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Relative length of longest paths and longest cycles in triangle-free graphs

Daniel Paulusma[1]
Department of Applied Mathematics
Faculty of Electrical Engineering, Mathematics and Computer Science
University of Twente, P.O. Box 217, 7500 AE Enschede, the Netherlands
d.paulusma@math.utwente.nl
and
Kiyoshi Yoshimoto[2]
Department of Mathematics, College of Science and Technology
Nihon University, Tokyo 101-8308, Japan
yosimoto@math.cst.nihon-u.ac.jp

Abstract

In this paper, we prove that if G is a triangle-free graph with minimum degree at least two and $\sigma_4(G) \geq |V(G)|+2$, then for any path P, there exists a cycle C such that $|V(P) \setminus V(C)| \leq 1$ or G is isomorphic to an exception.

Using this fact, easily we can show that for any set S of at most δ vertices, there is a cycle C such that $S \subset V(C)$ under same condition.

1 Introduction

The order of a graph is denoted by n throughout this paper and the minimum degree is written by δ, and let:

$$\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(x_i) \mid x_1, x_2, \ldots, x_k \text{ are independent}\},$$

where $d_G(x_i)$ is the degree of a vertex x_i. If the independence number of G is less than k, then we define $\sigma_k(G) = \infty$. For simplicity, we denote $G - V(H)$ by $G - H$ and a cycle C is called *dominating* if $G - C$ is edgeless.

Bondy [4] proved that if G is a 2-connected graph with $\sigma_3 \geq n+2$, then all longest cycles are dominating. This lower bound is best possible by $(K_k \cup K_k \cup K_k) * \overline{K_2}$.

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Enomoto et al. [7] generalized this fact as follows: if G is a 2-connected graph with $\sigma_3 \geq n+2$, then $p(G) - c(G) \leq 1$, where $p(G)$ and $c(G)$ are the length of longest paths and the circumference.

For triangle-free graphs, by the theorem of Broersma, Yoshimoto and Zhang [5], it holds that a 2-connected triangle-free graph with $\sigma_3 \geq (n+5)/2$ contains a longest cycle that is dominating. The lower bound is sharp, even for the existence of dominating cycles. In this theorem, longest cycles are not always dominating. However, if $\sigma_2 \geq (n+1)/2$, then all longest cycles are dominating [16]. This lower bound is almost best possible by examples due to Ash and Jackson [1]. The purpose of this paper is to show the following fact corresponding to the theorem by Enomoto et al.

Theorem 1. *Let G be a triangle-free graph with $\delta \geq 2$. If $\sigma_4 \geq n+2$, then for any path P, there exists a cycle C such that $|P - C| \leq 1$ or G is isomorphic to the graph in Figure 1i.*

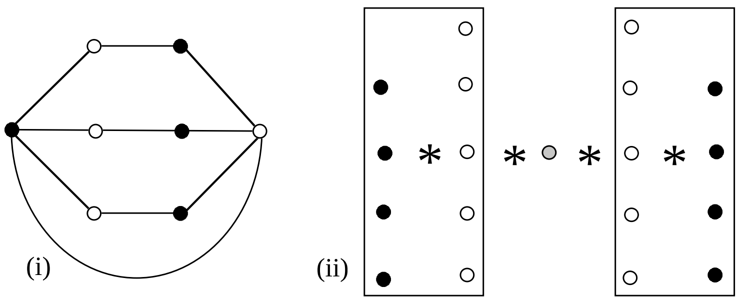

Figure 1:

The lower bound of σ_4 is best possible because the graph $H = \overline{K_{k-1}} * \overline{K_k} * K_1 * \overline{K_k} * \overline{K_{k-1}}$ contains a hamilton path and the minimum degree is $(n+1)/4$, and so $\sigma_4 = n+1$, however, the circumference is $(n+1)/2$. See Figure 1ii. Moreover, high connectivity is not useful for decreasing the lower bound of σ_4 because we can add edges between the left K_{k-1} and the right K_{k-1}.

