem 2** ([13]). *Let $G$ be a triangle-free graph with $\delta \geq 2$. If $\sigma_4 \geq n+2$, then for any set $S$ of at most $\delta$ vertices, there exists a cycle $C$ containing $S$.*

In this paper, we show the following related theorem.

**Theorem 3.** *Let $G$ be a triangle-free graph with $\delta \geq 2$ and $\sigma_4 \geq n+2$. Let $S \subset V(G)$ consist of less than $\sigma_4/4+1$ vertices. If all vertices of $S$ have degree at least three, then there exists a cycle $C$ containing $S$.*

The several bounds in these theorems are all tight. We show this by a number of counter examples. For these counter examples we use the following notations. We denote the *complement* of graph $G = (V, E)$ by $\overline{G} = (V, (V \times V) \backslash E)$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we denote their *union* by $G_1 \cup G_2 =$

$(V_1 \cup V_2, E_1 \cup E_2)$ and their *join* by $G_1 * G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup (V_1 \times V_2))$. A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on $n$ vertices is denoted by $K_n$. The *complete bipartite* graph $\overline{K_k} * \overline{K_\ell}$ is denoted by $K_{k,\ell}$.

- Consider the graph $\overline{K_{k-1}} * \overline{K_k} * K_1 * \overline{K_k} * \overline{K_{k-1}}$ with $\delta = (n+1)/4$ and $\sigma_4 = n+1$. If we choose two vertices from each $\overline{K_k}$, obviously there is no cycle containing the vertices. See Figure 2(i). Hence, in Theorem 2 and Theorem 3, the lower

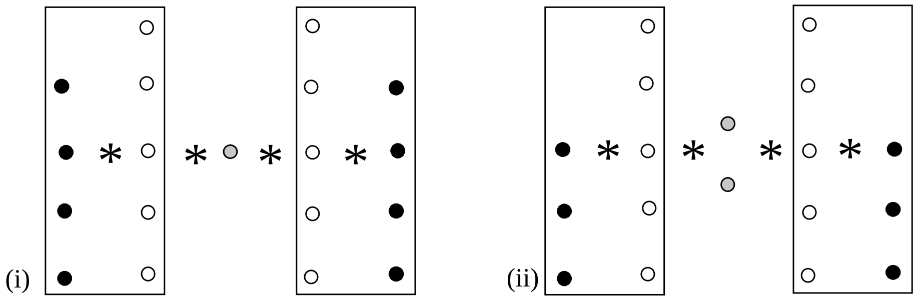


Figure 2:

bound on $\sigma_4$ is best possible.

- Consider the graph $\overline{K_{k-2}} * \overline{K_k} * \overline{K_2} * \overline{K_k} * \overline{K_{k-2}}$ with $\delta = (n+2)/4$ and $\sigma_4 = n+2$. There is no cycle containing all $k = (n+2)/4$ vertices of the left $\overline{K_k}$ and a vertex in the right $K_{k,k-2}$. See Figure 2(ii). Hence, in Theorem 2 and Theorem 3, the upper bound on $|S|$ is best possible.

We cannot relax the degree condition of vertices in $S$ in Theorem 3 into "all vertices of $S$ have degree at least two". For example in the graph in Figure 1, $\sigma_4/4 + 1 = 10/4 + 1$. So we can choose three vertices. However, if we choose the three white vertices of degree two in the graph, obviously there is no desired cycle. There is even a class of counter examples of large order as follows. Consider the graph $K_{k,k}$ for any $k \geq 3$. Let $x, x'$ be two vertices in the same partite set of this graph. Add two extra vertices $w, w'$ and add all edges between $\{x, x'\}$ and $\{w, w'\}$. This way we obtain a graph $G_k$ with $\sigma_4 = 2k + 4 = n + 2 \geq 10$. Now let $S \subset V(G_k)$ consist of the vertices $w, w'$ and some vertex $u$ not in $\{x, x'\}$. Then $|S| = 3 < 10/4 + 1 \leq \sigma_4/4 + 1$. However, the only cycle in $G_k$ that contains both $w$ and $w'$ is the cycle on the four vertices $x, x', w, w'$. This means that $G_k$ does not contain a cycle passing through $S$. We note that $S$ contains two vertices of degree two. The following conjecture seems to hold.

