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Even Subgraphs of Bridgeless Graphs and 2-Factors of Line Graphs

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Abstract

By Petersen's theorem, a bridgeless cubic multigraph has a 2-factor. H. Fleischner generalised this result to bridgeless multigraphs of minimum degree at least three by showing that every such multigraph has a spanning even subgraph. Our main result is that every bridgeless simple graph with minimum degree at least 3 has a spanning even subgraph in which every component has at least four vertices. We deduce that if G is a simple bridgeless graph with n vertices and minimum degree at least 3, then its line graph has a 2-factor with at most $\max\{1, (3n-4)/10\}$ components. This upper bound is best possible.

1 Introduction

All graphs considered are finite. We refer to graphs which and may contain loops and multiple edges as *multigraphs* and to graphs without loops and multiple edges as *simple graphs*. We denote the minimum degree of a graph G by $\delta(G)$. We refer to the number of vertices of G as the *order* of G and denote it by $|G|$. If no ambiguity can arise, we simply denote the order of G by n and the minimum degree by δ. We denote the number of components in G by $c(G)$ and the line graph of G by $L(G)$. A graph is said to be *even* if every vertex has positive even degree. All notation and terminology not explained in this paper is given in [3].

[1]This research was carried out while the second author was visiting Queen Mary, University of London.

Petersen [13] showed that every bridgeless cubic multigraph has a 2-factor. Fleischner [6] generalised this result to bridgeless multigraphs of minimum degree at least three by showing that every such graph has a spanning even subgraph. We extend these results in Section 3 for the special case of simple graphs by proving:

Theorem 1. *Every bridgeless simple graph G with $\delta \geq 3$ has a spanning even subgraph in which each component has order at least four.*

It is not true in general that every bridgeless multigraph with $\delta \geq 3$ has a spanning even subgraph in which every component has order at least four. Consider a bridgeless graph H with $\delta \geq 3$ which contains a 3-edge cut $\{e_1, e_2, e_3\}$, see Figure 1. Let G be obtained from H by inserting either a vertex incident to a loop, or

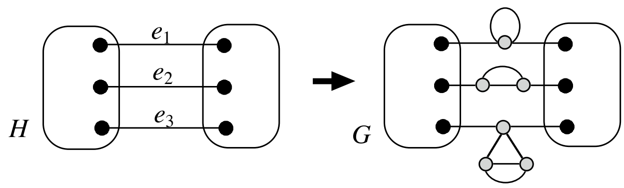

Figure 1:

two vertices joined by a mutiple edge, or a triangle with one edge replaced by a multiple edge, into each edge e_i, $1 \leq i \leq 3$. Then every spanning even subgraph of G contains at least one of the inserted loops, multiple edges, or triangles. We show in a forthcoming paper, however, that the conclusion of Theorem 1 holds in an even stronger form for 3-connected multigraphs.

Theorem 2 ([10])**.** *Every 3-connected multigraph on n vertices has a spanning even subgraph in which each component has order at least* $\min\{n, 5\}$.

Theorem 1 has the following immediate corollary.

Corollary 3. *Every bridgeless simple graph with $\delta \geq 3$ has a spanning even subgraph with at most $\lfloor n/4 \rfloor$ components.*

In the case that a bridgeless graph G has vertices of degree two, G does not necessarily have a spanning even subgraph. (Consider the graph obtained by subdividing the edges in a 3-edge-cut in a bridgeless graph.) We will show, however, that G has an even subgraph containing all vertices of degree at least three in G, and obtain an upper bound on the number of components of such a subgraph.

Lemma 4. *Let G be a bridgeless simple graph, $V_2(G)$ the set of vertices of degree 2 in G and S the set of all vertices in $V_2(G)$ whose neighbours are not adjacent. If G is not K_4, then G has an even subgraph X such that $V(G-X) \subset S$ and*

$$c(X) + \frac{|G-X|}{2} \leq \min\{\frac{n+|V_2(G)|}{4}, \frac{3n-4+2|V_2(G)|}{10}\}.$$

We use the above results to obtain upper bounds on the minimum number of components in a 2-factor of the line graph of a simple graph G with $\delta(G) \geq 3$. Chartrand and Wall [1] showed that if G is connected, then $L(L(G))$ is hamiltonian. Although $L(G)$ is not always hamiltonian, $L(G)$ does always have a 2-factor. This fact follows from the results of Egawa and Ota [4], Choudum and Paulraj [2], or Nishimura [12]. Fujisawa et al. [7] consider line graphs of graphs of minimum degree at least two, and their results imply that $L(G)$ has a 2-factor with at most $(3|G|-2)/8$ components. We use Lemma 4 to prove a stronger result in Section 4.

Theorem 5. *If G is a simple graph with $\delta \geq 3$, then $L(G)$ has a 2-factor with at most $\max\{1, (3|G|-4)/10\}$ components.*

This result resolves the case $\delta = 3$ of a conjecture from [7], which will be explained in Section 5. We also describe examples from [7] which show that the upper bound in Theorem 5 is in some sense best possible.

