A relationship between Thomassen's conjecture and Bondy's conjecture

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Abstract

In 1986, Thomassen posed the following conjecture; every 4-connected line graph has a Hamiltonian cycle. As a possible approach to the conjecture, many researchers have considered statements that are equivalent or related to it. One of them is the conjecture by Bondy; there exists a constant c_0 with $0 < c_0 \leq 1$ such that every cyclically 4-edge-connected cubic graph H has a cycle of length at least $c_0|V(H)|$. It is known that Thomassen's conjecture implies Bondy's conjecture, but nothing about the converse has been shown. In this paper, we show that Bondy's conjecture implies a slightly weaker version of Thomassen's

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This work was partially supported by Grant CE-ITI P202/12/G061 of Grant Agency of Czech Republic.

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This work was supported by JSPS KAKENHI Grant Number 22540152.

conjecture; every 4-connected line graph with minimum degree at least 5 has a Hamiltonian cycle.

Keywords: Hamiltonian cycles, line graphs, Thomassen's conjecture, dominating cycles, Bondy's conjecture,

1 Introduction

The motivation of this paper is the following well-known conjecture due to Thomassen.

Conjecture 1 (Thomassen [18]) *Every 4-connected line graph has a Hamiltonian cycle.*

As a possible approach to Conjecture 1, many researchers have considered statements that are equivalent or related to it. For example, Ryjáček [16] showed that Conjecture 1 is equivalent to the conjecture by Matthews and Sumner [15] stating that every 4-connected claw-free graph has a Hamiltonian cycle. The conjecture by Ash and Jackson [1] stating that every cyclically 4-edge-connected cubic graph has a dominating cycle, is also known to be equivalent to Conjecture 1, see the paper by Fleischner and Jackson [8]. Recall that for an integer k, a graph G is called *cyclically k-edge-connected* if deleting any $k-1$ edges from G does not create two components having a cycle. A *dominating* cycle C of a graph G is one such that for any edge e of G, at least one of the end vertices of e is contained in C. See [3, 7, 13, 14, 17] for other results and conjectures, and also a survey [5].

In addition to those conjectures that are equivalent to Conjecture 1, it is known that Conjecture 1 implies the following conjecture.

Conjecture 2 (Bondy, see [8]) *There exists a constant c_0 with $0 < c_0 \leq 1$ such that every cyclically 4-edge-connected cubic graph H has a cycle of length at least $c_0|V(H)|$.*

Proposition 3 *If Conjecture 1 is true, then Conjecture 2 is also true.*

For the proof of Proposition 3, see Section 3 in a survey [5]. Indeed, we can prove Proposition 3, by combining the argument on the relation between line graphs and preimage graphs (see Section 2 in this paper), the result in [8] (see Theorem 7 in this paper), and an observation that every dominating cycle in a cubic graph G has length at least $\frac{3}{4}|V(G)|$. (Recall that for a line graph G of a graph H, H is called the *preimage* graph of G, or sometimes called the *root* graph of G.) Then if Conjecture 1 is true, then Conjecture 2 is also true with $c_0 = \frac{3}{4}$.

Although Proposition 3 holds, nothing about the converse has been shown. In fact, the converse of Proposition 3 was posed in [5] as an open problem. In this paper, we focus on this problem, and show that Conjecture 2 implies the following conjecture, which is a slightly weaker version of Conjecture 1.

Conjecture 4 *Every 4-connected line graph with minimum degree at least 5 has a Hamiltonian cycle.*

Theorem 5 *If Conjecture 2 is true, then Conjecture 4 is also true.*

Zhan [20], and independently Jackson [11] proved that every 7-connected line graph has a Hamiltonian cycle, and several researchers [10, 19] have shown results on Hamiltonicity of 6-connected line graphs with additional conditions on the set of vertices of degree exactly 6. Recently, Kaiser and Vrána [12] improved this result by showing that every 5-connected line graph with minimum degree at least 6 has a Hamiltonian cycle. Theorem 5 suggests that Conjecture 2 is at least more difficult than the result by Kaiser and Vrána [12].