**Conjecture 4.** *Let $G$ be a triangle-free graph with $\delta \geq 2$ and $\sigma_4 \geq n+2$. Let $S \subset V(G)$ consist of less than $\sigma_4/4+1$ vertices. If $S$ contains at most one vertex of degree 2, then there exists a cycle $C$ containing $S$.*

Finally, we give some additional definitions and notations. The set of all the neighbours of a vertex $x \in V(G)$ is denoted by $N_G(x)$ or simply $N(x)$, and its cardinality by $d_G(x)$ or $d(x)$. For a subgraph $H$ of $G$, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. For simplicity, we denote $|V(H)|$ by $|H|$ and "$u_i \in V(H)$" by "$u_i \in H$". The set of neighbours $\bigcup_{v \in H} N_G(v) \setminus V(H)$ is written by $N_G(H)$ or $N(H)$, and for a subgraph $F \subset G$, $N_G(H) \cap V(F)$ is denoted by $N_F(H)$. Especially, for an edge $e = xy$, we denote $N(e) = (N(x) \cup N(y)) \backslash \{x, y\}$ and $d(e) = |N(e)|$.

Let $C = v_1 v_2 \ldots v_p v_1$ be a cycle with a fixed orientation. The segment $v_i v_{i+1} \ldots v_j$ is written by $v_i \overrightarrow{C} v_j$ where the subscripts are to be taken modulo $|C|$. The converse segment $v_j v_{j-1} \ldots v_i$ is written by $v_j \overleftarrow{C} v_i$. The successor of $u_i$ is denoted by $u_i^+$ and the predecessor by $u_i^-$. For a subset $A \subseteq V(C)$, we write $\{u_i^+ \mid u_i \in A\}$ and $\{u_i^- \mid u_i \in A\}$ by $A^+$ and $A^-$, respectively.

## 2 The Proof of Theorem 3

In the proof we make use of the following lemma. A cycle $C$ in a graph $G$ is called a *swaying* cycle of a subset $S \subseteq V(G)$ if $|C \cap S|$ is maximum in all cycles of $G$.

**Lemma 5.** *Let $G$ be a connected graph such that for any path $P$, there exists a cycle $C$ such that $|P - C| \leq 1$. Let $S \subset V(G)$. Then for any longest swaying cycle $C$ of $S$, $S \subset V(C)$ or $N(x) \subset C$ for any $x \in S - C$.*

*Proof.* Let $S \subset V(G)$ and $C$ a longest swaying cycle of $S$. Suppose $S - C \neq \emptyset$. For any vertex $x \in S - C$, there is a path $Q$ joining $x$ and $C$. Let $P$ be a longest path containing $V(C \cup Q)$. Then there exists a cycle $D$ such that $|P - D| \leq 1$. If $x$ has neighbours in $G - C$, then $|P| \geq |C| + 2$ and so $|D| \geq |C| + 1$. Because $|D \cap S| \geq |C \cap S|$, this contradicts the assumption that $C$ is a longest swaying cycle. Hence $N_{G-C}(x) = \emptyset$. □

Now let $G$ be a graph with $\delta \geq 2$ and $\sigma_4 \geq n+2$. Let $S \subset V(G)$ be a set of less than $\sigma_4/4+1$ vertices that all have degree at least three. Let $\mathcal{C}$ be the set of all longest swaying cycles of $S$. Suppose a cycle in $\mathcal{C}$ does not contain all vertices in $S$.

