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Contents

CHAPTER 1

Introduction

1. Introduction

In this thesis, we consider Hölder continuity for weak solutions to non-linear degenerate parabolic equations. The porous medium equation or the p-harmonic heat flow equation is one of the non-linear degenerate parabolic equations. It is well-known that the solution of the non-linear degenerate parabolic equation is not generally smooth. For example, the Barenblatt solution is an explicit solution to the porous medium equation which is not differentiable.

The Hölder continuity of solutions of these equations was firstly studied by Caffarelli-Friedman [**11**], DiBenedetto-Friedman [**20**] and Wiegner [**51**]. Their results however did not include the case of equations with an external force. It is important to consider the non-linear degenerate parabolic equation with an external force in the application. If we consider the non-linearly perturbed problem, the external term is necessarily treated and it is worth to consider regularity theory for the non-linear degenerate parabolic problem with an external force.

In the presented thesis, we mainly focus on the regularity problem for the non-linear degenerate parabolic equation with an external force and we apply the regularity result to study the large time asymptotic behavior of the solution for the non-linearly perturbed problem.

1.1. Hölder estimates for solutions of the porous medium equations with external forces. In Chapter 2, we study interior Hölder regularity of weak solutions for the Cauchy problem of the porous medium equation with external forces:

$$
\begin{cases} \partial_t u - \Delta u^\alpha = \operatorname{div} f + g, & t > 0\,, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}
$$

where $u = u(t,x) : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is the unknown function, $f = f(t,x) : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $g = g(t,x) : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ and $u_0 = u_0(x) : \mathbb{R}^n \to [0,\infty)$ are given external and initial data and $\alpha > 1$ is a constant. When $f, g \equiv 0$, the equation (1.1) is called a porous medium equation. The porous medium equation is described as the model of gas flow through a porous medium, non-linear heat transfer, ground water flow and population dynamics (cf. Vázquez [**50**]). The porous medium equation is one of the non-linear degenerate parabolic equations, namely the diffusion coefficient $\alpha u^{\alpha-1}$ may vanish. On the point that the diffusion coefficient vanishes, the porous medium equation behaves like the hyperbolic equation. Meanwhile, on the point that the diffusion coefficient does not vanish, the porous medium equation can be regarded as the parabolic equation. Therefore, when we study the porous medium equation, we need to take into account both properties.

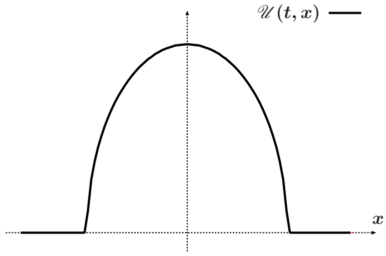

FIGURE 1.1. the Barenblatt solution (t: fixed, $\alpha = 3$)

The solution of the porous medium equation is not necessarily smooth and the solution is not generally differentiable in a classical sense. Therefore, we introduce the notion of weak solutions of the porous medium equation in the sense of distribution.

DEFINITION 1.1. For $u_0 \in L^1(\mathbb{R}^n)$ and for $f, g \in L^1(0,\infty\,; L^1(\mathbb{R}^n))$, we call u a weak solution of (1.1) if there exists $T > 0$ such that

(1) $u(t,x) \geq 0$ for almost all $(t,x) \in [0,T) \times \mathbb{R}^n$;
(2) $u \in L^\infty(0,T\,; L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n))$ with $\nabla u^\alpha \in L^2((0,T) \times \mathbb{R}^n)$;
(3) u satisfies (2.1) in the sense of distribution: For all $\varphi \in C^1(0,T\,; C^1_0(\mathbb{R}^n))$, and for almost all $0 < t < T$,

$$\int_{\mathbb{R}^n} u(t)\varphi(t)\,dx - \int_0^t \int_{\mathbb{R}^n} u\partial_t\varphi\,d\tau dx + \int_0^t \int_{\mathbb{R}^n} \nabla u^\alpha \cdot \nabla\varphi\,d\tau dx$$
$$= \int_{\mathbb{R}^n} u_0\varphi(0)\,dx - \int_0^t \int_{\mathbb{R}^n} f \cdot \nabla\varphi\,d\tau dx + \int_0^t \int_{\mathbb{R}^n} g\varphi\,d\tau dx.$$

The existence of the weak solution is firstly shown by Oleĭnik-Kalašinkov-Čžou [**43**]. J. L. Lions [**31**] also showed by using the Galerkin method (see also Ôtani [**44**]).

From the properties of the parabolic equations, one may expect that the weak solution has certain weaker regularity. On the other hand, since the equation is degenerate, the weak solution is not smooth in general. In fact, if $f, g \equiv 0$, the following Barenblatt solution is one of the explicit weak solution of (1.1):

$$\mathscr{U}(t,x) = \frac{1}{(1+\sigma t)^{\frac{n}{\sigma}}} \left(A - \frac{\alpha-1}{2\alpha} \frac{|x|^2}{(1+\sigma t)^{\frac{2}{\sigma}}} \right)_+^{\frac{1}{\alpha-1}}, \tag{1.2}$$

where $\sigma = n(\alpha-1)+2$, $(h(t,x))_+ := \max\{h(t,x), 0\}$ and $A > 0$ is a constant. In view of (1.2), the solution to (1.1) generally fails the smooth regularity, while one may expect the weak solution have weak regularity such as Hölder continuity. Indeed, the Barenblatt solution (1.2) is Hölder continuous in t and x. Thus we are interested in the Hölder regularity of the weak solution of (1.1) with presence of f and g.