As an application of Theorem 1, the following is shown in Section 3:

Theorem 2. *Let G be a triangle-free graph with $\delta \geq 2$. If $\sigma_4 \geq n+2$, then for any set S of at most δ vertices, there exists a cycle C such that $S \subset V(C)$.*

From this fact, it holds that a triangle-fee graph with $\delta \geq 2$ and $\sigma_4 \geq n+2$ is 2-connected. On the other hand, the graph H has a cut vertex, and so the lower bound $n+2$ of σ_4 for the 2-connectivity is best possible. Because of the proof of Theorem 2, we shall show a triangle-free graph with $\delta \geq 2$ and $\sigma_4 \geq n+1$ is connected. This lower bound is also sharp due to $K_{k,k} \cup K_{k,k}$.

Finally, we give some additional definitions and notations. The set of all the neighbours of a vertex $x \in V(G)$ is denoted by $N_G(x)$ or simply $N(x)$, and its cardinality by $d_G(x)$ or $d(x)$. For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. For simplicity, we denote $|V(H)|$ by $|H|$ and "$u_i \in V(H)$" by "$u_i \in H$". The set of neighbours $\bigcup_{v \in H} N_G(v) \setminus V(H)$ is written by $N_G(H)$ or $N(H)$, and for a subgraph $F \subset G$, $N_G(H) \cap V(F)$ is denoted by $N_F(H)$.

Let $C = v_1 v_2 \dots v_p v_1$ be a cycle with a fixed orientation. The segment $v_i v_{i+1} \dots v_j$ is written by $v_i \overrightarrow{C} v_j$ where the subscripts are to be taken modulo $|C|$. The converse segment $v_j v_{j-1} \dots v_i$ is written by $v_j \overleftarrow{C} v_i$. For a path $P = u_1 u_2 \dots u_p$, also we denote $u_i \overrightarrow{P} u_j = u_i u_{i+1} \dots u_j$ and $u_j \overleftarrow{P} u_i = u_j u_{j-1} \dots u_i$. The successor of u_i is denoted by u_i^+ and the predecessor by u_i^-. For a vertex subset A in C, we write $\{u_i^+ \mid u_i \in A\}$ and $\{u_i^- \mid u_i \in A\}$ by A^+ and A^-, respectively.

All notation and terminology not explained here is given in [6].

2 The Proof of Theorem 1

For a vertex subset S, if a path P is longest in all paths containing S, then we call P a *maximal* path for S, and the set of all the maximal paths is denoted by $\mathcal{P}(S)$. At first, we show the following lemma.

Lemma 3. *If G is a triangle-free graph with $\delta \geq 2$, then for any path R, there exists a path in $\mathcal{P}(V(R))$ such that the degree sum of the ends is at least $\sigma_4/2$ or a cycle C such that $|R - C| \leq 1$ or G is isomorphic to the graph in Figure 1i.*

Proof. Let R be any path in G and $P = u_1 u_2 \dots u_p \in \mathcal{P}(V(R))$ such that the degree sum of the ends is maximal in $\mathcal{P}(V(R))$. Notice that $N(u_1) = N_P(u_1)$ and $N(u_p) = N_P(u_p)$. If there exist vertices $u_i \in N(u_1) \setminus u_2$ and $u_j \in N(u_p) \setminus u_{p-1}$ such that $i \leq j$, then $\{u_1, u_{i-1}, u_{j+1}, u_p\}$ is an independent set; otherwise there is a triangle or a cycle containing $V(R)$, i.e., the cycle is a desired cycle. Because

$d(u_1) + d(u_{i-1}) + d(u_{j+1}) + d(u_p) \geq \sigma_4$, one of $d(u_1) + d(u_p)$ and $d(u_{i-1}) + d(u_{j+1})$ is at least $\sigma_4/2$. Therefore P or the path $u_{i-1}\overleftarrow{P}u_1u_i\overrightarrow{P}u_ju_p\overleftarrow{P}u_{j+1}$ is a desired path.