Line graphs are examples of claw-free graphs. There are several results concerning the minimum number of components in a 2-factor of a claw-free graph. Faudree et al. [5] showed that a simple claw-free graph G with $\delta \geq 4$ has a 2-factor with at most $6n/(\delta+2)-1$ components. Moreover, Gould and Jacobson [9] proved that if $\delta \geq (4n)^{\frac{2}{3}}$, then G has a 2-factor with at most n/δ components. In general the second upper bound is too strong. The second author gave examples of simple claw-free graphs in which every 2-factor contains more than n/δ components in [16]. In particular, he constructs a family $\{G_i\}$ of claw-free graphs with $\delta = 4$ such that

$$\frac{f_2(G_i)}{|G_i|} \to \frac{5}{18} \quad (|G_i| \to \infty),$$

where $f_2(G_i)$ is the minimum number of components in a 2-factor of G_i. We shall use Theorems 1 and 2 to show that highly connected claw-free graphs have 2-factors with fewer components.

Theorem 6. *Every 2-connected simple claw-free graph with $\delta \geq 4$ has a 2-factor with at most $(n+1)/4$ components.*

Theorem 7. *Every 3-connected simple claw-free graph with $\delta \geq 4$ has a 2-factor with at most $2n/15$ components.*

It is conceivable that every bridgeless simple claw-free graph with $\delta \geq 4$ has a 2-factor with at most $n/4$ components. The second named author proves a related result in [16]: if G is a simple claw-free graph with $\delta \geq 4$ and every edge of G lies in a triangle, then G has a 2-factor with at most $(n-1)/4$ components.

2 Notation and Preliminary Results

The set of all the neighbours of a vertex x in a graph G is denoted by $N_G(x)$, or simply $N(x)$, and its cardinality by $d_G(x)$, or $d(x)$. For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. The set of neighbours $\bigcup_{v \in H} N_G(v) \setminus V(H)$ is written by $N_G(H)$ or $N(H)$, and for a subgraph $F \subset G$, $N_G(H) \cap V(F)$ is denoted by $N_F(H)$. For simplicity, we denote $|V(H)|$ by $|H|$ and "$u_i \in V(H)$" by "$u_i \in H$" and "$G - V(H)$" by "$G - H$". The set of edges incident to a vertex v is denoted by $E(v)$. For a connected subgraph H of G, we denote by G/H the graph obtained from G by contracting every edge in H and use $[H]$ to denote the vertex of G/H corresponding to H. The set of all vertices of degree k in G is denoted by $V_k(G)$ and we put $V_{\geq k}(G) = \bigcup_{i \geq k} V_i(G)$. The maximum degree of G is denoted by $\Delta(G)$ and the minimum order of a component of G by $\sigma(G)$.

Given two distinct edges $e_1 = vx_1, e_2 = vx_2$ incident to a vertex v in a graph G, let $G_v^{e_1,e_2}$ be the graph obtained from $G - \{e_1, e_2\}$ by adding a new vertex v' and new edges $e_1' = x_1v', e_2' = x_2v'$. We say that $G_v^{e_1,e_2}$ has been obtained by *splitting* the vertex v. We will need the following elementary result on splitting in 2-edge-connected graphs.

Lemma 8. *Let G be a 2-edge-connected graph, $v \in V(G)$ with $d(v) \geq 4$ and $e_1 \in E(v)$. Then*
(a) there exists an edge $e_2 \in E(v) - e_1$ such that $G_v^{e_1,e_2}$ is 2-edge-connected.
(b) If $d(v) = 4$ then there exists at most one edge $e_3 \in E(v) - e_1$ such that $G_v^{e_1,e_3}$ is not 2-edge-connected.

Proof. Part (a) is well known, see for example [11] for a generalisation to k-edge-conectivity. To prove (b) we suppose that $E(v) = \{e_1, e_2, e_3, e_4\}$. Using (a) we may assume that $G_v^{e_1,e_2}$ is 2-edge-connected. Then there exist two edge-disjoint $v'v$-paths in $G_v^{e_1,e_2}$, say $P = v'x_1x_2 \dots x_rv$ and $Q = v'y_1y_2 \dots y_tv$. Without loss of generality $e_1' = v'x_1$, $e_2' = v'x_2$, $e_3 = x_rv$ and $e_4 = y_tv$. Then P and $Q' = v'y_ty_{t-1} \dots y_1v$ are two edge-disjoint $v'v$-paths in $G_v^{e_1,e_4}$. Hence $G_v^{e_1,e_4}$ is also 2-edge-connected. □

3 Even subgraphs

We need the following lemma to prove Theorem 1.

Lemma 9. *If G is a bridgeless cubic simple graph, then G has a triangle-free 2-factor.*

Proof. We proceed by contradiction. Choose a counterexample G with n as small as possible. Clearly G is 2-edge-connected. If G has no triangles, then the lemma holds by Petersen's Theorem. Thus G contains a triangle T. If $|N(T)| = 1$, then, since G is cubic and 2-edge-connected, G is isomorphic to K_4 and the lemma holds. If $|N(T)| = 3$, then $G' = G/T$ is still simple, bridgeless and cubic. By induction G' has a triangle-free 2-factor X'. It is easy to extend X' to the required triangle-free 2-factor of G. Thus for all triangles T in G, $|N(T)| = 2$. In this case, it is easy to see that no 2-factor of G can contain a triangle of G, and hence the lemma holds by Petersen's Theorem. $\square$

3.1 Proof of Theorem 1.

Our proof uses the vertex splitting operation to reduce to the cubic case. Note that when we apply this operation we must ensure that the new graph remains simple and bridgeless in order to apply Lemma 9.