Caffarelli-Friedman [**11**] and Caffarelli-Vázquez-Wolanski [**10**] showed the Hölder continuity of the weak solution of (1.1) under the assumption $f, g \equiv 0$. They essentially used the Aronson-Benilan type pointwise estimate [**2**] and the comparison principle for the weak solution. The Aronson-Benilan estimate is not generally known for the solutions of the porous medium equation with external forces. Furthermore, if the equation is vector valued or involves non-local effect such as the system with other equations, the comparison principle does not generally hold. Therefore, it is worth to derive the Hölder regularity of the weak solution of (1.1) *without using the comparison principle.*

DiBenedetto-Friedman [**19, 20**] and Wiegner [**51**] showed Hölder continuity for the gradient of the weak solution of the p-Laplace evolution equation:

$$\begin{cases} \partial_t v - \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, & t > 0\,, x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

where $p > 2$. Their method does not rely on the comparison principle. If we let $u := |\nabla v|$, then u solves

$$\partial_t u - \operatorname{div}(u^{p-2}\nabla u) = F(\nabla v, D^2 v, D^3 v)$$

for some function F and their method may be applicable to obtain the Hölder continuity of the weak solution of the porous medium equation. In fact, DiBenedetto-Friedman mentioned the Hölder continuity of the weak solution of (1.1) with the external force $f \in L^q\big(0,\infty\,; L^p(\mathbb{R}^n)\big)$ and $g \in L^{\frac{q}{2}}\big(0,\infty\,; L^{\frac{p}{2}}(\mathbb{R}^n)\big)$ satisfying $\frac{2}{q} + \frac{n}{p} < 1$. Our main result is to give the explicit proof of the Hölder estimate of the weak solutions, especially, we show the explicit representation of the Hölder continuity of the weak solution on external forces.

To specify the class of external forces, we introduce weak L^p spaces.

Definition 1.2. For a domain $\Omega \subset \mathbb{R}^n$ and an exponent $p > 1$, a function $f \in L^1_{\mathrm{loc}}(\Omega)$ belongs to $L^p_{\mathrm{w}}(\Omega)$ if

$$\|f\|_{L^p_{\mathrm{w}}(\Omega)} := \sup_{k \subset \Omega\,:\,\text{compact}} \frac{1}{|K|^{1-\frac{1}{p}}} \int_K |f|\,dx < \infty.$$

Our main theorem is the following:

Theorem 1.3. *Let $\alpha > 1$ and let u be a bounded weak solution of* (1.1). *Assume $f \in L^q\big(0,\infty\,; L^p_{\mathrm{w}}(\mathbb{R}^n)\big)$ and $g \in L^{\frac{q}{2}}\big(0,\infty\,; L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n)\big)$ for some $p, q > 2$ satisfying $\frac{2}{q} + \frac{n}{p} < 1$. Then, for all $\varepsilon > 0$, the weak solution u is uniformly Hölder continuous on $(\varepsilon, \infty) \times \mathbb{R}^n$ and there exist constants $C, \gamma > 0$ such that for all $(t,x), (s,y) \in (\varepsilon,\infty) \times \mathbb{R}^n$, we have*

$$|u(t,x) - u(s,y)| \le C(|t-s|^{\frac{\gamma}{2}} + |x-y|^{\gamma}),$$

where $\gamma > 0$ is depending only on n, α, p, q and $C > 0$ is depending only on $n, \alpha, p, q, \varepsilon, f, g$ and $\sup_{(0,\infty)\times\mathbb{R}^n} u$.

We remark that the explicit dependence of the constant C in Theorem 1.3 may be obtained in terms of f, g and $\sup_{(0,\infty)\times\mathbb{R}^n} u$ (cf. Theorem 2.4). We further remark that the pressure function $u^{\alpha-1}$ may be Lipschitz continuous (cf. Caffarelli-Vázquez-Wolanski [**10**]).

The proof of Theorem 1.3 is based on the intrinsic scaling argument and the alternative method by DiBenedetto-Friedman [**20**]. They introduced the modified parabolic cylinders whose dimensions are intrinsically scaled to reflect the degeneracy in (1.1). Since they use the local oscillation of the solution as the intrinsic scaling, it seems difficult to obtain the explicit Hölder estimate of the weak solution. We use the local maximum of the solution as the intrinsic scaling and reconstruct the alternative selection argument. In the course of the proof, the Caccioppoli estimate plays an important role. Including all the term of external forces into the Caccioppoli estimate, we obtain much generalized condition on the external force.

1.2. Regularity and asymptotic behavior for the Keller-Segel system of degenerate type with critical non-linearity. In Chapter 3, we consider the Cauchy problem for the Keller-Segel system of degenerate type:

$$
\begin{cases} \partial_t u - \Delta u^\alpha + \operatorname{div}(u\nabla\psi) = 0, & t>0\,, x\in\mathbb{R}^n, \\ -\Delta\psi + \psi = u, & t>0\,, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x) \geq 0, & x\in\mathbb{R}^n, \end{cases} \tag{1.4}
$$

where $\alpha > 1$. Keller and Segel [**27**] gave a semi-linear parabolic system as the model of chemotaxis. Considering that the diffusion of organisms is depending on the density and taking the zero relaxation time limit, we obtain the degenerate Keller-Segel system (1.4). Since $\psi = (-\Delta+1)^{-1}u$ is given by the Bessel potential of u, the Keller-Segel system (1.4) can be reduced to a single non-linear degenerate parabolic equation with a non-local term. As the case of the porous medium equation, we define the weak solution of (1.4) as in the sense of distribution since the regularity of solutions is not generally available.

Nagai and Mimura [**38**] firstly considered the degenerated model (1.4) as the model of the population dynamics with $n = 1$ (see also Díaz-Galiano-Jüngel [**16, 17**]). Ôtani [**44**] showed the existence of the local solution of (1.4). Sugiyama [**46**] and Sugiyama-Kunii [**48**] studied the existence or non-existence of the global solution of the Keller-Segel system (1.4). They showed that if $\alpha \leq 2 - \frac{2}{n}$ and the initial data u_0 is sufficiently small in some sense, then there exists a global decaying solution. Since the solution goes to zero, we may regard the non-linear term $\operatorname{div}(u\nabla\psi)$ in (1.4) as a small perturbation and the solution of (1.4) asymptotically converges to the Barenblatt solution of the porous medium equation without external forces. In fact, Luckhaus-Sugiyama [**32**] showed the asymptotic profile of the solution in L^p spaces when $1 < \alpha \leq 2 - \frac{2}{n}$, $n \geq 3$ and $1 \leq p \leq \infty$. Ogawa [**40**] showed that if $1 < \alpha < 2 - \frac{2}{n}$, then the algebraic convergence rate is obtained for the remainder term in L^1 space. He used the forward self-similar transform and the Hölder continuity of the rescaled solution. For the critical case $\alpha = 2 - \frac{2}{n}$, since the uniform Hölder regularity of the rescaled solution was not clear, we did not show the explicit convergence rate for the remainder term. Using the uniform Hölder estimate in Chapter 2, we obtain the algebraic convergence rate of the remainder term of the solution for the case of critical non-linearity as $t \to \infty$.