Assume that:

$$i > j \text{ for any vertices } u_i \in N(u_1) \setminus u_2 \text{ and } u_j \in N(u_p) \setminus u_{p-1}. \tag{1}$$

Suppose there is a vertex $u_s \in N_P(u_1) \setminus \{u_2, u_{p-2}\}$, and let $u_t \in N(u_p) \setminus u_{p-1}$. Then $P' = u_{t+1}\overrightarrow{P}u_su_1\overrightarrow{P}u_tu_p\overleftarrow{P}u_{s+1} \in \mathcal{P}(V(R))$. The vertex u_1 is not adjacent to u_{t+1} nor u_{s+1}; otherwise there is a triangle or a cycle containing $V(R)$. And the vertex u_p is not adjacent to u_{t+1} nor u_{s+1} by the assumptions (1) and $u_s \neq u_{p-2}$. Thus $\{u_1, u_{t+1}, u_{s+1}, u_p\}$ is an independent set, and hence one of the paths P and P' is a desired path as in the previous case. Therefore $N(u_1) = \{u_2, u_{p-2}\}$ and, by symmetry, $N(u_p) = \{u_3, u_{p-1}\}$. Furthermore, by the maximality of the degree sum of the ends of P:

$$\text{the degree of an end of any path in } \mathcal{P}(V(R)) \text{ is two.}$$

Because the path $u_1u_2u_3u_p\overleftarrow{P}u_4$ is in $\mathcal{P}(V(R))$, the vertex u_1 has to be adjacent to $u_4^{++} = u_6$; otherwise, as in the above case, we can obtain a desired cycle or path. Therefore $u_6 = u_{p-2}$, i.e., $p = 8$, and so any vertex in $\{u_1, u_2, u_4, u_5, u_7, u_8\}$ is the end of some path in $\mathcal{P}(V(R))$, and has degree two. As G is triangle-free, the vertices u_1, u_5 and u_7 are mutually disjoint. If $G - P$ is not empty, then for any $x \in G - P$, $\{x, u_1, u_5, u_7\}$ is an independent set. Hence:

$$d(x) \geq \sigma_4 - (d(u_1) + d(u_5) + d(u_7)) \geq n + 2 - 6 = n - 4.$$

However, x is adjacent to none of $\{u_1, u_2, u_4, u_5, u_7, u_8\}$ because these degrees are two. Thus $d(x) \leq n - 7$, a contradiction. Therefore $G - P = \emptyset$ and $n = 8$. As u_3 is adjacent to none of u_1, u_5 nor u_7, the vertex u_3 has to be adjacent to u_6; otherwise $d(u_1) + d(u_3) + d(u_5) + d(u_7) = 9 < n + 2$. Hence G is isomorphic to the graph in Figure 1i. □

Assume that G is not isomorphic to the graph in Figure 1i. By the previous lemma, we can suppose the independence number of G is at least four; otherwise we are done. Let R be any path in G and $P = u_1u_2 \ldots u_p \in \mathcal{P}(V(R))$ such that:

$$\text{the degree sum of the ends is maximal in } \mathcal{P}(V(R)). \tag{2}$$

Then from Lemma 3, $d(u_1) + d(u_p) \geq \sigma_4/2$. Notice that we may assume that there is no path in $\mathcal{P}(V(R))$ whose ends are adjacent; otherwise obviously there exists a cycle containing $V(R)$.

If there is $u_l \in N_P(u_1) \cap N_P(u_p)^+$, then the cycle $u_1 \overrightarrow{P} u_l^- u_p \overleftarrow{P} u_l u_1$ is a desired cycle. Thus we can suppose $N_P(u_1) \cap N_P(u_p)^+ = \emptyset$. Similarly, we get $N_P(u_1) \cap N_P(u_p)^{++} = \emptyset$ and $N_P(u_1)^- \cap N_P(u_p)^+ = \emptyset$. If $N_P(u_1)^- \cap N_P(u_p)^{++}$ is also empty, then $N_P(u_1), N_P(u_1)^-, N_P(u_p)^+$ and $(N_P(u_p) \setminus u_p)^{++}$ are mutually disjoint. Hence:

$$\begin{aligned} n \geq |P| &\geq |N_P(u_1)| + |N_P(u_1)^-| + |N_P(u_p)^+| + |(N_P(u_p) \setminus u_p)^{++}| \\ &\geq 2d(u_1) + 2d(u_p) - 1 \geq \sigma_4 - 1 > n. \end{aligned}$$

This is a contradiction. Therefore $N_P(u_1)^- \cap N_P(u_p)^{++} \neq \emptyset$.

Let $u_i \in N_P(u_1)^- \cap N_P(u_p)^{++}$.