**Claim 1.** *If there exists a swaying cycle $D$ of $S$ and $v \in S - D$ such that $N(v) \subset V(D)$, then $d(v) \leq |D \cap S|$, and so $d(v) < \sigma_4/4$.*

*Proof.* If $d(v) > |D \cap S|$, then there exist $y, z \in N(v)$ such that $y^+ = z$ or $y^+ \overrightarrow{D} z^- \cap S = \emptyset$ because $N(v) \subset V(D)$. Then the cycle $yvz\overrightarrow{D}y$ contains $|D \cap S|+1$ vertices in $S$. This contradicts the assumption that $D$ is a swaying cycle. Hence $d(v) \leq |D \cap S| \leq |S|-1 < \sigma_4/4$. □

Note that our statement holds if $G$ is isomorphic to the graph in Figure 1. Hence Claim 1 together with Theorem 1 and Lemma 5 implies that

$$d(v) < \sigma_4/4 \text{ for any } D \in \mathcal{C} \text{ and } v \in S - D. \tag{1}$$

Let $C = u_1 u_2 \cdots u_{|C|} \in \mathcal{C}$ such that $\max\{d(v) \mid v \in S - C\}$ is maximum in $\mathcal{C}$, and let $x \in S - C$ such that $d(x)$ is maximum in $S - C$. Then $d(x) < \sigma_4/4$ by (1). Let $N(x) = \{u_{\tau(1)}, u_{\tau(2)}, \ldots, u_{\tau(d(x))}\}$ which occur on $C$ in the order of their indices. Then clearly:

$$N(x)^+ \text{ is an independent set;} \tag{2}$$

otherwise there is a cycle containing $|C \cap S|+1$ vertices of $S$. As $G$ is triangle-free, a vertex $u^+_{\tau(l)} \in N(x)^+$ is not adjacent to $x$. If $u^+_{\tau(l)}$ is adjacent to a vertex $y \in G - (C \cup x)$, then the order of the path $yu^+_{\tau(l)}\overrightarrow{C}u_{\tau(l)}x$ is $|C|+2$. By Theorem 1, there is a cycle $D'$ such that $|D' \cap S| \geq |C \cap S|$ and $|D'| \geq |C|+1$. This is a contradiction. Therefore:

$$N(u^+_{\tau(l)}) \subset V(C) \text{ for } u^+_{\tau(l)} \in N(x)^+. \tag{3}$$

Let $I_l = u^+_{\tau(l)}\overrightarrow{C}u_{\tau(l+1)}$ and $J_l = u^+_{\tau(l+1)}\overrightarrow{C}u_{\tau(l)}$ and:

$$L = \{u^+_{\tau(i)} \mid d(u^+_{\tau(i)}) \text{ is maximum in } N(x)^+\}.$$

Because $\sigma_4/4 > d(x) \geq 3$ and $N(x)^+ \cup x$ is an independent set, there is a vertex in $N(x)^+$ whose degree is at least $\sigma_4/4$. Hence the degree of a vertex in $L$ is greater than $\sigma_4/4$. If $u^{++}_{\tau(i)} \in L^+$ is adjacent to $u^+_{\tau(j)} \in (N(x) \setminus u_{\tau(i)})^+$, then the cycle

$u^{++}_{\tau(i)}u^{+}_{\tau(j)}\overrightarrow{C}u_{\tau(i)}xu_{\tau(j)}\overleftarrow{C}u^{++}_{\tau(i)}$ and $u^{+}_{\tau(i)} \in S$ contradict (1). If $u^{++}_{\tau(i)}x \in E(G)$, then the cycle $u_{\tau(i)}xu^{++}_{\tau(i)}\overrightarrow{C}u_{\tau(i)}$ and $u^{+}_{\tau(i)}$ contradict (1). Hence:

$$u^{++}_{\tau(i)} \in L^{+} \text{ is adjacent to none of } (N(x) \setminus u_{\tau(i)})^{+} \cup x. \tag{4}$$

For each $u^{+}_{\tau(l)} \in N(x)^{+}$, we denote the edge $u^{+}_{\tau(l)}u^{++}_{\tau(l)}$ by $e_l$.