This paper is organized as follows. In Section 2, we give several definitions and lemmas for the proof of Theorem 5. The proof of Theorem 5 appears in Section 3, and is divided into two theorems (Theorems 11 and 12). We will prove them in Sections 4 and 5, respectively. In the last section of this paper (Section 6), we give a conclusion and open problems concerning Theorem 5.

2 Preliminaries

In this paper, we consider only finite graphs that may have multiple edges, but no loops. For terminology and notation not defined in this paper, we refer readers to [6].

Let H be a graph. A closed trail T in H is called a *dominating closed trail* in H if for any edge e of H, at least one of the end vertices of e is contained in T. (Note that in case that T is a cycle, we call T a *dominating cycle*.) In [9], it is shown that for a connected graph H with $|E(H)| \geq 3$, H has a dominating closed trail if and only if the line graph of H has a Hamiltonian cycle. For a closed trail T in a graph H, $\mathrm{dom}_H(T)$ denotes the number of edges e in H such that at least one of the end vertices of e is contained in T. Specifically, T is a dominating closed trail in H if and only if $\mathrm{dom}_H(T) = |E(H)|$.

An *edge-cut* of a graph H is an inclusionwise minimal set of edges whose removal makes H disconnected. An *essential edge-cut* X (resp. *cyclic edge-cut*) of a graph H is an edge-cut such that $H - X$ has exactly two components of orders at least 2 (resp. exactly two components having a cycle). For a positive integer k, a graph H is called *essentially k-edge-connected* if H has no essential edge-cut X with $|X| \leq k - 1$. It is known that a graph H is essentially k-edge-connected if and only if the line graph of H is k-connected or H is a complete graph, see Section 3 in [5]. Recall that a graph H is called *cyclically k-edge-connected* if H has no cyclic edge-cut X with $|X| \leq k-1$.

The *edge degree* of an edge e in a graph H is defined as the number of edges adjacent with e. Hence the edge degree of e in H corresponds to the degree of e in the line graph of H.

The above arguments directly imply that Conjecture 4 is equivalent to the following conjecture.

Conjecture 6 *Every essentially 4-edge-connected graph with minimum edge degree at least 5 has a dominating closed trail.*

Let H be a graph. An edge e of H is called a *pendant edge* if one of the end vertices of e has degree exactly 1 in H. For a vertex v in H, we denote, by $\deg_H(v)$ and $\mathrm{pen}_H(v)$, the degree of v in H and the number of pendant edges that are incident with v in H, respectively. For a vertex v of degree exactly 2 in a graph H, *suppressing* v is an operation to replace the path u_1vu_2 in H by an edge connecting u_1 and u_2, where u_1 and u_2 are the neighbors of v. Note that suppressing a vertex may create multiple edges. For an integer k, we denote, by $V_k(H)$, $V_{\geq k}(H)$ and $V_{\leq k}(H)$, the set of vertices v of degree exactly k, at least k and at most k in H, respectively.

Let H be a graph and $v \in V_{\geq 4}(H)$, and let $u^1, u^2, \dots, u^d$ ($d = \deg_H(v)$) be an ordering of neighbors of v (we allow repetition in case of parallel edges). Then the graph obtained from the disjoint union of $H - v$ and the cycle $C_v = v^1v^2\dots v^dv^1$ by adding the edges u^iv^i for each $1 \leq i \leq d$ is called an *inflation of H at v*. If $\delta(H) \geq 3$, then, by successively taking an inflation at each vertex of degree greater than 3, we obtain a cubic graph H^I, called a *cubic inflation* of H. An inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of v) and it might happen that the operation decreases the edge-connectivity of the graph. However, the following was proven in [8].

Theorem 7 (Fleischner and Jackson [8]) *Let H be an essentially 4-edge-connected graph with $\delta(H) \geq 3$. Then some cubic inflation of H is cyclically 4-edge-connected.*

We also need the following lemma in Section 4.