THEOREM 1.4. *Let $\alpha = 2 - \frac{2}{n}$ and $n \geq 3$. Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ sufficiently small in some sense and $|x|^a u_0 \in L^1(\mathbb{R}^n)$ for some $a > n$. Then, there exist $C > 0$ and $\nu > 0$ such that the corresponding global weak solution u of* (1.4) *satisfies*

$$
\|u(t) - \mathscr{U}(t)\|_{L^1(\mathbb{R}^n)} \leq C(1+\sigma t)^{-\nu}, \quad t > 0,
$$

where $\sigma = n(\alpha-1)+2$ and

$$
\mathscr{U}(t,x) = (1+\sigma t)^{-\frac{n}{\sigma}} \left(A - \frac{\alpha-1}{2\alpha} \frac{|x|^2}{(1+\sigma t)^{\frac{2}{\sigma}}} \right)_+^{\frac{1}{\alpha-1}}
$$

is the Barenblatt solution with the constant $A > 0$ satisfying $\|\mathscr{U}\|_1 = \|u_0\|_1$.

To prove Theorem 1.4, we show the decay estimate of the entropy functional of the rescaled solution. We follow the basic strategy to derive the uniform convergence rate by the method due to Carrillo-Toscani [**15**]. The key idea is to derive the decay estimate of the entropy functional and use the relative entropy estimate. To this end, we need to differentiate the entropy functional with respect to the time variable t. Since the equation (1.4) is degenerate, the rescaled solution is not generally differentiable. To overcome this difficulties, we use the uniform Hölder continuity of the rescaled solution and apply it to obtain the decay rate of the entropy functional.

By the explicit decay rate of the remainder term, we obtain the optimal decay rate of the solution of (1.4). Indeed, we find that the optimal decay rate of the solution is the same decay rate as the decay rate of the Barenblatt solution of the porous medium equation.

1.3. Hölder continuity for solutions of the p-harmonic heat flow. In Chapter 4, we consider the following p-harmonic heat flow equation:

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, & t>0,\ x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.5}$$

where $p>2$ is a constant, $u:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ is unknown function, $f:(0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ and $u_0:\mathbb{R}^n\to\mathbb{R}$ are given external and initial data.

It is well-known that the p-harmonic operator $-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is derived by the Euler-Lagrange equation of a p-energy functional $\|\nabla u\|_{L^p}^p$. The evolution equation (1.5) is described as the gradient flow of the p-energy functional with the lower order term.

We are interested in how the regularity of f is reflected in the regularity of solutions. For the case $p=2$, the Hölder and more higher regularity for the solution of (1.5) is well-known (cf. Giaquinta [**23**] and the references there in). For the case $p>2$ and $f\equiv 0$, the Hölder continuity of the gradient of solutions was established by DiBenedetto-Friedman [**19, 20**] and Wiegner [**51**]. Misawa [**33**] showed the gradient Hölder estimate of the solution of (1.5) with the external force f. He assumed the Hölder continuity of the external force with respect to t and x. We generalize his results and give more suitable condition of the external force for the Hölder continuity of the gradient of solutions.

THEOREM 1.5. *Let u be a weak solution of (1.5) satisfying $\nabla u\in L^\infty((0,\infty)\times\mathbb{R}^n)$. Assume that for some constant $K>0$ and $\gamma_0>n+2-\frac{p}{p-1}$, the external force f satisfies*

$$\int_{t_0-R^2}^{t_0}\int_{\{x\in\mathbb{R}^n:|x-x_0|<R\}} |\nabla f|^{\frac{p}{p-1}}\,dxdt \le KR^{\gamma_0} \tag{1.6}$$

for all $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n$ and $0<R<1$ satisfying $(t_0-R^2,t_0)\times\{x\in\mathbb{R}^n:|x-x_0|<R\}\subset(0,\infty)\times\mathbb{R}^n$. Then ∇u is Hölder continuous, namely for all $\varepsilon>0$,

$$|\nabla u(t,x)-\nabla u(s,y)|\le C(|t-s|^{\frac{\gamma}{2}}+|x-y|^{\gamma})$$

for $(t,x),(s,y)\in(\varepsilon,\infty)\times\mathbb{R}^n$, where the constant $\gamma>0$ depends only on n,p,γ_0 and the constant $C>0$ depends only on n,p,γ_0,ε.

To prove Theorem 1.5, we consider the time dependent mean oscillation of f and we show the decay estimate of the mean oscillation of ∇u using the perturbation argument. It is well-known that we need to show the decay estimate of the mean oscillation of ∇u to obtain the Hölder continuity (cf. Campanato [**14**]). By the Morrey type regularity of ∇f, we obtain the decay estimate of the mean oscillation of ∇u. If the external force f is Hölder continuous with respect to t and x, then we have the Morrey type regularity (1.6) of ∇f. Hence our results cover the results of Misawa [**33**].

In Appendix A, we consider some semi-linear parabolic equation related to the mean curvature flow. We study the Harnack inequality of non-negative solutions for the Cauchy problem of the semi-linear parabolic equation. Particularly, we give the explicit dependence of the Harnack inequality on the coefficient of the semi-linear parabolic equation.

2. Notation

In this thesis, we use the following notation. We denote a set of nonnegative integer by $\mathbb{N}_0$. For $\rho, \theta_0 > 0$ and $t_0 \in \mathbb{R}$, we write open intervals $I_\rho(t_0) = (t_0 - \rho^2, t_0)$ and $I_\rho^{\theta_0}(t_0) = (t_0 - \frac{\theta_0}{2}\rho^2, t_0)$. For $\rho > 0$ and $x_0 \in \mathbb{R}^n$, we denote the n-dimensional open ball with radius ρ and center x_0 by $B_\rho(x_0)$. We also denote the open cube with length ρ and center $x_0 = (x_{0,i})_i$ by $K_\rho(x_0) = \{y = (y_i)_i \in \mathbb{R}^n : \max_{1\le i\le n} |x_{0,i} - y_i| < \rho\}$. We define parabolic cylinders $Q_\rho(t_0, x_0)$, $Q_\rho^{\theta_0}(t_0, x_0)$ by $Q_\rho(t_0, x_0) = I_\rho(t_0) \times B_\rho(x_0)$ and $Q_\rho^{\theta_0}(t_0, x_0) = I_\rho^{\theta_0}(t_0) \times B_\rho(x_0)$. For a parabolic cylinder $Q_\rho(t_0, x_0)$, we define a parabolic boundary $\partial_p Q_\rho(t_0, x_0)$ by

$$\partial_p Q_\rho(t_0, x_0) := \Big(\{t_0 - \rho^2\} \times B_\rho(x_0)\Big) \cup \Big(I_\rho(t_0) \times \partial B_\rho(x_0)\Big).$$