**Claim 2.** *For any $u^{+}_{\tau(i)} \in L$, it holds that:*

1. $N_{I_i}(e_i)^{-} \cap N_{I_i}(u^{+}_{\tau(i+1)}) = \emptyset$.
2. $N_{J_i}(x)^{+} \cap N_{J_i}(e_i) = \emptyset$.
3. $N_{J_i}(e_i) \cap N_{J_i}(u^{+}_{\tau(i+1)})^{-} = \emptyset$.

*Proof.* Suppose there is a vertex $u_l \in N_{I_i}(e_i)^{-} \cap N_{I_i}(u^{+}_{\tau(i+1)})$, and let $y \in V(e_i) \cap N(u^{+}_{l})$. Then the cycle:

$$D = y\overrightarrow{C}u_l u^{+}_{\tau(i+1)}\overrightarrow{C}u_{\tau(i)}xu_{\tau(i+1)}\overleftarrow{C}u^{+}_{l}y$$

contains all vertices of $V(C) \cup x$ if $y = u^{+}_{\tau(i)}$, i.e, $|D| = |C \cap S| + 1$. See Figure 3(i). This contradicts the assumption that $C \in \mathcal{C}$. If $y = u^{++}_{\tau(i)}$, then $D \in \mathcal{C}$ and $d(u^{+}_{\tau(i)}) \geq$

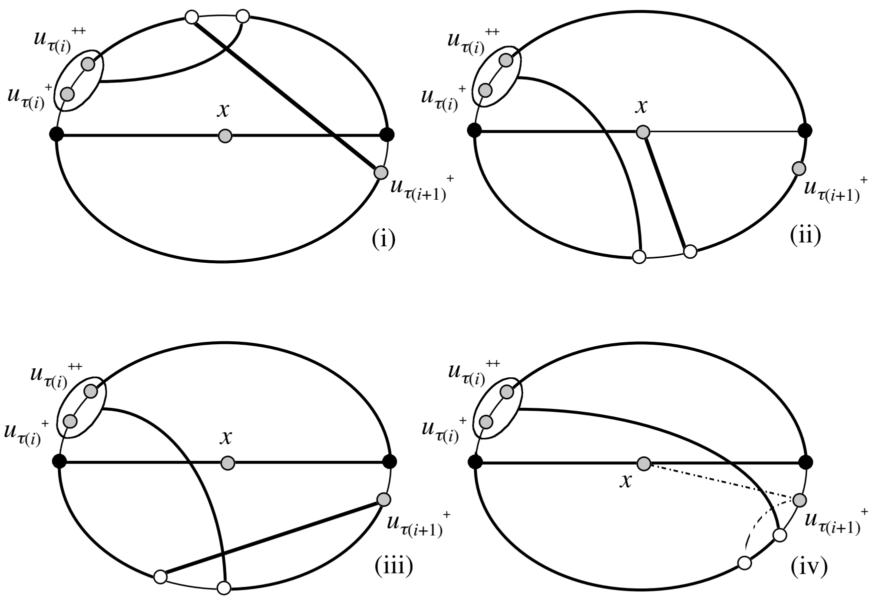


Figure 3:

$\sigma_4/4$. This contradicts (1). Hence $N_{I_i}(e_i)^{-} \cap N_{I_i}(u^{+}_{\tau(i+1)}) = \emptyset$. Similarly, we can show the other statements. See Figure 3(ii)-(iii). □

Let $\alpha_i = |N_{J_i}(x)^+ \cap N_{J_i}(u^+_{\tau(i+1)})^-|$. By this number, we will divide our argument into three cases, and in each case, the following claim will be used.