Lemma 8 *Let H be an cyclically 4-edge-connected cubic graph. Let u_1u_2 and v_1v_2 be two edges in H such that $u_i \neq v_j$ for $i, j = 1, 2$. Let H' be the graph obtained from H by subdividing the edges u_1u_2 and v_1v_2, and adding a new edge connecting w and z, where w and z are the vertices obtained by subdivision of the edges u_1u_2 and v_1v_2, respectively. Then H' is also a cyclically 4-edge-connected cubic graph.*

Proof of Lemma 8. Suppose not. Then there exists a cyclic edge-cut X' of H' with $|X'| \leq 3$. Let X be the set of edges of H obtained from X' by deleting the edge wz, replacing the edge wu_i with u_1u_2 for $i = 1, 2$, and replacing the edge zv_j with v_1v_2 for $i = 1, 2$, if X' contains them, respectively. Since X' is an edge-cut of H', it follows from the construction of H' and X that X is an edge-cut of H. Since $|X| \leq |X'| \leq 3$ and H is cyclically 4-edge-connected, at least one component D of $H - X$ cannot contain a cycle. By the construction of X, $H' - X'$ has the component D' containing all vertices in D. Since X' is a cyclic edge-cut of H', D' has a cycle. However, since D contains no cycle, we obtain $w, z \in V(D')$ and D' has exactly one cycle, which passes through the edge wz. In particular, D' has exactly $|V(D')|$ edges, since D' is connected and has exactly one cycle. On the other hand, since $u_i \neq v_j$ for $i, j = 1, 2$, D' has at least four vertices, that is, w, z, u_i and v_j for some $i, j = 1, 2$. Hence

$$|X'| = \sum_{v \in V(D')} \deg_{H'}(v) - 2|E(D')| = 3|V(D')| - 2|V(D')| \geq 4,$$

contradicting the assumption that $|X'| \leq 3$. This completes the proof of Lemma 8. $\square$

3 Proof of Theorem 5

The proof of Theorem 5 is divided into two parts. To do that, we need the following two "intermediate" conjectures and two theorems, which might be interesting themselves, see Section 6.

Conjecture 9 *There exists a constant c_1 with $0 < c_1 \leq 1$ such that every essentially 4-edge-connected graph H has a closed trail T with $\mathrm{dom}_H(T) \geq c_1|E(H)|$.*

Conjecture 10 *There exists a constant c_2 with $0 < c_2 \leq 1$ such that every essentially 4-edge-connected graph H with minimum edge degree at least 5 has a closed trail T with $\mathrm{dom}_H(T) \geq c_2|E(H)|$.*

Theorem 11 *If Conjecture 2 is true, then Conjecture 9 is also true.*

Theorem 12 *If Conjecture 10 is true, then Conjecture 4 is also true.*

Here we prove Theorem 5 assuming Theorems 11 and 12. We will show Theorems 11 and 12 in Sections 4 and 5, respectively.

Proof of Theorem 5. Suppose that Conjecture 2 is true. Then by Theorem 11, Conjecture 9 is also true, that is, there exists a constant c_1 with $0 < c_1 \leq 1$ such that every essentially 4-edge-connected graph H has a closed trail T with $\mathrm{dom}_H(T) \geq c_1|E(H)|$. Indeed, the constant c_1 satisfies that $0 < c_1 \leq 1$ and every essentially 4-edge-connected graph H with minimum edge degree at least 5 has a closed trail T with $\mathrm{dom}_H(T) \geq c_1|E(H)|$. Hence Conjecture 10 is also true for $c_2 = c_1$. Then by Theorem 12, Conjecture 4 is true. $\square$

4 Proof of Theorem 11

In order to prove Theorem 11, we further divide the proof into two parts, using the following conjecture.

Conjecture 13 *There exists a constant c_1' with $0 < c_1' \leq 1$ such that every essentially 4-edge-connected graph H without vertices of degree 2 has a closed trail T with $\mathrm{dom}_H(T) \geq c_1'|E(H)|$.*

Theorem 14 *If Conjecture 2 is true, then Conjecture 13 is also true.*

Theorem 15 *If Conjecture 13 is true, then Conjecture 9 is also true.*

It is easy to prove Theorem 11 assuming Theorems 14 and 15. Hence we omit the proof of Theorem 11, and it is enough to prove Theorems 14 and 15.