Proof. Suppose the theorem is false and choose a counterexample G with $\Delta = \Delta(G)$ as small as possible and, subject to this condition, such that the number of vertices of degree Δ is as small as possible. Clearly G is connected and hence 2-edge-connected. We have $\Delta \geq 4$ by Lemma 9. Choose a vertex v of G with $d(v) = \Delta$. If $N(v)$ induces a complete subgraph in G then the facts that G is connected and has maximum degree Δ imply that G is complete, and hence hamiltonian. Thus we may choose edges $e = vw, f = vx \in E(v)$ such that $wx \notin E(G)$.

Claim 1. $\Delta = 4$.

Proof. Suppose that $\Delta \geq 5$. Let G_1 be the graph obtained from $G_v^{e,f}$ by suppressing v'. Then G_1 is simple. If G_1 is bridgeless then, by induction, G_1 has a spanning even subgraph X_1 with $\sigma(X_1) \geq 4$. Now X_1 readily gives rise to the required even subgraph of G. Thus G_1, and hence also $G_v^{e,f}$, contains a bridge e_0. Let H_1, H_2 be the components of $G_v^{e,f} - e_0$. Since G is 2-edge-connected, we may suppose that $w, x, v' \in V(H_1)$ and $v \in V(H_2)$. Relabelling w, x if necessary, we may suppose further that w is not incident with e_0. By Lemma 8, $G_v^{e,h}$ is 2-edge-connected for some $h = vz \in E_G(v)$. Then $z \in V(H_2)$ and hence $wz \notin E(G)$. Let G_1' be the graph obtained from $G_v^{e,h}$ by suppressing v'. We may apply induction to G_1' to obtain a contradiction, as above. $\square$

Claim 2. *For some $h \in E_G(v) - \{e\}$, the graph obtained from $G_v^{e,h}$ by adding the edge $v'v$ is both simple and bridgeless.*

Proof. Let G_2 be obtained from $G_v^{e,f}$ by adding the edge $v'v$. Since $wx \notin E(G)$, G_2 is simple. Suppose G_2 has a bridge e_0. Since G is 2-edge-connected, we must have $e_0 = v'v$, and hence $G_v^{e,f}$ is disconnected. Let H_1, H_2 be the components of $G_v^{e,f}$. Since G is 2-edge-connected, we may suppose that $w, x, v' \in V(H_1)$ and $v \in V(H_2)$. Choose $h = vz \in E_G(v)$ with $z \in V(H_2)$. By Lemma 8(b), $G_v^{e,h}$ is 2-edge-connected. Clearly $wz \notin E(G)$ and hence $G_v^{e,h}$ is also simple. $\square$

Relabelling f and h if necessary, we may assume that $G_2 = G_v^{e,f} + v'v$ is bridgeless and simple. Let $N(v) = \{w, x, y, z\}$. By induction G_2 has a spanning even subgraph X_2 with $\sigma(X_2) \geq 4$. If $v'v \notin E(X_2)$ then X_2 readily gives rise to the required even subgraph of G. Hence $v'v \in E(X_2)$. Let D be the component of X_2 which contains $v'v$. Since $X/v'v$ is a spanning even subgraph of G and G is a counterexample to the theorem, D must be a 4-cycle. Relabelling w and x, and y and z, if necessary, we may suppose that $T = D/vv' = vwyv$ is a triangle in G. Let H be the subgraph of G induced by $\{w, x, y, z\}$ and $\overline{H}$ be the complement of H.

Claim 3. *$\overline{H}$ has a 1-factor.*

Proof. Suppose not. Since $wx \notin E(G)$ we have $wx \in E(\overline{H})$. Since $\overline{H}$ has no 1-factor, we must have $yz \notin E(\overline{H})$, and hence $yz \in E(G)$. We also have $yw \in E(G)$ by the preceding paragraph.

Suppose $yx \in E(G)$. Then $d_G(y) = 4 = d_G(v)$ and the edge vy is a chord in the 4-cycle $vxywv$ of G. Thus $G - vw$ satisfies the hypotheses of the theorem. Applying induction we deduce that $G - vw$, and hence also G, contains the required even subgraph of G.

Thus $yx \notin E(G)$. Then $yx \in E(\overline{H})$. Since $\overline{H}$ has no 1-factor, we must have $wz \notin E(\overline{H})$, and hence $wz \in E(G)$. Hence $\{v, w, y, z\}$ induces a K_4 in G. Furthermore, since G is 2-edge-connected, some vertex $u \in \{w, y, z\}$ is adjacent to a vertex of $V(G) - \{v, w, y, z\}$ in G. Then v, u both have degree four in G and we may now apply induction to $G - vu$ as in the preceding paragraph. $\square$

Using Claim 3 and relabelling if necessary, we may suppose that $wx, yz \notin E(G)$.

By symmetry, we may suppose that this relabelling has also been done in such a way that $T = vwyv$ continues to be a triangle in G. Let G_3 be the graph obtained from $G_v^{e,f}$ by suppressing both v' and v.

Suppose G_3 is bridgeless. Then, by induction, G_3 has a spanning even subgraph X_3 with $\sigma(X_3) \geq 4$. Let X_3' be the even subgraph of G corresponding to X_3. If $v \in V(X_3')$ then X_3' is the required even subgraph of G. Hence $v \notin V(X_3')$. Then $E(T) \cap E(X_3') \subseteq \{wy\}$. Let $Z = X_3' \cup T$ if $wy \notin E(X_3')$ and $Z = (X_3' \cup T) - wy$ if $wy \in E(X_3')$. Then Z is the required even subgraph of G.