Claim 1. *If $d(u_i) + d(u_{i-1}) > n/2$, then there is a desired cycle.*

Proof. Let $e_0 = x_1x_2 = u_{i-1}u_i$ and:

$$C = u_1 \overrightarrow{P} u_{i-2} u_p \overleftarrow{P} u_{i+1} u_1 = v_1 v_2 \dots v_{p-2} v_1$$

which occur on C in the order of their indices. Notice that $N(e_0) = N(x_1) \cup N(x_2) \setminus \{x_1, x_2\} \subset V(C)$ because P is a maximal path for $V(R)$.

If $N(e_0)$ and $N(e_0)^+$ are not disjoint, then there exists a triangle or a desired cycle. Hence $N(e_0) \cap N(e_0)^+ = \emptyset$. In the set of segments $C - N(e_0)$, there are two segments $v_s^+ \overrightarrow{C} v_{s'}^-$ and $v_t^+ \overrightarrow{C} v_{t'}^-$ such that $\{v_s, v_{t'}\} \subset N(x_1)$ and $\{v_{s'}, v_t\} \subset N(x_2)$. Then $v_{s+2}, v_{t+2} \notin N_C(e_0) \cup N_C(e_0)^+$; otherwise there is a desired cycle. Therefore:

$$\begin{aligned} n - 2 \geq |C| &\geq |N(e_0)| + |N(e_0)^+| + |\{v_{s+2}, v_{t+2}\}| \\ &= |N_C(x_1)| + |N_C(x_1)^+| + |N_C(x_2)| + |N_C(x_2)^+| + |\{v_{s+2}, v_{t+2}\}| \\ &= 2(d(x_1) - 1) + 2(d(x_2) - 1) + 2 = 2(d(x_1) + d(x_2)) - 2 > n - 2. \end{aligned}$$

This is a contradiction. □

If $\delta \geq (n+2)/4$, then our proof is completed now by this claim. We divide our argument into two cases.

Case 1. $|N_P(u_1)^- \cap N_P(u_p)^{++}| = 1$

Let $\{u_i\} = N_P(u_1)^- \cap N_P(u_p)^{++}$. We show that $d(u_i) + d(u_{i-1}) > n/2$. Because:

$$\begin{aligned} n \geq |P| &\geq |N_P(u_1)| + |N_P(u_1)^-| + |N_P(u_p)^+| + |(N_P(u_p) \setminus u_{p-1})^{++}| \\ &\quad -|N_P(u_1)^- \cap N_P(u_p)^{++}| \\ &= 2d(u_1) + 2d(u_p) - 1 - 1 \geq \sigma_4 - 2 \geq n, \end{aligned}$$

it holds that:

$$V(G) = V(P) = N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \setminus u_{p-1})^{++} \quad (3)$$

and:

$$d(u_1) + d(u_p) = \frac{n}{2} + 1. \quad (4)$$

Hence the order n is even.

Because:

$$u_{i-3}\overleftarrow{P}u_1u_{i+1}u_iu_{i-1}u_{i-2}u_p\overleftarrow{P}u_{i+2} \in \mathcal{P}(V(R)),$$

we have $u_{i-3}u_{i+2} \notin E(G)$. If $u_{i-3}u_1 \in E(G)$, then:

$$u_{i-2} \notin N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \setminus u_{p-1})^{++}.$$

See Figure 2i. This contradicts (3). Thus $u_{i-3}u_1 \notin E(G)$. Especially, u_{i-3} is not u_2.

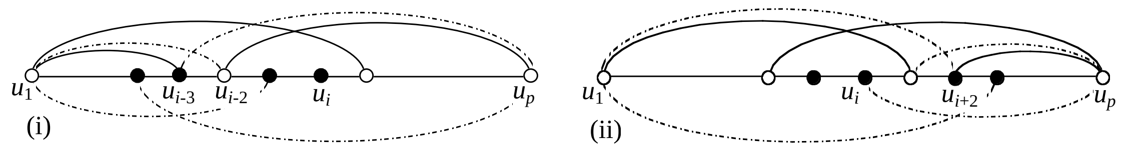

Figure 2:

Similarly, if $u_{i+2}u_p \in E(G)$, then

$$u_{i+2} \notin N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \setminus u_{p-1})^{++}.$$

See Figure 2ii. This also contradicts (3). Hence, $u_{i+2}u_p \notin E(G)$ and especially $u_{i+2} \neq u_{p-1}$. As $u_1u_p \notin E(G)$, $\{u_1, u_{i-3}, u_{i+2}, u_p\}$ is an independent set.