Proof of Theorem 14. Suppose that Conjecture 2 is true. Then there exists a constant c_0 with $0 < c_0 \leq 1$ such that every cyclically 4-edge-connected cubic graph H has a cycle of length at least $c_0|V(H)|$.

Let

$$c_1' = \frac{c_0}{6}.$$

We will show that every essentially 4-edge-connected graph H without vertices of degree 2 has a closed trail T with $\mathrm{dom}_H(T) \geq c_1'|E(H)|$. Let H be an essentially 4-edge-connected graph without vertices of degree 2. If H is a star, then the center of it can form a closed trail T with $\mathrm{dom}_H(T) = |E(H)|$. Hence we may assume that H is not a star. Let H_0 be the graph obtained from H by deleting all pendant edges of H. Note that H_0 is essentially 4-edge-connected, and $\delta(H_0) \geq 3$, since H has no vertices of degree 2. Since H is essentially 4-edge-connected, $\mathrm{pen}_H(v) = 0$ for $v \in V_3(H_0)$; Otherwise the set of edges in H_0 incident with v is an essential edge-cut of size 3, a contradiction. Notice also that for each $v \in V(H)$, $\deg_H(v) = \deg_{H_0}(v) + \mathrm{pen}_H(v)$, in particular, $\deg_H(v) = \deg_{H_0}(v)$ if $\deg_H(v) = 3$. This implies that

$$|E(H)| = |E(H_0)| + \sum_{v \in V_{\geq 4}(H_0)} \mathrm{pen}_H(v). \tag{1}$$

Now from H, we construct new graphs H_1 and H_2 as follows; First using Theorem 7 to H_0, we obtain a cubic inflation H_1 of H_0 such that H_1 is cyclically 4-edge-connected. Clearly, $|E(H_1)| \geq |E(H_0)|$. Let $v \in V_{\geq 4}(H_0)$. Note that v corresponds to the cycle C_v in H_1 of length exactly $\deg_{H_0}(v)$. Let v^1, v^2, v^3, v^4 be four consecutive vertices of C_v and let $p = \mathrm{pen}_H(v)$. We subdivide the edges v^1v^2 and v^3v^4 exactly p times, and obtain the paths $v^1v_1v_2\ldots v_pv^2$ and $v^3v_{p+1}v_{p+2}\ldots v_{2p}v^4$, respectively. Then we add an edge connecting v_i and v_{2p+1-i} for $1 \leq i \leq p$. See Figure 1. We perform the above operation to all vertices v in $V_{\geq 4}(H_0)$, and let H_2 be the obtained graph. Using Lemma 8 repeatedly (more precisely, using Lemma 8 $\sum_{v \in V_{\geq 4}(H_0)} \mathrm{pen}_H(v)$ times), we see that H_2 is a cyclically 4-edge-connected cubic graph.

Let $D_v = V(C_v) \cup \{v_i : 1 \leq i \leq 2p\}$ for every vertex $v \in V_{\geq 4}(H_0)$. For simplifying the argument, we let $D_v = \{v\}$ for a vertex v in $V_3(H_0)$. Then for each vertex v in H_0,

$$|D_v| = \begin{cases} \deg_{H_0}(v) + 2\,\mathrm{pen}_H(v) & \text{if } \deg_{H_0}(v) \geq 4, \\ 1 & \text{if } \deg_{H_0}(v) = 3. \end{cases} \tag{2}$$

Clearly from the construction, for each $v \in V_{\geq 4}(H_0)$, there are at least $\mathrm{pen}_H(v)$ edges inside of D_v. Hence by equality (1),

$$\begin{aligned} |E(H_2)| &\geq |E(H_1)| + \sum_{v \in V_{\geq 4}(H_0)} \mathrm{pen}_H(v) \\ &\geq |E(H_0)| + \sum_{v \in V_{\geq 4}(H_0)} \mathrm{pen}_H(v) \quad = |E(H)|. \end{aligned} \tag{3}$$