For $(t, x), (s, y) \in \mathbb{R} \times \mathbb{R}^n$, we write a parabolic distance $\operatorname{dist}_p\big((t, x), (s, y)\big)$ by

$$\operatorname{dist}_p\big((t, x), (s, y)\big) := \max\big\{|t - s|^{\frac{1}{2}}, |x - y|\big\}.$$

For $A, B \subset \mathbb{R} \times \mathbb{R}^n$, we write a parabolic distance $\operatorname{dist}_p(A, B)$ by

$$\operatorname{dist}_p(A, B) := \inf_{z\in A, z'\in B} \operatorname{dist}_p(z, z').$$

We denote the set of infinitely differentiable functions with compact support in Ω by $C_0^\infty(\Omega)$. We denote the space of p-th integrable functions in Ω by $L^p(\Omega)$. We denote the norm of $L^p(\Omega)$ by $\|f\|_{L^p(\Omega)}$ and if there is no confusion, we write $\|f\|_p = \|f\|_{L^p(\Omega)}$ for short. For $k \in \mathbb{N}$ and $1 \le p \le \infty$, we write the Sobolev space by

$$W^{k,p}(\Omega) := \Big\{u \in L^p(\Omega) : \|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)} < \infty\Big\}.$$

As the Sobolev space $W^{k,2}(\Omega)$ is the Hilbert space, we denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$. The completion $C_0^\infty(\Omega)$ in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$. We define the weak L^p spaces $L^p_{\mathrm{w}}(\Omega)$ by

$$L^p_{\mathrm{w}}(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{L^p_{\mathrm{w}}(\Omega)} := \sup_{k\subset\Omega\,:\,\mathrm{compact}} \frac{1}{|K|^{1-\frac{1}{p}}} \int_K |f|\,dx < \infty \right\}.$$

For $a \in \mathbb{R}$, we define the weighted L^p space $L^p_a(\Omega)$ by

$$L^p_a(\Omega) := \{f \in L^1_{\mathrm{loc}}(\Omega) : |x|^a f \in L^p(\Omega)\}.$$

For a Banach space X and time interval $I \subset \mathbb{R}$, we denote the set of X-valued p-th powered integrable functions in I by $L^p(I\,; X)$ and the set of X-valued essentially bounded maps in I by $L^\infty(I\,; X)$, endowed with a norm

$$\|u\|_{L^p(I\,;X)} := \left(\int_I \|u(t)\|_X^p\,dt\right)^{\frac{1}{p}}, \quad \|u\|_{L^\infty(I\,;X)} := \operatorname*{ess.sup}_{t\in I} \|u(t)\|_X.$$

A Banach-valued function space $L^q(I_{\rho,M}\,; L^p_{\mathrm{w}}(B_\rho))$ is abbreviated to $L^q(L^p_{\mathrm{w}})(Q_{\rho,M})$ and another function spaces are also same. For a set A, we denote the characteristic function by χ_A, namely

$$\chi_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

For a function f on a set A, we denote the oscillation of f in A by $\operatorname{osc}_A f := \sup_A f - \inf_A f$. We denote the positive part of f and the negative part of f by $f_+ := \max\{0, f\}$ and $f_- := \max\{0, -f\}$, respectively. We remark that a superscript plus or minus is different of the positive part or the negative part. For a constant $k \in \mathbb{R}$ and a function f on a set Ω, we let

$$\{f > k\} := \{x \in \Omega : f(x) > k\}$$

and other level sets such as $\{f < k\}$ are defined in a similar manner. For a measurable set $A \subset \mathbb{R}^n$ and an integrable function f on A, we denote an integral mean by

$$(f)_A := \frac{1}{|A|} \int_A f \, dx.$$

We denote a constant depending on $\alpha, \beta, \dots$ by $C(\alpha, \beta, \dots)$. The same letter C will be used to denote different constants. We use subscript numbers if we distinguish between the constants.

CHAPTER 2

Hölder estimates for solutions of the porous medium equation with external forces

1. The porous medium equation with external forces

We consider the following degenerate parabolic equation:

$$
\begin{cases} \partial_t u - \Delta u^\alpha = \operatorname{div} f + g, & t > 0\,, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{2.1}
$$

where $\alpha > 1$ is a constant, $u = u(t,x) : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is unknown, $u_0 = u_0(x) : \mathbb{R}^n \to [0,\infty)$, $f = f(t,x) : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g = g(t,x) : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ are given data. For $f, g \equiv 0$, the equation (2.1) is called the porous medium equation. The equation (2.1) is a degenerate parabolic equation since the diffusion coefficient $\alpha u^{\alpha-1}$ may vanish. It is well-known that solutions of the degenerate parabolic equation (2.1) are not generally smooth even if the initial data u_0 is enough smooth. Thus we introduce the notion of weak solutions.

Definition 2.1. For $u_0 \in L^1(\mathbb{R}^n)$ and for $f, g \in L^1(0,\infty\,; L^1(\mathbb{R}^n))$, we call u a weak solution of (2.1) if there exists $T > 0$ such that

(1) $u(t,x) \geq 0$ for almost all $(t,x) \in [0,T) \times \mathbb{R}^n$;
(2) $u \in L^\infty(0,T\,; L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n))$ with $\nabla u^\alpha \in L^2((0,T) \times \mathbb{R}^n)$;
(3) u satisfies (2.1) in the sense of distribution, namely for all $\varphi \in C^1(0,T\,; C^1_0(\mathbb{R}^n))$ and for almost all $0 < t < T$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} u(t)\varphi(t)\,dx - \int_0^t \int_{\mathbb{R}^n} u\partial_t\varphi\,d\tau dx + \int_0^t \int_{\mathbb{R}^n} \nabla u^\alpha \cdot \nabla\varphi\,d\tau dx \\
= \int_{\mathbb{R}^n} u_0\varphi(0)\,dx - \int_0^t \int_{\mathbb{R}^n} f \cdot \nabla\varphi\,d\tau dx + \int_0^t \int_{\mathbb{R}^n} g\varphi\,d\tau dx.
\end{aligned}
$$

The existence of weak solutions of (2.1) is shown by Oleĭnik-Kalašinkov-Čžou [**43**] and J. L. Lions [**31**] (cf. Ôtani [**44**]). Our aim in this chapter is to obtain Hölder estimates for weak solutions of (2.1).