Thus G_3, and hence also $G_v^{e,f}$, has a bridge e_0. Let H_1, H_2 be the components of $G_v^{e,f} - e_0$. Since G is 2-edge-connected, we necessarily have $e_0 = wy$ and, relabelling if necessary, $w, x \in V(H_1)$ and $y, z \in V(H_2)$. Then $wz, xy \notin E(G)$. Let $h = vz$. By Lemma 8(b), $G_v^{e,h}$ is 2-edge-connected. We may now apply the argument in the preceding paragraph to $G_v^{e,h}$. □

3.2 Proof of Lemma 4

Proof. For an even subgraph X' of a graph G', let

$$\varphi(G', X') = c(X') + \frac{|G' - X'|}{2},$$

$$\psi_1(G') = \frac{|G'| + |V_2(G')|}{4} \quad \text{and} \quad \psi_2(G') = \frac{3|G'| - 4 + 2|V_2(G')|}{10}.$$

Note that $\psi_1(G') \leq \psi_2(G')$ whenever $|G'| - |V_2(G')| \geq 8$. Let $S(G')$ be the set of all vertices in $V_2(G')$ whose neighbours are not adjacent.

Suppose the lemma is false and choose a counterexample G such that n is as small as possible, and subject to this, $|V_2(G)|$ is as small as possible. Since G is not K_4, it is easy to check that $\psi_1(G), \psi_2(G) \geq 1$. Hence,

$$G \textit{ has no spanning connected even subgraph.} \tag{1}$$

Suppose $V_2(G) = \emptyset$. Then, by Corollary 3, there exists a spanning even subgraph X of G with at most $n/4$ components, and so $\varphi(G, X) \leq \psi_1(G)$. If $n \geq 8$, then $\psi_1(G) \leq \psi_2(G)$, and so $\varphi(G, X) \leq \psi_2(G)$ holds. If $n \leq 7$, then $c(X) = 1$, which contradicts (1). Hence

$$V_2(G) \neq \emptyset.$$

Also, if $n = 3$ or 4, then G is hamiltonian. So $n \geq 5$.

Claim 4. *G is connected.*

Proof. Suppose G is disconnected, and let G_1 be a component of G and $G_2 = G - G_1$. If there is no component isomorphic to K_4, then each G_j has an even subgraph X_j such that $G_j - X_j \subset S(G_j)$ and $\varphi(G_j, X_j) \leq \min\{\psi_1(G_j), \psi_2(G_j)\}$. Then,

$$G - (X_1 \cup X_2) = (G_1 - X_1) \cup (G_2 - X_2) \subset S(G_1) \cup S(G_2) = S(G)$$

and

$$\varphi(G, X_1 \cup X_2) = \varphi(G_1, X_1) + \varphi(G_2, X_2) \leq \psi_i(G_1) + \psi_i(G_2) \leq \psi_i(G),$$

for each $i \in \{1, 2\}$. This contradicts the choice of G.

Assume that G has a component isomorphic to K_4, say G_1. If G_2 is also isomorphic to K_4, then G has a 2-factor with two components and $\psi_i(G) = 2$ for each $i \in \{1, 2\}$, which contradicts the choice of G. Hence $G_2 \neq K_4$. In this case,

$$G - (D \cup X_2) = G_2 - X_2 \subset S(G_2) = S(G),$$

where D is a hamilton cycle of G_1. Hence

$$\begin{aligned} \varphi(G, D \cup X_2) = \varphi(G_2, X_2) + 1 &\leq \psi_1(G_2) + 1 = \frac{|G_2| + |V_2(G)|}{4} + 1 \\ &= \frac{(|G| - 4) + |V_2(G)|}{4} + 1 = \frac{|G| + |V_2(G)|}{4} = \psi_1(G) \end{aligned}$$

and

$$\begin{aligned} \varphi(G, D \cup X_2) = \varphi(G_2, X_2) + 1 &\leq \psi_2(G_2) + 1 = \frac{3(|G| - 4) - 4 + 2|V_2(G)|}{10} + 1 \\ &= \frac{3|G| - 4 + 2|V_2(G)| - 2}{10} < \psi_2(G), \end{aligned}$$

a contradiction. $\square$

Now we divide our argument into two cases.

Case 1. $S(G) \neq \emptyset$**.**

Let $u \in S(G)$ and $N(u) = \{x, y\}$. Since $xy \notin E(G)$, $G' = G/uy$ is simple. See Figure 2. As $d_{G'}(x) = d_G(x)$ and $d_{G'}(y) = d_G(y)$, $|V_2(G')| = |V_2(G)| - 1$ and

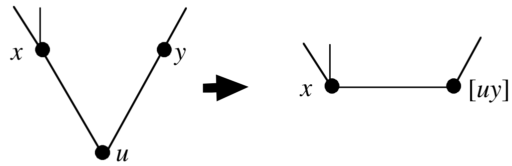

Figure 2:

$|G'| = n - 1 \geq 4$. Therefore,

$$\psi_1(G') + \frac{1}{2} = \psi_1(G) \text{ and } \psi_2(G') + \frac{1}{2} = \psi_2(G).$$

If $G' = K_4$, then G is hamiltonian, which contradicts (1). Thus $G' \neq K_4$. By induction, G' has an even subgraph X' such that

$$G' - X' \subset S(G') \text{ and } \varphi(G', X') \leq \psi_i(X') \text{ for each } i \in \{1, 2\}.$$

Suppose the edge $x[uy]$ is used by X'. Then $X = (X' \setminus \{x[uy]\}) \cup \{xuy\}$ is an even subgraph of G such that $c(X) = c(X')$ and $G - X = G' - X' \subset S(G') \subset S(G)$, and so $\varphi(G, X) = \varphi(G', X')$. Hence,

$$\varphi(G, X) = \varphi(G', X') < \psi_i(G') + \frac{1}{2} = \psi_i(G),$$

for each $i \in \{1, 2\}$. This contradict the choice of G.