**Claim 3.** *For any $u^+_{\tau(i)} \in L$, $n \geq d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) - 2 - \alpha_i$. Especially if the equality holds, then $u^{++}_{\tau(i+1)} \in N(e_i)$ and $u^+_{\tau(i+1)} \in S$ and:*

$$J_i = (N_{J_i}(x) \setminus u_{\tau(i)})^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u^+_{\tau(i+1)})^-.$$

*Proof.* By the previous claim, we have:

$$\begin{aligned}
|I_i| &\geq |N_{I_i}(e_i)^- \cup N_{I_i}(u^+_{\tau(i+1)}) \cup \{u^+_{\tau(i)}\}| \\
&\geq |N_{I_i}(e_i)^-| + |N_{I_i}(u^+_{\tau(i+1)})| + |\{u^+_{\tau(i)}\}| \\
&= d_{I_i}(e_i) + d_{I_i}(u^+_{\tau(i+1)}) + 1 \\
|J_i| &\geq |(N_{J_i}(x) \setminus u_{\tau(i)})^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u^+_{\tau(i+1)})^-| \\
&\geq |(N_{J_i}(x) \setminus u_{\tau(i)})^+| + |N_{J_i}(e_i)| + |N_{J_i}(u^+_{\tau(i+1)})^-| - \alpha_i \\
&= d_{J_i}(x) - 1 + d_{J_i}(e_i) + d_{J_i}(u^+_{\tau(i+1)}) - \alpha_i.
\end{aligned}$$

Therefore:

$$\begin{aligned}
n &\geq |C| + d_{G-C}(e_i) + |\{x\}| = |I_i| + |J_i| + d_{G-C}(e_i) + 1 \\
&\geq (d_{I_i}(e_i) + d_{I_i}(u^+_{\tau(i+1)})) + (d_{J_i}(x) + d_{J_i}(e_i) + d_{J_i}(u^+_{\tau(i+1)}) - \alpha_i) + d_{G-C}(e_i) + 1 \\
&= (d_{I_i}(e_i) + d_{J_i}(e_i) + d_{G-C}(e_i)) + (d_{I_i}(u^+_{\tau(i+1)}) + d_{J_i}(u^+_{\tau(i+1)})) + (d_{J_i}(x) + 1) - \alpha_i \\
&= d(e_i) + d(u^+_{\tau(i+1)}) + d(x) - \alpha_i \\
&= d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) - 2 - \alpha_i.
\end{aligned}$$

If equalities hold in the above inequalities, then:

$$|J_i| = |(N_{J_i}(x) \setminus u_{\tau(i)})^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u^+_{\tau(i+1)})^-|$$

also holds and so:

$$J_i = (N_{J_i}(x) \setminus u_{\tau(i)})^+ \cup N_{J_i}(e_i) \cup N_{J_i}(u^+_{\tau(i+1)})^-.$$

Because $G$ is triangle-free, $u^{++}_{\tau(i+1)} \notin N(u^+_{\tau(i+1)})^-$ and $u^{++}_{\tau(i+1)} \notin N(x)^+$, and so $u^{++}_{\tau(i+1)} \in N(e_i)$.

Let $y = V(e_i) \cap N(u^{++}_{\tau(i+1)})$ and:

$$C' = y\overrightarrow{C}u_{\tau(i+1)}xu_{\tau(i)}\overleftarrow{C}u^{++}_{\tau(i+1)}y.$$

Suppose $u^+_{\tau(i+1)} \notin S$. Because $C'$ does not contain $|C \cap S| + 1$ vertices of $S$, we have $y \neq u^+_{\tau(i)}$ and $u^+_{\tau(i)} \in S$. Therefore $N(u^+_{\tau(i)}) \subset V(C')$ by (2) and (3). This contradicts Claim 1 as $u^+_{\tau(i)} \in L$. See Figure 3(iv). Hence $u^+_{\tau(i+1)} \in S$. □

For $u^+_{\tau(i)} \in L$, if the vertex $u^+_{\tau(i+1)}$ is adjacent to $u^{++}_{\tau(s)} \in (N_{J_i}(x) \setminus u_{\tau(i)})^{++}$, then the cycle

$$C' = u^+_{\tau(i+1)} u^{++}_{\tau(s)} \overrightarrow{C} u_{\tau(i+1)} x u_{\tau(s)} \overleftarrow{C} u^+_{\tau(i+1)}$$

is a longest swaying cycle of $S$. Hence $u^+_{\tau(s)} \in S$; otherwise $|C' \cap S| \geq |C \cap S| + 1$. Therefore from (1), it holds that:

$$u^+_{\tau(s)} \in S \text{ and } d(u^+_{\tau(s)}) < \sigma_4/4 \text{ for all } u^+_{\tau(s)} \in N_{J_i}(x)^+ \cap N_{J_i}(u^+_{\tau(i+1)})^-. \tag{5}$$