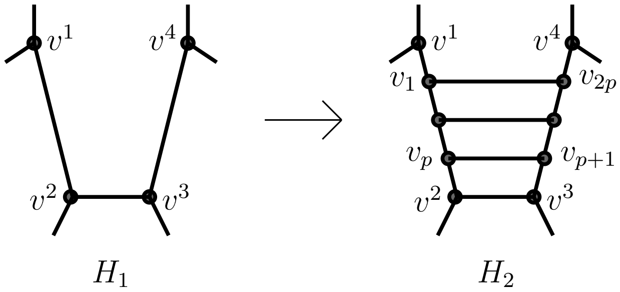

Figure 1: A cubic inflation H_1 of H_0 and the graph H_2.

Since H_2 is a cubic graph,

$$|E(H_2)| = \frac{3}{2}|V(H_2)|. \tag{4}$$

Since we assumed that Conjecture 2 is true, H_2 has a cycle T_2 of length at least $c_0|V(H_2)|$, that is,

$$|V(T_2)| \geq c_0|V(H_2)|. \tag{5}$$

Let $U_{T_2}(H_0)$ be the set of vertices v of H_0 with $D_v \cap V(T_2) \neq \emptyset$. Note that $V(T_2) \subset \bigcup_{v \in U_{T_2}(H_0)} D_v$, and hence $|V(T_2)| \leq \sum_{v \in U_{T_2}(H_0)} |D_v|$. Let T be the subgraph of H obtained from T_2 by contracting all vertices in D_v into one vertex for each $v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)$ and deleting all loops (but we remain multiple edges if exist). Note that T is a closed trail of H and $V(T) = U_{T_2}(H_0)$. Since each edge is dominated by T from at most two end vertices of it, we have $2 \cdot \mathrm{dom}_H(T) \geq \sum_{v \in V(T)} \deg_H(v)$. Then it follows from the above arguments and (in)equalities (2)–(5) that

$$\begin{aligned}
\mathrm{dom}_H(T) &\geq \frac{1}{2} \sum_{v \in U_{T_2}(H_0)} \deg_H(v) \\
&= \frac{1}{2} \sum_{v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)} \Big(\deg_{H_0}(v) + \mathrm{pen}_H(v)\Big) + \frac{1}{2} \sum_{v \in U_{T_2}(H_0) \cap V_3(H_0)} \deg_{H_0}(v) \\
&\geq \frac{1}{4} \sum_{v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)} \Big(\deg_{H_0}(v) + 2\ \mathrm{pen}_H(v)\Big) + \frac{3}{2} \sum_{v \in U_{T_2}(H_0) \cap V_3(H_0)} 1 \\
&= \frac{1}{4} \sum_{v \in U_{T_2}(H_0) \cap V_{\geq 4}(H_0)} |D_v| + \frac{3}{2} \sum_{v \in U_{T_2}(H_0) \cap V_3(H_0)} |D_v| \\
&\geq \frac{1}{4} \sum_{v \in U_{T_2}(H_0)} |D_v| \quad \geq \frac{1}{4}|V(T_2)| \\
&\geq \frac{c_0}{4}|V(H_2)| \ = \ \frac{c_0}{6}|E(H_2)| \ \geq \ \frac{c_0}{6}|E(H)| \ = \ c_1'|E(H)|.
\end{aligned}$$

Then T is a closed trail of H with $\mathrm{dom}_H(T) \geq c_1'|E(H)|$. This holds for every essentially 4-edge-connected graph H without vertices of degree 2, and hence Conjecture 13 is also true. This completes the proof of Theorem 14. $\square$

Proof of Theorem 15. Suppose that Conjecture 13 is true. Then there exists a constant c_1' with $0 < c_1' \leq 1$ such that every essentially 4-edge-connected graph H without vertices of degree 2 has a closed trail T with $\mathrm{dom}_H(T) \geq c_1'|E(H)|$. Let

$$c_1 = \frac{c_1'}{2}.$$

Note that $0 < c_1 \leq 1$. We will show that every essentially 4-edge-connected graph H has a closed trail T with $\mathrm{dom}_H(T) \geq c_1|E(H)|$.