Thus $x[uy] \notin E(X')$. Then $X = X'$ is an even subgraph of G such that $c(X) = c(X')$ and $G - X = (G' - X') \cup \{u\} \subset S(G') \cup \{u\} = S(G)$, and so $\varphi(G, X) = \varphi(G', X') + 1/2$. Therefore for each $i \in \{1, 2\}$,

$$\varphi(G, X) = \varphi(G', X') + \frac{1}{2} \leq \psi_i(G') + \frac{1}{2} = \psi_i(G),$$

a contradiction.

Case 2. $S(G) = \emptyset$.

Let $u \in V_2(G)$ and $N(u) = \{x, y\}$. By symmetry, we may assume $d(x) \leq d(y)$. Consider the following subcases.

1. $N(x) \cap N(y) = \{u\}$**.**

In this subcase, $G' = G/xuy$ is simple. See Figure 3. Because $|G'| = n - 2 \geq 3$ and

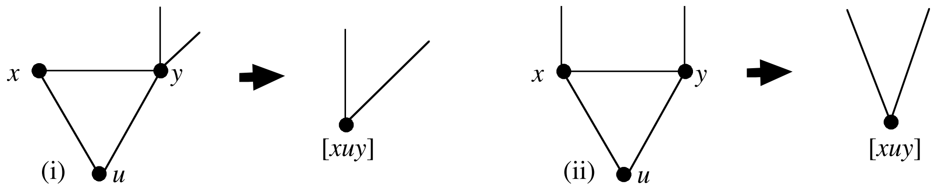

Figure 3:

$|V_2(G')| \leq |V_2(G)|$,

$$\psi_1(G') + \frac{1}{2} \leq \psi_1(G) \text{ and } \psi_2(G') + \frac{3}{5} \leq \psi_2(G).$$

If $G' = K_4$, then G' has a hamilton cycle D such that $D \cup xuyx$ is a spanning connected even subgraph of G. This contradicts (1). Thus $G' \neq K_4$. Then G' has an even subgraph X' such that $G' - X' \subset S(G') \subset \{[xuy]\}$ and $\varphi(G', X') \leq \psi_i(G')$ for each $i \in \{1, 2\}$.

Suppose $[xuy] \notin X'$. Then the triangle $xuyx$ is a new component in the even subgraph $X = X' \cup xuyx$ of G, and so $c(X) = c(X') + 1$ and $|G - X| = |(G' - X') \setminus \{[xuy]\}| = |G' - X'| - 1$. Hence the following inequalities hold for each $i \in \{1, 2\}$.

$$\varphi(G, X) \leq \varphi(G', X') + \frac{1}{2} \leq \psi_i(G') + \frac{1}{2} \leq \psi_i(G). \tag{2}$$

This contradicts the choice of G.

Thus $[xuy] \in X'$, i.e., X' is spanning G'. If there are an even number of edges of X' incident to x in G, let $X = X' \cup xuyx$. On the other hand, if there are an odd number of edges of X' incident to x in G, then let $X = X' \cup xuy$. In both cases X is a spanning even subgraph of G, $c(X) = c(X')$ and $G - X = G' - X' = \emptyset$. Hence $\varphi(G, X) = \varphi(G', X') \leq \psi_i(G') \leq \psi_i(G)$ for each $i \in \{1, 2\}$.

2. $N(x) \cap N(y) \neq \{u\}$**.**

Let $z \in (N(x) \cap N(y)) \setminus \{u\}$.

Suppose $d(y) \geq 4$, and let $G' = G - u$. See Figure 4(i). Then $V_2(G') \subset$

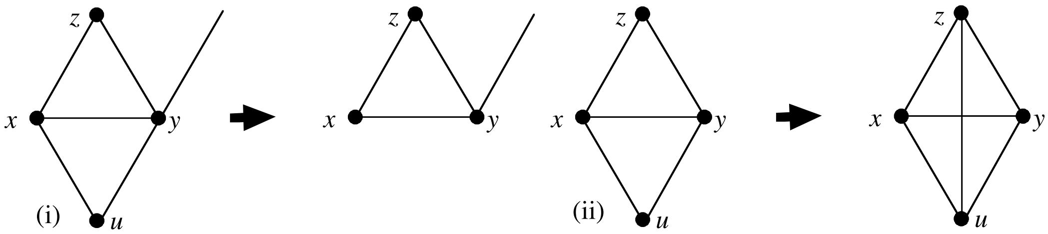

Figure 4:

$(V_2(G) \setminus \{u\}) \cup \{x\}$, and so $|V_2(G')| \le |V_2(G)|$ and $|G'| = n - 1$. Thus

$$\psi_1(G') + \frac{1}{4} \le \psi_1(G) \quad \text{and} \quad \psi_2(G') + \frac{3}{10} \le \psi_2(G).$$

Because $xz, yz \in E(G')$, $x, y \notin S(G')$ and so $S(G') = S(G) = \emptyset$. If $G' = K_4$, then G has a hamilton cycle, which contradicts (1). Thus $G' \ne K_4$ and there exists a spanning even subgraph X' of G' such that $\varphi(G', X') \le \psi_i(G')$ for each $i \le 2$. If $xy \in E(X')$, then let $X = (X' \setminus \{xy\}) \cup xuy$. Otherwise let $X = X' \cup xuyx$. In either case, X is a spanning even subgraph of G and $c(X) = c(X')$. Hence

$$\varphi(G, X) = \varphi(G', X') \le \psi_i(G') < \psi_i(G)$$

for each $i \in \{1, 2\}$.