If there are three vertices in $N_{J_i}(x)^+ \cap N_{J_i}(u^+_{\tau(i+1)})^-$, then the three vertices and $x$ are independent by (2), however, the sum of these degrees are less than $\sigma_4$. Therefore, $\alpha_i \leq 2$. Now we divide our argument.

*Case 1. There is $u^+_{\tau(i)} \in L$ such that $\alpha_i = 1$.*

Let $\{u^+_{\tau(s)}\} = N_{J_i}(x)^+ \cap N_{J_i}(u^+_{\tau(i+1)})^-$. By (5), $d(u^+_{\tau(s)}) < \sigma_4/4 \leq d(u^+_{\tau(i)})$, and by (2) and (4), $\{u^{++}_{\tau(i)}, u^+_{\tau(i+1)}, u^+_{\tau(s)}, x\}$ is an independent set. Hence by Claim 3, it holds that:

$$\begin{aligned}
n &\geq d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) - 2 - 1 \\
&\geq d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(u^+_{\tau(s)}) + d(x) \\
&\quad + (d(u^+_{\tau(i)}) - d(u^+_{\tau(s)})) - 3 \\
&\geq \sigma_4 + (d(u^+_{\tau(i)}) - d(u^+_{\tau(s)})) - 3 \\
&\geq (n+2) + 1 - 3 = n.
\end{aligned}$$

Therefore all equalities have to hold in the above inequalities, and so we have

$$n = d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) - 3 \tag{6}$$

$$d(u^+_{\tau(i)}) = d(u^+_{\tau(s)}) + 1, \tag{7}$$

Because $u^+_{\tau(i+1)} u^{++}_{\tau(s)} \overrightarrow{C} u_{\tau(i+1)} x u_{\tau(s)} \overleftarrow{C} u^+_{\tau(i+1)} \in \mathcal{C}$, we have $d(x) \geq d(u^+_{\tau(s)})$ by the maximality of $d(x)$. Then $d(x) + 1 \geq d(u^+_{\tau(s)}) + 1 = d(u^+_{\tau(i)}) \geq \sigma_4/4$ by (7). On the other hand, $d(x) + 1 \leq |C \cap S| + 1 \leq |S| < \sigma_4/4 + 1$ by Claim 1. Thus:

$$\frac{\sigma_4}{4} \leq d(x) + 1 \leq |S| < \frac{\sigma_4}{4} + 1,$$

i.e., $|S| = d(x) + 1$. Therefore $|u^+_{\tau(l)}\overrightarrow{C}u^-_{\tau(l+1)} \cap S| = 1$ for all $l \le d(x)$; otherwise we can easily obtain a cycle containing $|C \cap S| + 1$ vertices of $S$ as in the proof of Claim 1. However, by (6) and Claim 3, $u^+_{\tau(i+1)} \in S$, and by (5), $u^+_{\tau(s)} \in S$, and hence:

$$u^{++}_{\tau(i+1)}\overrightarrow{C}u^-_{\tau(i+2)} \cap S = u^{++}_{\tau(s)}\overrightarrow{C}u^-_{\tau(s+1)} \cap S = \emptyset.$$

Then, the cycle $u^+_{\tau(i+1)}u^{++}_{\tau(s)}\overleftarrow{C}u_{\tau(i+2)}xu_{\tau(s+1)}\overrightarrow{C}u^+_{\tau(i+1)}$ contains $|C \cap S| + 1$ vertices in $S$. See Figure 4. This contradicts the assumption that $C$ is a swaying cycle.