Let H be an essentially 4-edge-connected graph. We construct the new graph $\widetilde{H}$ by suppressing all vertices of degree 2 in H. Note that $\widetilde{H}$ is an essentially 4-edge-connected graph without vertices of degree 2. Hence by the assumption that Conjecture 13 is true, $\widetilde{H}$ has a closed trail $\widetilde{T}$ with $\mathrm{dom}_{\widetilde{H}}(\widetilde{T}) \geq c_1'|E(\widetilde{H})|$. We obtain the closed trail T of H by subdividing all suppressed edges in $\widetilde{T}$. Note that $\mathrm{dom}_H(T) \geq \mathrm{dom}_{\widetilde{H}}(\widetilde{T})$. On the other hand, since H is essentially 4-edge-connected, there are no two consecutive vertices of degree 2 in H. Hence each edge of $\widetilde{H}$ is obtained by suppressing a vertex of degree 2 in H at most once, and hence $|E(\widetilde{H})| \geq \frac{1}{2}|E(H)|$. These imply that

$$\begin{aligned} \mathrm{dom}_H(T) &\geq \mathrm{dom}_{\widetilde{H}}(\widetilde{T}) \geq c_1'|E(\widetilde{H})| \\ &\geq \frac{c_1'}{2}|E(H)| = c_1|E(H)|, \end{aligned}$$

which completes the proof of Theorem 15. $\square$

Remark: As mentioned before, combining Theorems 14 and 15, we obtain Theorem 11. Indeed, if Conjecture 2 is true for some c_0, then Conjecture 9 is also true with $c_1 = \frac{1}{12}c_0$.

5 Proof of Theorem 12

Proof of Theorem 12. By the argument in Section 2, it is enough to show that if Conjecture 6 is false, then Conjecture 10 is also false. Suppose that Conjecture 6 is false. Then there exists an essentially 4-edge-connected graph H with minimum edge degree at least 5 such that H has no dominating closed trail. If there exists an edge e of H connecting two vertices in $V_{\leq 3}(H)$, then the edge degree of e is at most 4, contradicting the minimum edge degree condition on H. Hence there exists no such an edge e of H, which implies that every vertex in $V_{\leq 3}(H)$ has a neighbor in $V_{\geq 4}(H)$. Then if H has a closed trail T that passes through all vertices in $V_{\geq 4}(H)$, then T is a dominating closed trail in H, contradicting the choice of H. Hence we have the following claim.

Claim 1 *For any closed trail T in H, there exists a vertex v in $V_{\geq 4}(H)$ such that v is not visited by T.*

We construct an infinite sequence of graphs $H_0, H_1, \ldots$ as follows; Let $H_0 = H$, and take any vertex v in $V_{\geq 4}(H)$. For $i \geq 1$, the graph H_i is obtained from H_{i-1} and $|V_{\geq 4}(H_{i-1})|$ copies of H by identifying each vertex in $V_{\geq 4}(H_{i-1})$ and the vertex v in a copy of H. Since $\deg_H(v) \geq 4$, for $i \geq 0$, H_i is an essentially 4-edge-connected graph with minimum edge degree at least 5. Notice also that $|V_{\geq 4}(H_i)| = h \cdot |V_{\geq 4}(H_{i-1})|$, where $h := |V_{\geq 4}(H)|$. Since $|V_{\geq 4}(H_0)| = h$,

$$|V_{\geq 4}(H_i)| = h^{i+1}. \tag{6}$$

For a graph H', let $f_{\geq 4}(H')$ be the maximum number of vertices v in $V_{\geq 4}(H')$ such that v is visited by a closed trail T', where T' is taken over all closed trails in H'. The following plays a crucial role in the proof of Theorem 12.

Claim 2 $f_{\geq 4}(H_i) \leq (h-1)^{i+1}$.