Thus $d(y) = 3$. Since $d(x) \le d(y)$ we have $d(x) = 3$. This implies that z is a cut vertex of G. Let $G' = G \cup \{uz\}$. Then $|G'| = n$ and $|V_2(G')| + 1 = |V_2(G)|$ and so

$$\psi_1(G') + \frac{1}{4} = \psi_1(G) \quad \text{and} \quad \psi_2(G') + \frac{1}{5} = \psi_2(G).$$

See Figure 4(ii). Since $|V_2(G')| < |V_2(G)|$ and $S(G') = S(G) = \emptyset$, by our assumption, G' has a spanning even subgraph X' such that $\varphi(G', X') \le \psi_i(G')$ for each $i \in \{1, 2\}$.

Suppose $zu \notin E(X')$. Then $X = X'$ is a spanning even subgraph of G such that

$$\varphi(G, X) = \varphi(G', X') \le \psi_i(G') < \psi_i(G), \tag{3}$$

for each $i \in \{1, 2\}$.

Thus $zu \in E(X')$. By symmetry, we may assume $xu \in E(X')$, and then $xy \in E(X')$; otherwise $y \notin X'$. Hence, $yz \in E(X')$, $X = (X' \setminus \{zu, xy\}) \cup \{xz, uy\}$ is a spanning even subgraph of G, and the inequalities (3) hold. This completes the proof of Lemma 4. $\square$

4 2-factors in line graphs

4.1 Proof of Theorem 5

Let G be a simple graph with $\delta \geq 3$ and let $\mathcal{S}$ be a set of mutually edge-disjoint connected even subgraphs and stars. If each star has at least three edges and every edge in $E(G) \setminus \bigcup_{L \in \mathcal{S}} E(L)$ is incident to an even subgraph in $\mathcal{S}$, then $\mathcal{S}$ is called a *system that dominates* G.

We shall use the following result of Gould and Hynds.

Lemma 10. *[8] Let G be a simple graph. Then $L(G)$ has a 2-factor with c components if and only if there is a system that dominates G with c elements.*

Let X be an even subgraph in G such that $G - X$ is a forest. Let H be obtained from $G - X$ by deleting all its isolated vertices and let (A, B) be a bipartition of H with $|A| \leq |B|$. For $v \in A$, let $St(v)$ be the star with edge set $\{uv \ : \ u \in N_G(v)\}$. Let $\mathcal{S}$ be the set whose elements are each of the components of X and each of the stars $St(v)$ for $v \in A$. Then $\mathcal{S}$ is a system that dominates G and $|\mathcal{S}| \leq c(X) + |G - X|/2$. Therefore Theorem 5 is an easy consequence of the following lemma because if we choose an even subgraph X of G such that $c(X) + |G - X|/2$ is as small as possible, then $G - X$ must be a forest.

Lemma 11. *If G is a simple graph with $\delta \geq 3$, then $G \simeq K_4$ or G has an even subgraph X such that*

$$c(X) + \frac{|G - X|}{2} \leq \frac{3n - 4}{10}.$$

Proof. Let $\varphi(G, X) = c(X) + |G - X|/2$. We suppose the lemma is false and choose a counterexample G such that n is as small as possible. As in the proof of Lemma 4, we can see that G is connected.

Suppose G is bridgeless. Then, by Corollary 3, G has a spanning even subgraph X with $c(X) \leq n/4$. If $n \geq 8$ then $n/4 \leq (3n - 4)/10$ so we must have $n \leq 7$. This is impossible since any such G has a spanning connected even subgraph. Thus G has a bridge.

Claim 5. *For any bridge e, one of the components of $G - e$ is isomorphic to K_4.*

Proof. Let G_1, G_2 be the components of $G - e$ and $e = u_1u_2$ and $u_i \in G_i$. Suppose neither G_1 nor G_2 is K_4. Let $L_i \simeq K_4$ and $x_i \in L_i$ for each $i \in \{1, 2\}$. Notice that $|G_1|, |G_2| \geq 5$ as $\delta \geq 3$. Then the minimum degree of the graph

$$G_i' = G_i \cup \{u_ix_i\} \cup L_i$$

is at least three and $|G_i'| = |G_i| + 4 \leq n - 1$. Hence, by induction, G_i' has an even subgraph X_i' such that:

$$\varphi(G_i', X_i') \leq \frac{3(|G_i| + 4) - 4}{10}$$

for each $i \leq 2$. Choosing X_i' such that $\varphi(G_i', X_i')$ is smallest, X_i' must contain a hamilton cycle D_i of L_i as a component. Let $X_i = X_i' - D_i$. Then, $X_1 \cup X_2$ is an even subgraph of G and $\varphi(G, X_1 \cup X_2)$ is at most

$$\frac{3(|G_1| + 4) - 4}{10} - 1 + \frac{3(|G_2| + 4) - 4}{10} - 1 = \frac{3(|G_1| + |G_2|) - 4}{10} = \frac{3n - 4}{10}.$$

This contradicts the choice of G, and so at least one of G_1 and G_2 is K_4. □

Let $e^1, \ldots, e^t$ be the all bridges in G, let G_1^i, G_2^i be the components $G - e^i$, and let $e^i = u_1^iu_2^i$. We may assume that $u_2^i \in G_2^i \simeq K_4$ for all $1 \leq i \leq t$, and let D^i be a hamilton cycle of G_2^i. Then, $H = G - \bigcup_{i=1}^t G_2^i$ is bridgeless since a bridge of H is a bridge of G as well. Clearly $H \neq K_4$; otherwise G has a 2-factor without triangles, which contradicts our assumption that G is a counterexample.

