$\boxed{12.2}$

$$\frac{\partial \rho}{\partial x_i}(t,x) = -\frac{2x_i}{4t}\frac{1}{(4\pi t)^{\frac{n-1}{2}}}\exp\left(-\frac{\|x\|^2}{4t}\right)$$

$$= -\frac{x_i}{2t}\rho(t,x) \qquad —(*)$$

より

$$\frac{\partial^2 \rho}{\partial x_i^2}(t,x) = \frac{\partial}{\partial x_i}\left(-\frac{x_i}{2t}\rho(t,x)\right)$$

$$= -\frac{1}{2t}\rho(t,x) - \frac{x_i}{2t}\frac{\partial \rho}{\partial x_i}(t,x)$$

$$= \left(-\frac{1}{2t} + \frac{x_i^2}{4t^2}\right)\rho(t,x) \qquad (\because (*))$$

他方

$$\frac{\partial \rho}{\partial t}(t,x) = -\frac{n}{2t}\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{\|x\|^2}{4t}\right) + \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(\frac{\|x\|^2}{4t^2}\right)\exp\left(-\frac{\|x\|^2}{4t}\right)$$

$$= \left(-\frac{n}{2t} + \frac{\|x\|^2}{4t^2}\right)\rho(t,x)$$

より

$$\frac{\partial \rho}{\partial t}(t,x) = \frac{\partial^2 \rho}{\partial x_1^2}(t,x) + \cdots + \frac{\partial^2 \rho}{\partial x_n^2}(t,x)$$

$\boxed{12.4}$

(1) $\dfrac{\partial f}{\partial x_i}(x) = \dfrac{\partial}{\partial x_i}(g(r)) = \dfrac{\partial g}{\partial r}(r)\dfrac{\partial r}{\partial x_i}$　(連鎖公式)

$$= g'(r)\frac{x_i}{r}$$

だから

$$\boxed{\nabla f(x) = g'(r)\frac{x}{r}}$$

(2) $\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial f}{\partial x_i}(x)\right) = \dfrac{\partial}{\partial x_i}\left(\dfrac{g'(r)}{r}x_i\right)$

$$= \left(\frac{\partial}{\partial x_i}x_i\right)\frac{g'(r)}{r} + x_i\frac{\partial}{\partial x_i}\left(\frac{g'(r)}{r}\right)$$

$$= \frac{g'(r)}{r} + x_i\left(\frac{g'(r)}{r}\right)'\frac{x_i}{r} \quad (\because (1))$$

$$= \frac{g'(r)}{r} + \frac{x_i^2}{r^2}g''(r) - \frac{x_i^2}{r^3}g'(r)$$

となる. 従って

$$\Delta f(x) = \sum_{i=1}^{n}\left(\frac{g'(r)}{r} + \frac{x_i^2}{r^2}g''(r) - \frac{x_i^2}{r^3}g'(r)\right)$$

$$= \frac{ng'(r)}{r} + \frac{r^2}{r^2}g''(r) - \frac{r^2}{r^3}g'(r)$$

$$= \boxed{g''(r) + \frac{n-1}{r}g'(r)}$$