Proof. First we show that $f_{\geq 4}(H_i) \leq (h-1) \cdot f_{\geq 4}(H_{i-1})$. Let T_i be any closed trail in H_i. Let T_{i-1} be the closed trail in H_{i-1} such that T_{i-1} is the restriction of T_i on H_{i-1}. By the definition of $f_{\geq 4}(H_{i-1})$, T_{i-1} visits at most $f_{\geq 4}(H_{i-1})$ vertices in $V_{\geq 4}(H_{i-1})$. Let u be a vertex in $V_{\geq 4}(H_{i-1})$ that is visited by T_{i-1}. By the above argument, we have at most $f_{\geq 4}(H_{i-1})$ choices for such a vertex u. Let H_u be the copy of H that is added to u when we construct H_i from H_{i-1}, and let T_u be the closed trail in H_u such that T_u is the restriction of T_i on H_u. By Claim 1, at most $h-1$ vertices in $V_{\geq 4}(H_u)$ can be visited by T_u. Hence T_i can visit at most $(h-1) \cdot f_{\geq 4}(H_{i-1})$ vertices in $V_{\geq 4}(H_i)$. This implies that $f_{\geq 4}(H_i) \leq (h-1) \cdot f_{\geq 4}(H_{i-1})$.

Since $f_{\geq 4}(H_0) \leq h-1$ by Claim 1, we obtain $f_{\geq 4}(H_i) \leq (h-1)^{i+1}$. This completes the proof of Claim 2. $\square$

Now we are ready to show that Conjecture 10 does not hold. Let c_2 be any constant with $0 < c_2 \leq 1$. Since

$$\lim_{i \to \infty} \frac{1 + (h-1)^{i+1}}{1 + h^{i+1}} = \lim_{i \to \infty} \frac{\frac{1}{h^{i+1}} + \left(1 - \frac{1}{h}\right)^{i+1}}{\frac{1}{h^{i+1}} + 1} = 0,$$

there exists an integer i such that

$$\frac{1 + (h-1)^{i+1}}{1 + h^{i+1}} < c_2. \tag{7}$$

For a non-negative integer t, let $H_i(t)$ be the graph obtained from H_i by adding t pendant edges to all vertices in $V_{\geq 4}(H_i)$. Since we added pendant edges only to vertices in $V_{\geq 4}(H_i)$, $H_i(t)$ is still essentially 4-edge-connected, and moreover, the minimum edge degree of $H_i(t)$ is at least 5 if $t \geq 2$. Note that $\left|E\big(H_i(t)\big)\right| = |E(H_i)| + t \cdot |V_{\geq 4}(H_i)|$.

Let T be any closed trail in $H_i(t)$. By the definition of $f_{\geq 4}(H_i)$, T can pass through at most $f_{\geq 4}(H_i)$ vertices in $V_{\geq 4}(H_i)$. Hence T can dominate at most $t \cdot f_{\geq 4}(H_i)$ added pendant edges. This implies that for any closed trail T in $H_i(t)$, we have $\mathrm{dom}_{H_i(t)}(T) \leq |E(H_i)| + t \cdot f_{\geq 4}(H_i)$.

Let $H' = H_i(m)$, where $m = |E(H_i)|$. Then by equality (6), Claim 2, and inequality (7), for each closed trail T' in H',

$$\begin{aligned}\frac{\mathrm{dom}_{H'}(T')}{|E(H')|} &\leq \frac{|E(H_i)| + m \cdot f_{\geq 4}(H_i)}{|E(H_i)| + m \cdot |V_{\geq 4}(H_i)|} \\ &\leq \frac{1 + (h-1)^{i+1}}{1 + h^{i+1}} \quad < \; c_2.\end{aligned}$$

This means that for each constant c_2 with $0 < c_2 \leq 1$, there exists an essentially 4-edge-connected graph H' with minimum edge degree at least 5 such that any closed trail T' in H' satisfies $\mathrm{dom}_{H'}(T') < c_2|E(H')|$. So Conjecture 10 does not hold. This completes the proof of Theorem 12. □