Let $x_1x_2 = u_{i-1}u_i$ and $w_1 = u_{i-3}$ and $w_2 = u_{i+2}$. Because $d(u_1)+d(u_p)+d(w_1)+d(w_2) \geq \sigma_4 \geq n+2$, we have:

$$d(w_1) + d(w_2) = \frac{n}{2} + 1$$

by (2) and (4). Notice that none of u_1, u_p, w_1, w_2 are adjacent to x_1 nor x_2; otherwise easily we can find a triangle or a desired cycle. Hence for each i, j,

$$d(u_1) + d(u_p) + d(x_i) + d(w_j) \geq n + 2.$$

Assume that $n/2$ is even, say $2l$. Then $d(u_1) + d(u_p) = d(w_1) + d(w_2) = 2l + 1$. By symmetry, we can suppose that $d(w_1) \leq l$. Because:

$$d(u_1) + d(u_p) + d(x_i) + d(w_1) \geq 4l + 2,$$

we have $d(x_i) \geq l + 1$ for $i = 1, 2$. Hence $d(x_1) + d(x_2) \geq 2l + 2 > n/2$.

Suppose $n/2$ is odd, say $2l + 1$. Then $d(u_1) + d(u_p) = d(w_1) + d(w_2) = 2l + 2$. By symmetry, we may assume that $d(w_1) \leq l + 1$. Because:

$$d(u_1) + d(u_2) + d(w_1) + d(x_i) \geq 4l + 4,$$

we have $d(x_i) \geq l + 1$ for $i = 1, 2$. Thus $d(x_1) + d(x_2) \geq 2l + 2 > n/2$.

Therefore, in either cases, $d(u_i) + d(u_{i-1}) > n/2$, and hence we are done by Claim 1.

Case 2. $|N_P(u_1)^- \cap N_P(u_p)^{++}| \geq 2$.

Let $u_i, u_j \in N_P(u_1)^- \cap N_P(u_p)^{++}$ $(i > j)$. If u_{i-1} is adjacent to u_{j-1}, then the cycle $u_1 \overrightarrow{P} u_{j-1} u_{i-1} u_i u_i^+ \overrightarrow{P} u_p u_{i-2} \overleftarrow{P} u_j^+ u_1$ is a desired cycle. See Figure 3i. Therefore

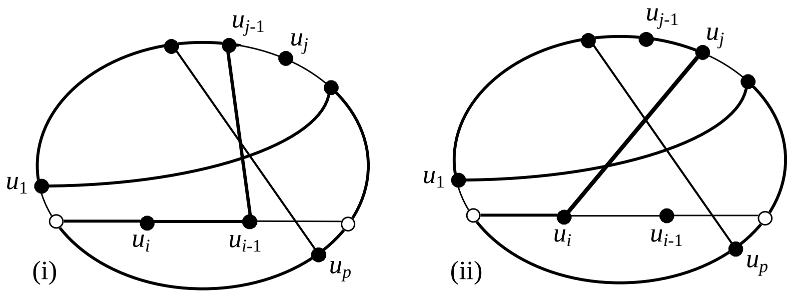

Figure 3:

$u_{i-1} u_{j-1} \notin E(G)$. Similarly we can obtain $u_i u_j \notin E(G)$. See Figure 3ii. Hence:

$$\begin{aligned} & (d(u_1) + d(u_p) + d(u_{i-1}) + d(u_{j-1})) + (d(u_1) + d(u_p) + d(u_i) + d(u_j)) \\ \geq \;& \sigma_4 + \sigma_4 \geq 2n + 4. \end{aligned}$$