Caffarelli-Friedman [**11**] and Caffarelli-Vázquez-Wolanski [**10**] showed Hölder continuity for solutions of the porous medium equation (2.1). They essentially use a point-wise estimate for the derivative of solutions given by Aronson-Benilan [**2**] and the comparison principle for the porous medium equation. The Aronson-Benilan type estimate is not known for the general case with the external force. In addition, if the equation involves non-local effect such as the system with other equations, the comparison principle does not generally hold. Therefore, it is worth to derive the regularity of the weak solution of (2.1) without using the comparison principle.

This chapter is based on the paper [**35**].

On the other hand, DiBenedetto-Friedman [**20**], Wiegner [**51**] considered the p-Laplace evolution equation:

$$\begin{cases} \partial_t v - \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, & t > 0\,, x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

The p-Laplace evolution equation is also the degenerate parabolic equation. They showed the Hölder continuity for the gradient of solutions to (2.2) by using the alternative method and the intrinsic scaling argument. Misawa [**33**] showed the gradient Hölder estimate for more general p-Laplace evolution equations. We remark that their methods do not rely on the comparison principle for the p-Laplace evolution equation (2.2). Roughly speaking, the gradient of the solution can be regarded to satisfy (2.1) with $f, g \equiv 0$ and it seems possible to apply their methods for solutions of the problem (2.1). In fact, DiBenedetto-Friedman [**20**] showed Hölder continuity for solutions of (2.1) with $f, g \equiv 0$ and $\alpha > 1$. They mentioned the Hölder continuity of the weak solution of (2.1) involving the external force $f \in L^q\big(0,\infty\,; L^p(\mathbb{R}^n)\big)$ and $g \in L^{\frac{q}{2}}\big(0,\infty\,; L^{\frac{p}{2}}(\mathbb{R}^n)\big)$ with $\frac{2}{q} + \frac{n}{p} < 1$. In this chapter, we extend their results and for more general external forces f, g, we show the Hölder continuity for the solutions of (2.1). In addition, we obtain Hölder estimates of solutions.

Denoting the main theorem, we introduce weak L^p spaces.

Definition 2.2. For a domain $\Omega \subset \mathbb{R}^n$ and an exponent $p > 1$, a function $f \in L^1_{\mathrm{loc}}(\Omega)$ belongs to $L^p_{\mathrm{w}}(\Omega)$ if

$$\|f\|_{L^p_{\mathrm{w}}(\Omega)} := \sup_{k \subset \Omega\,:\,\mathrm{compact}} \frac{1}{|K|^{1-\frac{1}{p}}} \int_K |f|\,dx < \infty.$$

Remark 2.3. By the Hölder inequality, we find $L^p(\Omega) \subset L^p_{\mathrm{w}}(\Omega)$. In fact, $L^p_{\mathrm{w}}(\Omega)$ is strictly larger than $L^p(\Omega)$ since $|x|^{-\frac{n}{p}} \notin L^p(\mathbb{R}^n)$ but is belonging to $L^p_{\mathrm{w}}(\mathbb{R}^n)$.

Now, we state our main theorem.

Theorem 2.4. *Let $\alpha > 1$ and let u be a bounded weak solution of* (2.1)*. Assume that $f \in L^q(0,\infty\,; L^p_{\mathrm{w}}(\mathbb{R}^n))$ and $g \in L^{\frac{q}{2}}(0,\infty\,; L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))$ for some $p, q > 2$ satisfying $\frac{2}{q} + \frac{n}{p} < 1$. Then, for all $\varepsilon > 0$, the solution u is uniform Hölder continuous with respect to (t,x) in $(\varepsilon,\infty) \times \mathbb{R}^n$. Precisely, there exist constants $C, \gamma > 0$ such that*

$$\begin{aligned} |u(t,x) - u(s,y)| \le C\Big(&\|u\|_{L^\infty((0,\infty)\times\mathbb{R}^n)} \\ &+ \|u\|^{\frac{1}{q}(1-\frac{1}{\alpha})}_{L^\infty((0,\infty)\times\mathbb{R}^n)} \big\|f\big\|^{\frac{1}{\alpha}}_{L^q(0,\infty\,;L^p_{\mathrm{w}}(\mathbb{R}^n))} + \|u\|^{\frac{2}{q}(1-\frac{1}{\alpha})}_{L^\infty((0,\infty)\times\mathbb{R}^n)} \|g\|^{\frac{1}{\alpha}}_{L^{\frac{q}{2}}(0,\infty\,;L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))}\Big) \\ &\times \big(\|u\|^{\frac{\gamma}{2}(1-\frac{1}{\alpha})}_{L^\infty((0,\infty)\times\mathbb{R}^n)} |t-s|^{\frac{\gamma}{2}} + |x-y|^\gamma\big) \end{aligned}$$

for all $(t,x), (s,y) \in (\varepsilon,\infty) \times \mathbb{R}^n$, where $\gamma > 0$ depends only on n, α, p, q and $C > 0$ depends only on $n, \alpha, p, q, \varepsilon$.

Remark 2.5. Our result is also valid for interior Hölder estimates for solutions. Indeed, for $f \in L^q_{\mathrm{loc}}\big(0,\infty\,; L^p_{\mathrm{w,loc}}\big)$ and $g \in L^{\frac{q}{2}}_{\mathrm{loc}}\big(0,\infty\,; L^{\frac{p}{2}}_{\mathrm{w,loc}}\big)$ with $\frac{2}{q} + \frac{n}{p} < 1$, we obtain interior Hölder continuity for solutions.