Suppose $|H| \geq 3$. Then by Lemma 4 there exists an even subgraph X' in H such that

$$\varphi(H, X') \leq \frac{3|H| - 4 + 2|V_2(H)|}{10}.$$

Then $X = X' \cup \bigcup_{i=1}^t D^i$ is an even subgraph of H. Since $|V_2(H)| \leq t$ we have

$$\varphi(G, X) \leq \frac{3|H| - 4 + 2|V_2(H)|}{10} + t \leq \frac{3(n - 4t) - 4 + 2t + 10t}{10} = \frac{3n - 4}{10}.$$

This contradicts the choice of G.

Thus $|H| \le 2$. Because H is bridgeless, $|H| \ne 2$. Hence $|H| = 1$ and $G - X = H = K_1$. Thus $n = 4t + 1$ and, since $\delta \ge 3$, $t \ge 3$. Then $X = \bigcup_{i=1}^{t} D^i$ is an even subgraph of G and

$$\varphi(G, X) = t + \frac{1}{2} \le \frac{12t - 1}{10} = \frac{3(4t + 1) - 4}{10} = \frac{3n - 4}{10},$$

a contradiction. □

4.2 Proofs of Theorems 6 and 7

Let G be a claw-free graph. For each vertex x of G, $N_G(x)$ induces a subgraph with at most two components. Furthermore, if this subgraph has two components, both of them must be cliques. In the case that the subgraph induced by $N(x)$ is connected, we add edges joining all pairs of nonadjacent vertices in $N(x)$. This operation is called *local completion* of G at x. The *closure* $cl(G)$ of G is a graph obtained by recursively repeating the local completion operation, as long as this is possible. Ryjáček [14] showed that the closure of G is uniquely determined and G is hamiltonian if and only if $cl(G)$ is hamiltonian. The latter result was extended to 2-factor as follows.

Theorem 12 (Ryjáček, Saito and Shelp [15]). *Let G be a claw-free graph. If $cl(G)$ has a 2-factor with k components, then G has a 2-factor with at most k components.*

Since G is a spanning subgraph of $cl(G)$, Theorem 12 implies that

$$f_2(G) = f_2(cl(G)),$$

where $f_2(G)$ is the minimum number of components in a 2-factor of G. Ryjáček also proved:

Theorem 13 (Ryjáček [14]). *If G is a claw-free graph, then there is a triangle-free simple graph H such that*

$$L(H) = cl(G).$$

Theorems 12 and 13 imply that we can obtain general upper bounds on the mimimum number of components in a 2-factor of claw-free graphs by considering the special case of line graphs of triangle-free simple graphs.

A graph H is *essentially* k-edge-connected if for any edge set E_0 of at most $k-1$ edges, $H \setminus E_0$ contains at most one component with edges. The *edge-degree* of an edge xy is defined as $d(x) + d(y) - 2$. Clearly $L(H)$ is k-connected if and only if H is essentially k-edge-connected and the minimum degree of $L(H)$ is at least four if and only if the minimum edge-degree of H is at least four. Thus, for Theorem 6, it is sufficent to prove the following.

Lemma 14. *Let H be an essentially 2-edge-connected triangle-free simple graph with minimum edge-degree at least four. Then there is a set $\mathcal{T}$ of mutually vertex-disjoint even subgraphs and stars such that*

1. *every component in $\mathcal{T}$ contains at least four vertices.*
2. *every edge in $E(H) \setminus \bigcup_{L \in \mathcal{T}} E(L)$ is incident to an even subgraph or the central vertex of some star in $\mathcal{T}$.*

Clearly the cardinality of $\mathcal{T}$ in this lemma is at most $|H|/4$. We may modify $\mathcal{T}$ to create a system which dominates H as follows. For each $e \in E(H) \setminus \bigcup_{L \in \mathcal{T}} E(L)$ which is not incident with an even subgraph in $\mathcal{T}$ choose a star $S \in \mathcal{T}$ with e incident to the central vertex of S, and add e to S. The resulting system $\mathcal{S}$ has $|\mathcal{S}| = |\mathcal{T}| \leq |H|/4$. Thus $L(H)$ has a 2-factor with at most $|H|/4$ components by Lemma 10. As $|L(H)| = |E(H)| \geq |H| - 1$, we obtain Theorem 6.