By symmetry, without loosing generality, we may assume:

$$d(u_1) + d(u_p) + d(u_{i-1}) + d(u_i) \geq n + 2. \tag{5}$$

Let $e_0 = x_1x_2 = u_{i-1}u_i$ and C be the cycle $u_1\overrightarrow{P}u_{i-2}u_p\overleftarrow{P}u_{i+1}u_1 = v_1v_2\ldots v_{p-2}v_1$ which occur on C in the order of their indices. Notice that a vertex in $N_C(e_0)^+ \cup \{x_1, x_2\}$ has no neighbours in $G - P$; otherwise P is not maximal. Let $v_s \in N_C(x_2)$ and $v_t \in N_C(x_1)$ and $I_s = v_s^+\overrightarrow{C}v_t$ and $I_t = v_t^+\overrightarrow{C}v_s$. If there is a vertex $v_l \in N_{I_s}(v_s^+)^- \cap N_{I_s}(v_t^+)$, then the cycle: $v_s^+\overrightarrow{C}v_lv_t^+\overrightarrow{C}v_sx_2x_1v_t\overleftarrow{C}v_l^+v_s^+$ is a desired cycle. See Figure 4i. Hence $N_{I_s}(v_s^+)^- \cap N_{I_s}(v_t^+) = \emptyset$. Similarly, we have that:

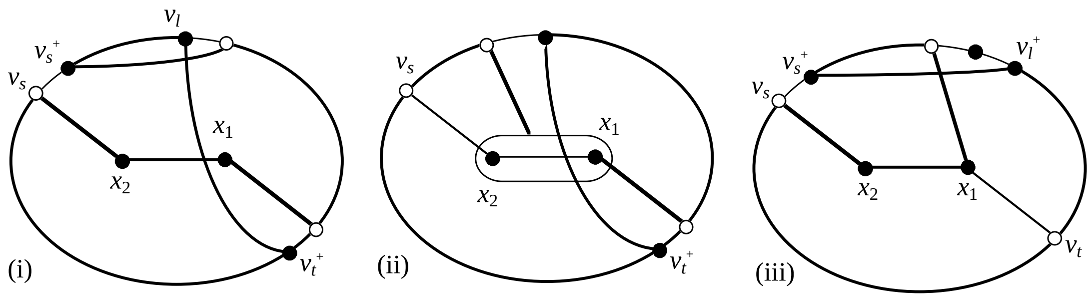

Figure 4:

$$N_{I_s}(e_0)^+ \cap N_{I_s}(v_t^+) = \emptyset \text{ and } N_{I_s}(v_s^+)^- \cap N_{I_s}(x_1)^+ = \emptyset.$$

See Figure 4ii-iii. Hence:

$$|I_s| \geq |N_{I_s}(v_s^+)^-| + |N_{I_s}(v_t^+)| + |(N_{I_s}(e_0) \setminus v_t)^+| - |N_{I_s}(v_s^+)^- \cap N_{I_s}(x_2)^+|.$$

Let $L = N_{I_s}(v_s^+)^- \cap N_{I_s}(x_2)^+$. If L is not empty, then for any vertex $v_l \in L$, $v_l^+ \notin N_{I_s}(v_s^+)^-$ because G is triangle-free. If $v_l^+v_t^+ \in E(G)$, then the cycle $v_l^-x_2x_1v_t\overleftarrow{C}v_l^+v_t^+\overrightarrow{C}v_l^-$ is a desired cycle. Since $v_l^+ \notin N_C(e_0)^+$,

$$v_l^+ \notin N_{I_s}(v_s^+)^- \cup N_{I_s}(v_t^+) \cup N_{I_s}(e_0)^+,$$

and so:

$$L^+ \cap (N_{I_s}(v_s^+)^- \cup N_{I_s}(v_t^+) \cup N_{I_s}(e_0)^+) = \emptyset.$$

Similarly, the vertex v_s^{++} is not contained in $N_{I_s}(v_s^+)^- \cup N_{I_s}(v_t^+) \cup N_{I_s}(e_0)^+$. Therefore:

$$\begin{aligned} |I_s| &\geq |N_{I_s}(v_s^+)^-| + |N_{I_s}(v_t^+)| + |(N_{I_s}(e_0) \setminus v_t)^+| - |L| + |L^+| + |\{v_s^{++}\}| \\ &\geq |N_{I_s}(v_s^+)| + |N_{I_s}(v_t^+)| + |N_{I_s}(e_0) \setminus v_t| + 1 \\ &= d_{I_s}(v_s^+) + d_{I_s}(v_t^+) + d_{I_s}(x_1) + d_{I_s}(x_2). \end{aligned}$$

By symmetry, we get $|I_t| \geq d_{I_t}(v_s^+) + d_{I_t}(v_t^+) + d_{I_t}(x_1) + d_{I_t}(x_2)$. By (5),

$$\begin{aligned} n-2 \geq |C| = |I_s| + |I_t| &\geq d_{I_s}(v_s^+) + d_{I_s}(v_t^+) + d_{I_s}(x_1) + d_{I_s}(x_2) \\ &\quad + d_{I_t}(v_s^+) + d_{I_t}(v_t^+) + d_{I_t}(x_1) + d_{I_t}(x_2) \\ &= d(v_s^+) + d(v_t^+) + (d(x_1) - 1) + (d(x_2) - 1) \geq n \end{aligned}$$

This is a contradiction. The proof is completed now.