Remark 2.6. The pressure function $u^{\alpha-1}$ may be Lipschitz continuous (cf. Caffarelli-Vázquez-Wolanski [**10**])

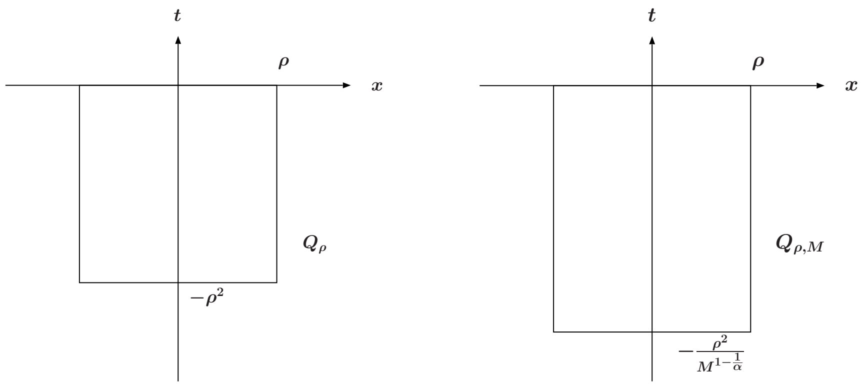

FIGURE 2.1. the usual parabolic cylinder and the modified parabolic cylinder

The basic strategy to prove Theorem 2.4 is to use the intrinsic scaling argument and the alternative method by DiBenedetto-Friedman [**20**]. Since they use the local oscillation of the solution as the intrinsic scaling, it seems difficult to obtain the Hölder estimate of the solution. On the other hand, we use the local maximum of the solution as the intrinsic scaling and we make the more exact Caccioppoli estimate. The Caccioppoli estimate plays the important role to show the alternative method. Reconstructing the iteration argument, we obtain the Hölder estimate of the solution.

For an application, we may consider the external force as the perturbation of the solution (cf. [**41**]). Applying our theorem, we do not need L^p integrability of the external force, but growth order of L^2 integral. Therefore, it is useful to study L^2 theory of non-linear degenerate parabolic equations. Furthermore, we can exactly estimate the Hölder norm of the solution by the external force and the maximum of the solution.

The chapter is organized as follows. In section 2, we firstly give an alternative lemma and we show Theorem 2.4 using the alternative lemma. The alternative lemma gives either the better lower bounds or the better upper bounds of the solution. We show the lower bounds of the solution in section 3 and the upper bounds of the solution in section 4.

At the end of this section, we introduce some notations in this chapter. For $\rho, M, \theta_0 > 0$ and $t_0 \in \mathbb{R}$, we define open intervals $I_{\rho,M}(t_0)$ and $I^{\theta_0}_{\rho,M}(t_0)$ by

$$I_{\rho,M}(t_0) := \Big(t_0 - \frac{\rho^2}{M^{1-\frac{1}{\alpha}}}, t_0\Big), \quad I^{\theta_0}_{\rho,M}(t_0) := \Big(t_0 - \frac{\theta_0}{2}\frac{\rho^2}{M^{1-\frac{1}{\alpha}}}, t_0\Big).$$

For $x_0 \in \mathbb{R}^n$, we define modified parabolic cylinders $Q_{\rho,M}(t_0, x_0)$ and $Q^{\theta_0}_{\rho,M}(t_0, x_0)$ by

$$Q_{\rho,M}(t_0, x_0) := I_{\rho,M}(t_0) \times B_\rho(x_0), \quad Q^{\theta_0}_{\rho,M}(t_0, x_0) := I^{\theta_0}_{\rho,M}(t_0) \times B_\rho(x_0).$$

We often abbreviate the center of parabolic cylinders (t_0, x_0). We put $\gamma_0 = 1 - \frac{2}{q} - \frac{n}{p}$ and $h(\rho, M, \omega) := \big\|f\big\|^2_{L^q(L^p_{\mathrm{w}})(Q_{\rho,M})} + \omega\|g\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}$. For a function f on a set A, we denote the oscillation of f in A by $\operatorname{osc}_A f := \sup_A f - \inf_A f$. For a set $A \subset \mathbb{R}^n$, we denote the n-dimensional Lebesgue measure by $|A|$. For an open interval $(a, b) \subset \mathbb{R}$ and an open ball $B_\rho(x_0) \subset \mathbb{R}^n$, we call $\eta = \eta(t, x)$ a cut-off function in $Q = (a, b) \times B_\rho(x_0)$ if $\eta \in C^\infty(Q)$ satisfies

$$\eta(t, x) \equiv 0 \quad a \le t \le b, \quad x \in \partial B_\rho(x_0) \quad \text{and} \quad \eta(a, x) \equiv 0 \quad x \in B_\rho(x_0).$$

2. Alternative lemma and proof of the main theorem

We hereafter replace u^α by u and we consider the following equation:

$$\partial_t u^{\frac{1}{\alpha}} - \Delta u = -\operatorname{div} f + g. \tag{2.3}$$

Let M and ω be an approximated supremum and oscillation of the weak solution u, namely

$$\sup_{Q_{\rho,M}(t_0,x_0)} u \le M \le 3 \sup_{Q_{\rho,M}(t_0,x_0)} u, \tag{2.4}$$

and

$$\frac{3}{4}\omega \le \operatorname*{osc}_{Q_{\rho,M}(t_0,x_0)} u \le \omega. \tag{2.5}$$

Lemma 2.7 (alternative lemma). *Let us assume* (2.4) *and* (2.5). *Then there exist constants $0 < \theta_0, \eta_0 < 1$ and $\delta_0 > 0$ depending only on n, α, p, q such that for all $\rho > 0$ satisfying $\rho^{\gamma_0} \le \delta_0 \omega M^{-\frac{1}{q}(1-\frac{1}{\alpha})} h(\rho, M, \omega)^{-\frac{1}{2}}$, where $h(\rho, M, \omega) := \big\|f\big\|^2_{L^q(L^p_{\mathrm{w}})(Q_{\rho,M})} + \omega \|g\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}$, we obtain the following estimates:*

i) (the lower bounds) Either if

$$\left|Q_{\rho,M}(t_0,x_0) \cap \left\{u < \inf_{Q_{\rho,M}(t_0,x_0)} u + \frac{\omega}{2}\right\}\right| \le \theta_0 |Q_{\rho,M}(t_0,x_0)|,$$

where $\left|Q_{\rho,M}(t_0,x_0) \cap \left\{u < \inf_{Q_{\rho,M}(t_0,x_0)} u + \frac{\omega}{2}\right\}\right|$ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$, then we have

$$u(t,x) \ge \inf_{Q_{\rho,M}(t_0,x_0)} u + \eta_0 \omega \quad \text{for } (t,x) \in Q_{\frac{\rho}{2},M}(t_0,x_0);$$

ii) (the upper bounds) otherwise, if

$$\left|Q_{\rho,M}(t_0,x_0) \cap \left\{u < \inf_{Q_{\rho,M}(t_0,x_0)} u + \frac{\omega}{2}\right\}\right| > \theta_0 |Q_{\rho,M}(t_0,x_0)|,$$

then we have

$$u(t,x) \le \sup_{Q_{\rho,M}(t_0,x_0)} u - \eta_0 \omega \quad \text{for } (t,x) \in Q^{\theta_0}_{\frac{\rho}{2},M}(t_0,x_0).$$

According to Lemma 2.7, we obtain

$$\operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho}{2},M}(t_0,x_0)} u \le \operatorname*{osc}_{Q_{\rho,M}(t_0,x_0)} u - \eta_0 \omega \le (1-\eta_0)\omega$$

provided $\rho^{\gamma_0} \le \delta_0 \omega M^{-\frac{1}{q}(1-\frac{1}{\alpha})} h(\rho, M, \omega)^{-\frac{1}{2}}$. We remark that we may take $\eta_0 \le \frac{1}{4}$.