Proof of Lemma 14. Since H is essentially 2-edge-connected, $F = H - V_1(H)$ is bridgeless. Hence, by Lemma 4, there exists an even subgraph X such that $V(F - X) \subset V_2(F)$. Let

$$S = V_2(F) \cap N_H(V_1(H)).$$

Since the minimum edge-degree in H is at least four, $d_H(x) \geq 5$ for all $x \in S$. Thus $|N_H(x) \cap V_1(H)| \geq 3$ for all $x \in S$. We also have

$$V_2(F) \setminus S = V_2(H).$$

For each $x \in S$, let $St^*(x)$ be the star with edge-set $\{xv \ : \ v \in N(x) \cap V_1(H)\}$, and let $\mathcal{T}$ be the set whose elements are each component of X and each star $St^*(x)$ for $x \in S$. As H is triangle-free, $\mathcal{T}$ satisfies condition 1 of our lemma.

Since $V_{\geq 3}(H) = V_{\geq 3}(F) \cup S \subset V(X) \cup S$, we have

$$H - (V(X) \cup S) \subset V_1(H) \cup V_2(H).$$

Clearly the subgraph induced by $V_1(H) \cup V_2(H)$ has no edges; otherwise there is an edge of edge-degree at most two. Hence $\mathcal{T}$ also satisfies condition 2 of the lemma. $\square$

Similarly, for Theorem 7, it is sufficent to prove the following.

Lemma 15. *Every essentially 3-edge-connected graph H contains a dominating even subgraph X such that $V_3(H) \subset V(X)$ and $c(X) \leq \max\{1, 2|E(H)|/15\}$.*

Proof. We proceed by contradiction. Choose a counterexample H with $|H|$ as small as possible. Suppose $V_1(H) \neq \emptyset$, and let $u \in V_1(H)$ and $v \in N(u)$. Since $H - u$ is essentially 3-edge-connected, $H - u$ has a dominating even subgraph X' with $V_3(H - u) \subset V(X')$ and $c(X') \leq \max\{1, 2(|E(H)| - 1)/15\}$. As H is essentially 3-edge-connected, $d_{H-u}(v) \geq 3$, and so $v \in X'$. Thus X' is the required subgraph of H. This contradicts the choice of H. Hence $V_1(H) = \emptyset$.

Suppose $V_2(H) \neq \emptyset$, and let $u \in V_2(H)$. The graph H' obtained from H by suppressing u has a desired subgraph X'. Since H is essentially 3-edge-connected, the degree of a neighbour of u is at least three. Hence as in the above case, X' is the required subgraph of H, and hence $\delta(H) \geq 3$. Then H is 3-edge-connected graph, and so by Theorem 2, H has a spanning even subgraph X with $\sigma(H) \geq \max\{|H|, 5\}$. This implies $c(X) \leq \max\{1, n/5\}$. Since $3|H|/2 \leq |E(H)|$, we obtain $c(X) \leq \max\{1, 2|E(H)|/15\}$. $\square$

5 Closing Remarks

The following example shows that the upper bound in Theorem 5 is, in some sense, best possible. Let P_{2m} be a path of length $2m - 1$. We add $2m + 2$ edges to

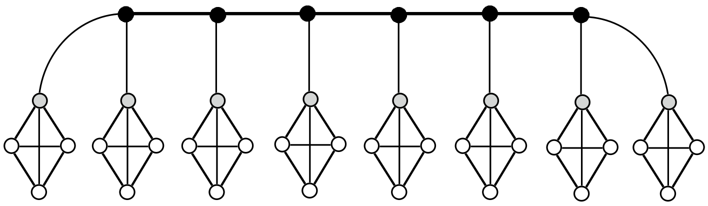

Figure 5:

$P_{2m} \cup (2m+2)K_4$. Figure 5 is the example when $m = 3$. Let $H_{2m,4}$ be the resultant graph. Then $n = |H_{2m,4}| = 10m+8$ and so $m = (n-8)/10$. It is easy to see that every system which dominates $H_{2m,4}$ has at least $3m+2$ elements. Thus, by Lemma 10, the number of components in a 2-factor of $L(H_{2m,4})$ is at least $3m+2 = (3n-4)/10$.

Fujisawa et al. [7] conjectured that if G is a simple graph with minimum degree $\delta \geq 3$, then its line graph has a 2-factor X such that $c(X) \leq \frac{(2\delta-3)n}{2(\delta^2-\delta-1)}$. If we allow an exception graph, the following conjectured upper bound would be sharper.

Conjecture 16. *If G is a simple graph with minimum degree $\delta \geq 3$, then $G \simeq K_{d+1}$ or $L(G)$ has a 2-factor X such that*

$$c(X) \leq \frac{(2\delta-3)n - 2\delta + 2}{2(\delta^2 - \delta - 1)}.$$

Theorem 5 resolves the case $d = 3$ of this conjecture. If true Conjecture 16 would be in some sense best possible. Consider the graph, given in [7], obtained from $H_{2m,4}$ by replacing each K_4 adjacent to internal vertices of P_{2m} by $(d-2)K_{d+1}$ and by replacing each $2K_4$ adjacent to the ends by $(d-1)K_{d+1}$, see Figure 6.

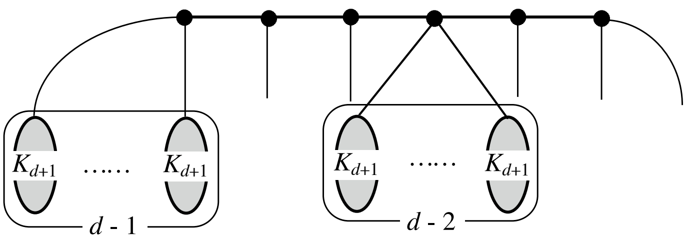

Figure 6:

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