3 The Proof of Theorem 2

By Theorem 1 and the following lemma, it is enough to show that G is connected. Notice that if a graph is isomorphic to the exception of Theorem 1, then obviously for any two vertices, there is a cycle containing the specified vertices.

Lemma 4 ([17])**.** *Let G be a connected graph such that for any path P, there exists a cycle C such that $|P - C| \leq 1$. Then for any set S with at most δ vertices, there exists a cycle C such that $S \subset V(C)$.*

Lemma 5. *Let G be a triangle-free graph and H a connected component of G. If $|H| \geq 3$, then there are non-adjacent vertices x, y in H such that $|H| \geq \max\{2d(x), 2d(y)\}$.*

Proof. Let $P = u_1 u_2 \dots u_p$ be a longest path of H. If $u_1 u_p \notin E(G)$, then $|P| \geq |N(u_1)| + |N(u_1)^-| + |\{u_p\}| = 2d(u_1) + 1$. Hence by symmetry, we have $|H| \geq \max\{2d(u_1) + 1, 2d(u_p) + 1\}$, and so $\{u_1, u_p\}$ is a desired pair. If $u_1 u_p \in E(G)$, then $u_1 u_{p-1} \notin E(G)$, and $V(H) = V(P)$ as P is longest. Then, we have

$$|P - u_p| \geq |N(u_{p-1}) \setminus u_p| + |(N(u_{p-1}) \setminus u_p)^+| + |u_1| = 2d(u_{p-1}) - 1.$$

Therefore $|H| \geq 2d(u_{p-1})$. As in the above case, we can have $|H| \geq 2d(u_1)$, and so $\{u_1, u_{p-1}\}$ is a desired pair. □

Lemma 6. *Let G be a triangle-free graph with $\delta \geq 2$. If $\sigma_4 \geq n+1$, then G is connected.*

Proof. Suppose G contains two connected components H_1 and H_2. Then the assumption that G is triangle-free and $\delta \geq 2$ implies $H_i \geq 3$ for $i = 1, 2$. Therefore there are non-adjacent vertices x_i, y_i in H_i such that $|H_i| \geq \max\{2d(x_i), 2d(y_i)\}$ for $i = 1, 2$ by the previous lemma. Hence $d(x_1) + d(y_1) + d(x_2) + d(y_2) \geq \sigma_4 \geq n+1$. By symmetry, we may assume $d(x_1) + d(x_2) \geq (n+1)/2$. Thus $n \geq |H_1| + |H_2| \geq 2(d(x_1) + d(x_2)) \geq n+1$. A contradiction. □

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A The Proof of Lemma 4

Proof. Let $S \subset V(G)$ and C a longest swaying cycle of S. Suppose $S - C \neq \emptyset$. For any vertex $x \in S - C$, there is a path Q joining x and C. Let P be a longest path containing $V(C \cup Q)$. Then there exists a cycle D such that $|P - D| \leq 1$. If x has neighbours in $G - C$, then $|P| \geq |C| + 2$ and so $|D| \geq |C| + 1$. Because $|D \cap S| \geq |C \cap S|$, this contradicts the assumption that C is a longest swaying cycle. Hence $N_{G-C}(x) = \emptyset$.

Because $|C \cap S| < \delta$ and $d_C(x) = d(x) \geq \delta$, there exist two vertices $v_i, v_j \in N(x)$ such that $v_{i+1} = v_j$ or $v_i^+ \overrightarrow{C} v_j^- \subset C - S$. Hence the cycle $v_i x v_j \overrightarrow{C} v_i$ contains at least $|C \cap S| + 1$ vertices of S. This contradicts the assumption that C is a swaying cycle. □