Remark 2.8. We explain an advantage to use the modified parabolic cylinder. For $\rho \ll 1$ and $M > 0$, we consider

$$\partial_t u^{\frac{1}{\alpha}} - \Delta u = -\operatorname{div} f + g \quad \text{in } Q_{\rho,M}.$$

Introducing the scale transform

$$t = \frac{\rho^2}{M^{1-\frac{1}{\alpha}}} s, \qquad x = \rho y,$$
$$u_{\rho,M}(s,y) = \frac{1}{M} u(t,x), \qquad f_{\rho,M}(s,y) = f(t,x), \qquad g_{\rho,M}(s,y) = g(t,x),$$

we obtain

$$\partial_s u_{\rho,M}^{\frac{1}{\alpha}} - \Delta_y u_{\rho,M} = -\operatorname{div}\Big(\frac{\rho}{M} f_{\rho,M}\Big) + \frac{\rho^2}{M} g_{\rho,M} \quad \text{in } Q_1.$$

Considering $M \cong \sup_{Q_{\rho,M}(t_0,x_0)} u$, which is corresponding to the assumption (2.4), we can regard the smoothing effect of the equation as uniformly. Furthermore, in view of

$$\begin{aligned}
\Big\| \frac{\rho}{M} f_{\rho,M} \Big\|_{L^q(L^p_{\mathrm{w}})(Q_1)} &= \rho^{1-\frac{2}{q}-\frac{n}{p}} M^{-1+\frac{1}{q}(1-\frac{1}{\alpha})} \|f\|_{L^q(L^p_{\mathrm{w}})(Q_{\rho,M}(t_0,x_0))}, \\
\Big\| \frac{\rho^2}{M} g_{\rho,M} \Big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_1)} &= \rho^{2(1-\frac{2}{q}-\frac{n}{p})} M^{-1+\frac{2}{q}(1-\frac{1}{\alpha})} \|g\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M}(t_0,x_0))},
\end{aligned}$$

the inequality $1 - \frac{2}{q} - \frac{n}{p} > 0$ is the sufficient condition to ignore the external force.

Proof of Theorem 2.4. We show Theorem 2.4 by temporary admitting the alternative lemma, Lemma 2.7. We put $Q = (0,\infty) \times \mathbb{R}^n$, $M_0 = \sup_Q u$ and $\omega_0 = M_0$. Let θ_0, δ_0 and η_0 be as in Lemma 2.7. We choose $0 < \rho_0 < \varepsilon$ satisfying

$$\rho_0^{\gamma_0} \le \delta_0 \omega_0 M_0^{-\frac{1}{q}(1-\frac{1}{\alpha})} \Big(\|f\|^2_{L^q(0,\infty;L^p_{\mathrm{w}}(\mathbb{R}^n))} + \omega_0 \|g\|_{L^{\frac{q}{2}}(0,\infty;L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))} \Big)^{-\frac{1}{2}}.$$

For $(t_0,x_0) \in (0,\infty) \times \mathbb{R}^n$, we denote $Q_0 = Q_{\rho_0,M_0}(t_0,x_0)$, $\mu_0^+ = \sup_{Q_0} u$ and $\mu_0^- = \inf_{Q_0} u$. Then, we find

$$\begin{cases}
\operatorname*{osc}_{Q_0} u \le \omega_0, \\
\sup_{Q_0} u \le \sup_{Q} u \le M_0, \\
\quad \rho_0^{\gamma_0} \le \delta_0 \omega_0 M_0^{-\frac{1}{q}(1-\frac{1}{\alpha})} h(\rho_0, M_0, \omega_0)^{-\frac{1}{2}},
\end{cases}$$

where $h(\rho_0,M_0,\omega_0) = \|f\|^2_{L^q(L^p_{\mathrm{w}})(Q_0)} + \omega_0 \|g\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_0)}$. We choose

$$r_0 := \min\Big\{ (1-\eta_0)^{\frac{1}{\gamma_0}},\ \frac{1}{2}\Big(\frac{1}{3}\Big)^{\frac{1}{2}(1-\frac{1}{\alpha})} \Big(\frac{\theta_0}{2}\Big)^{\frac{1}{2}} \Big\}$$

and choose sequences as follows: For $j \in \mathbb{N}$,

$$\begin{aligned}
\omega_j &:= (1-\eta_0)\omega_{j-1}, & \rho_j &:= r_0 \rho_{j-1}, \\
M_j &:= \max\{\mu_{j-1}^+,\ \omega_j\}, & Q_j &:= Q_{\rho_j,M_j}(t_0,x_0), \\
\mu_j^+ &:= \sup_{Q_j} u, & \mu_j^- &:= \inf_{Q_j} u.
\end{aligned} \tag{2.6}$$

LEMMA 2.9. *Let* $\{\omega_j, \rho_j, M_j, Q_j\}_{j=1}^{\infty}$ *is defined the above* (2.6)*. Then for* $0 < \delta_0 < 1$ *defined in Lemma 2.7 and for* $j \in \mathbb{N}$*, we obtain*

$$\begin{cases} \operatorname*{osc}_{Q_j} u \le \omega_j, \\ \sup_{Q_j} u \le \sup_{Q_{j-1}} u \le M_j, \\ \rho_j^{\gamma_0} \le \delta_0 \omega_j M_j^{-\frac{1}{q}(1-\frac{1}{\alpha})} h(\rho_j, M_j, \omega_j)^{-\frac{1}{2}}. \end{cases} \tag{2.7}$$

PROOF OF LEMMA 2.9. By the definition of M_j, we obtain $\sup_{Q_j} u \le M_j$. Since $r_0 \le (1-\eta_0)^{\frac{1}{\gamma_0}}$ and $\omega_j = (1-\eta_0)\omega_{j-1}$, we find $\rho_j^{\gamma_0} \le \delta_0 \omega_j M_j^{-\frac{1}{q}(1-\frac{1}{\alpha})} h(\rho_j, M_j, \omega_j)^{-\frac{1}{2}}$. We show $\operatorname{osc}_{Q_j} u \le \omega_j$.