Remark: In the proof of Theorem 12, assuming that Conjecture 6 is not true, we construct, for each constant c_2 with $0 < c_2 \leq 1$, the graph H' with $\frac{\mathrm{dom}_{H'}(T')}{|E(H')|} < c_2$. We here point out that if Conjecture 6 is not true, the correct magnitude of $\mathrm{dom}_{H'}(T')$ is at most $|E(H')|^\alpha$, where $\alpha = \log_h(h-1)$. Indeed, the proof of Theorem 12 also shows that if Conjecture 6 is not true, then there exist infinitely many essentially 4-edge-connected graphs H' with minimum edge degree at least 5 such that for any closed trail T' in H', $\mathrm{dom}_{H'}(T') \in O\big(|E(H')|^\alpha\big)$. Recall that $h = |V_{\geq 4}(H)|$ and H is a counterexample of Conjecture 6.

It should be mentioned here that Blinski, Jackson, Ma, and Yu [2] recently showed that every essentially 3-edge-connected graph H' has a closed trail T' with $\mathrm{dom}'_H(T') \geq \big(\frac{|E(H)|}{12}\big)^\beta + 2$, where $\beta \approx 0.753$, consider the preimage version of Theorem 1.2 in [2]. So, if Conjecture 6 is not true, the gap of bounds on $\mathrm{dom}_{H'}(T')$ between the essentially 4-edge-connected case and the essentially 3-edge-connected case would be only the difference between α and β.

6 Conclusion and open problems

In this paper, we have shown that Conjecture 2 implies Conjecture 4, which is a weaker version of Conjecture 1. Together with Proposition 3, we see the following situation;

$$\text{Conjecture 1} \xrightarrow{\text{Proposition 3}} \text{Conjecture 2} \xrightarrow{\text{Theorem 5}} \text{Conjecture 4}$$

However, we do not know about the converse of these two implications. Indeed, as mentioned in Section 1, the converse of Proposition 3 appeared in [5] as an open problem. In addition to that, we left an open problem on the converse of Theorem 5.

Problem 16 *Is Conjecture 4 equivalent to Conjecture 2, or moreover to Conjecture 1?*

On the other hand, now we point out that Theorem 12 gives a corollary concerning Conjecture 4. It shows the equivalence of Conjecture 4 and the following conjecture, which is the line graph version of Conjecture 10, see Section 2. This corollary might be interesting itself.

Conjecture 17 *There exists a constant c_3 with $0 < c_3 \leq 1$ such that every 4-connected line graph G with $\delta(G) \geq 5$ has a cycle of length at least $c_3|V(G)|$.*

Corollary 18 *Conjecture 4 is equivalent to Conjecture 17.*

Proof. It is easy to see that if Conjecture 4 is true, then Conjecture 17 is also true with $c_3 = 1$. On the other hand, suppose that Conjecture 17 is true. Since Conjecture 10 is the preimage version of Conjecture 17, Conjecture 10 is also true with $c_2 = c_3$. By Theorem 12, Conjecture 4 is also true. $\square$

Thus, by Corollary 18, in order to solve Conjecture 4, instead of a Hamiltonian cycle, it is enough to find a cycle of length c_3 times the order of a given graph, even for $c_3 = 1/1000000$. We hope that Corollary 18 gives a step to solve Conjecture 4.

Now we consider the above situation for Conjecture 1. It is shown in [4] that Conjecture 1 is equivalent to the following statement; there exists a function f such that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$, and every 4-connected line graph of order n has a cycle of length at least $n - f(n)$. So, in order to solve Conjecture 1, it is enough to find a cycle of length at least $n - f(n)$ in 4-connected line graphs. However, we do not know if it is enough to find a cycle of length linear on the order of a graph. Indeed, we can consider the following conjecture, which is analogous to Conjecture 17 and seemingly weaker than Conjecture 1. Considering Corollary 18, we expect that Conjecture 19 is equivalent to Conjecture 1, and left it as an open problem.

Conjecture 19 *There exists a constant c_4 with $0 < c_4 \leq 1$ such that every 4-connected line graph G has a cycle of length at least $c_4|V(G)|$.*

Problem 20 *Is Conjecture 19 equivalent to Conjecture 1?*

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