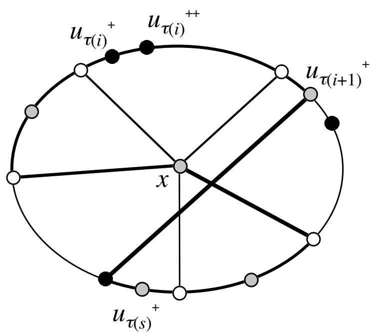


Figure 4:

*Case 2. There exists $u^+_{\tau(i)} \in L$ such that $\alpha_i = 2$.*

Let $\{u^{++}_{\tau(s)}, u^{++}_{\tau(t)}\} = N_{J_i}(x)^+ \cap N_{J_i}(u^+_{\tau(i+1)})^-$. By (2), $\{u^+_{\tau(s)}, u^+_{\tau(t)}, x, u^+_{\tau(i+1)}\}$ is an independent set. By (5), both of the degrees of $u^+_{\tau(s)}$ and $u^+_{\tau(t)}$ are less than $\sigma_4/4$, and so $d(u^+_{\tau(i+1)}) \ge \sigma_4/4$. Thus, it holds that:

$$d(u^+_{\tau(s)}) < \sigma_4/4 \le d(u^+_{\tau(i)}) \text{ and } d(u^+_{\tau(t)}) < \sigma_4/4 \le d(u^+_{\tau(i+1)}).$$

Therefore by Claim 3,

$$\begin{aligned}
n &\ge d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) - 2 - 2 \\
&\ge d(u^{++}_{\tau(i)}) + d(u^+_{\tau(s)}) + d(u^+_{\tau(t)}) + d(x) \\
&\quad + (d(u^+_{\tau(i)}) - d(u^+_{\tau(s)})) + (d(u^+_{\tau(i+1)}) - d(u^+_{\tau(t)})) - 4 \\
&\ge \sigma_4 + (d(u^+_{\tau(i)}) - d(u^+_{\tau(s)})) + (d(u^+_{\tau(i+1)}) - d(u^+_{\tau(t)})) - 4 \\
&\ge (n+2) + 1 + 1 - 4 = n
\end{aligned}$$

because $\{u^{++}_{\tau(i)}, u^+_{\tau(s)}, u^+_{\tau(t)}, x\}$ is an independent set by (2) and (4). Thus all equalities hold in the above inequalities, and we can use the same arguments as in Case 1.

*Case 3.* $\alpha_i = 0$ *for any* $u_{\tau(i)^+} \in L$.

For any $u^+_{\tau(s)} \in (N(x) \setminus \{u_{\tau(i)}, u_{\tau(i+1)}\})^+$,

$$\begin{aligned} n &\geq d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) - 2 \\ &\geq d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(u^+_{\tau(s)}) + d(x) \\ &\quad +(d(u^+_{\tau(i)}) - d(u^+_{\tau(s)})) - 2 \\ &\geq \sigma_4 + (d(u^+_{\tau(i)}) - d(u^+_{\tau(s)})) - 2 \\ &\geq (n+2) - 2 = n \end{aligned}$$

by Claim 3 because $\{u^{++}_{\tau(i)}, u^+_{\tau(i+1)}, u^+_{\tau(s)}, x\}$ is an independent set from (2) and (4). Therefore all equalities hold in the above inequalities, and so we have:

$$n = d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) - 2 = \sigma_4 - 2 \tag{8}$$

$$d(u^+_{\tau(i)}) = d(u^+_{\tau(s)}). \tag{9}$$

From (9), we obtain $u^+_{\tau(s)} \in L$, and so, by symmetry, $N(x)^+ \subset L$.

**Claim 4.** $u^{++}_{\tau(i)}$ *is adjacent to all of* $\{u^{++}_{\tau(s)} \mid s \neq i\}$.

*Proof.* By (8) and Claim 3, $u^{++}_{\tau(i+1)} \in N(e_i)$. Because $u^+_{\tau(i+1)} \in L$, $u^{++}_{\tau(i+1)}$ is not adjacent to $u^+_{\tau(i)}$ by (4). Hence $u^{++}_{\tau(i)} u^{++}_{\tau(i+1)} \in E(G)$.

Suppose the vertex $u^{++}_{\tau(i)}$ is not adjacent to $u^{++}_{\tau(s)}$ $(s \neq i, i+1)$. If $u^+_{\tau(i+1)} u^{+++}_{\tau(s)} \notin E(G)$, i.e., $u^{++}_{\tau(s)} \notin N(u^+_{\tau(i+1)})^-$, then $u^{++}_{\tau(s)} \in N(e_i)$ by (8) and Claim 3, and so $u^+_{\tau(i)} u^{++}_{\tau(s)} \in E(G)$. This contradicts (4) because $u^+_{\tau(s)} \in L$.

Assume $u^+_{\tau(i+1)} u^{+++}_{\tau(s)} \in E(G)$. By (4), (8) and (9) we have

$$\begin{aligned} &d(u^{++}_{\tau(s)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) \\ \geq\ &\sigma_4 \\ =\ &d(u^+_{\tau(i)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x) \\ =\ &d(u^+_{\tau(s)}) + d(u^{++}_{\tau(i)}) + d(u^+_{\tau(i+1)}) + d(x). \end{aligned}$$

Hence $d(u^{++}_{\tau(s)}) \geq d(u^+_{\tau(s)}) \geq \sigma_4/4$. Let:

$$D = u^+_{\tau(i+1)} \overrightarrow{C} u_{\tau(s)} x u_{\tau(i+1)} \overleftarrow{C} u^{+++}_{\tau(s)} u^+_{\tau(i+1)}.$$

By (3), $N(u^+_{\tau(s)}) \subset V(C)$. As $u^+_{\tau(s)} \in L$, the vertex $u^{++}_{\tau(s)}$ is not adjacent to $x$. If $u^{++}_{\tau(s)}$ is adjacent to the vertex $y \in G - C$, then the order of the path

$yu^{++}_{\tau(s)}\overleftarrow{C}u^{+}_{\tau(i+1)}u^{+++}_{\tau(s)}\overrightarrow{C}u_{\tau(i+1)}x$ is $|C|+2$. As in the proof of (3), this contradicts the assumption that $C \in \mathcal{C}$ by Theorem 1. Hence, we obtain $N(u^{++}_{\tau(s)}) \subset V(C)$. Thus $N(e_s) \subset V(D)$. Because $|D \cap S| \leq |C \cap S| < \sigma_4/4$,

$$d(e_s) \geq \sigma_4/2 - 2 \geq \sigma_4/4 > |S \cap D|.$$

Therefore, there exist vertices $y, z \in D \cap N(e_s)$ such that $y^+ = z$ or $y^+\overrightarrow{D}z^- \cap S = \emptyset$ and $y^+\overrightarrow{D}z^- \cap N(e_s) = \emptyset$. If $y$ and $z$ are adjacent to distinct ends of $e_s$, say $yu^+_{\tau(s)}, zu^{++}_{\tau(s)} \in E(G)$, then $yu^+_{\tau(s)}u^{++}_{\tau(s)}z\overrightarrow{D}y$ contains $|C \cap S|+1$ vertices of $S$. Hence, by symmetry, we may assume $u^+_{\tau(s)}$ is adjacent to both $y$ and $z$. Then the cycle $D' = yu^+_{\tau(s)}z\overrightarrow{D}y$ is a swaying cycle and $N(u^{++}_{\tau(s)}) \subset N(D')$. This contradicts Claim 1 because $d(u^{++}_{\tau(s)}) \geq \sigma_4/4$. □

By symmetry, the vertex $u^{++}_{\tau(i+1)}$ is adjacent to $u^{++}_{\tau(s)}$, and so there is the triangle $u^{++}_{\tau(i)}u^{++}_{\tau(i+1)}u^{++}_{\tau(s)}$. This is a contradiction.

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