To show $\operatorname{osc}_{Q_j} u \le \omega_j$, we make induction. We firstly consider the case $j = 1$. Either if $\operatorname{osc}_{Q_0} u \le \frac{3}{4}\omega_0$, then we find $Q_1 \subset Q_0$ since $r_0 \le (\frac{3}{4})^{\frac{1}{2}(1-\frac{1}{\alpha})}$ and

$$\frac{M_1}{M_0} \ge \frac{\omega_1}{M_0} = (1-\eta_0) \ge \frac{3}{4}.$$

For this reason, we obtain

$$\operatorname*{osc}_{Q_1} u \le \operatorname*{osc}_{Q_0} u \le \frac{3}{4}\omega_0 \le (1-\eta_0)\omega_0 = \omega_1.$$

Otherwise, if $\frac{3}{4}\omega_0 \le \operatorname{osc}_{Q_0} u$, we obtain $M_0 = \omega_0 \le \frac{4}{3}\mu_0^+$. Applying Lemma 2.7, we find

$$\operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho_0}{2}, M_0}(t_0, x_0)} u \le (1-\eta_0)\omega_0.$$

Since $r_0 \le \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{2}(1-\frac{1}{\alpha})}\left(\frac{\theta_0}{2}\right)^{\frac{1}{2}}$, we have $Q_1 \subset Q^{\theta_0}_{\frac{\rho_0}{2}, M_0}(t_0, x_0) \subset Q_0$ and hence

$$\operatorname*{osc}_{Q_1} u \le \operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho_0}{2}, M_0}(t_0, x_0)} u \le (1-\eta_0)\omega_0 = \omega_1.$$

In either case, we obtain (2.7) for $j = 1$. Next, we assume (2.7) for $j \le k$ and we show for $j = k+1$. First, we give the following inequality:

$$\mu_{k-1}^+ \le \max\left\{\frac{3}{2(1-\eta_0)}\omega_k \,,\, 3\mu_k^+\right\}. \tag{2.8}$$

To show (2.8), we consider the case $\mu_{k-1}^- \le \frac{1}{3}\mu_{k-1}^+$ first. Then

$$\mu_{k-1}^+ \le \operatorname*{osc}_{Q_{k-1}} u + \mu_{k-1}^- \le \omega_{k-1} + \frac{1}{3}\mu_{k-1}^+$$

and hence $\mu_{k-1}^+ \le \frac{3}{2}\omega_{k-1} = \frac{3}{2(1-\eta_0)}\omega_k$. For the other case $\mu_{k-1}^- > \frac{1}{3}\mu_{k-1}^+$, we have $\mu_{k-1}^+ < 3\mu_{k-1}^- \le 3\mu_k^- \le 3\mu_k^+$ and we obtain (2.8).

We show (2.7) for $j = k+1$. First, we consider the case $\operatorname{osc}_{Q_k} u \le \frac{3}{4}\omega_k$ and we show $Q_{k+1} \subset Q_k$. Either if $M_k = \omega_k$, then

$$\frac{M_{k+1}}{M_k} = \frac{M_{k+1}}{\omega_k} \ge \frac{(1-\eta_0)\omega_k}{\omega_k} = (1-\eta_0) \ge \frac{3}{4}.$$

Since $r_0 \le (\frac{3}{4})^{\frac{1}{2}(1-\frac{1}{\alpha})}$, we obtain $Q_{k+1} \subset Q_k$. Otherwise, if $M_k = \mu_{k-1}^+$, we obtain by (2.8),

$$\begin{aligned}\frac{M_{k+1}}{M_k} = \frac{M_{k+1}}{\mu_{k-1}^+} &\ge \frac{M_{k+1}}{\max\left\{\frac{3}{2(1-\eta_0)}\omega_k\,,\,3\mu_k^+\right\}} \\ &\ge \frac{1}{\max\left\{\frac{3}{2(1-\eta_0)}\frac{\omega_k}{M_{k+1}}\,,\,\frac{3\mu_k^+}{M_{k+1}}\right\}} \\ &\ge \frac{1}{\max\left\{\frac{3}{2(1-\eta_0)^2}\,,\,3\right\}} \ge \frac{1}{3}.\end{aligned}$$

Since $r_0 \le (\frac{1}{3})^{\frac{1}{2}(1-\frac{1}{\alpha})}$, we have $Q_{k+1} \subset Q_k$. In either case, we have $Q_{k+1} \subset Q_k$ and hence

$$\operatorname*{osc}_{Q_{k+1}} u \le \operatorname*{osc}_{Q_k} u \le \frac{3}{4}\omega_k \le \omega_{k+1}.$$

Second, we consider the case $\frac{3}{4}\omega_k \le \operatorname{osc}_{Q_k} u \le \omega_k$. Since $\omega_k \le \frac{4}{3}\mu_k^+$, we obtain

$$\mu_{k-1}^+ \le \max\left\{\frac{3}{2(1-\eta_0)}\omega_k\,,\,3\mu_k^+\right\} \le \max\left\{\frac{2}{(1-\eta_0)}\mu_k^+\,,\,3\mu_k^+\right\} \le 3\mu_k^+$$

and hence

$$M_k \le \max\left\{\frac{4}{3}\mu_k^+\,,\,3\mu_k^+\right\} \le 3\mu_k^+.$$

Hence we apply Lemma 2.7 and we obtain

$$\operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho_k}{2},M_k}(t_0,x_0)} u \le (1-\eta_0)\omega_k = \omega_{k+1}.$$

Since $r_0 \le \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{2}(1-\frac{1}{\alpha})}\left(\frac{\theta_0}{2}\right)^{\frac{1}{2}}$ and $\frac{M_{k+1}}{M_k} \ge \frac{\mu_k^+}{3\mu_k^+} = \frac{1}{3}$, we have $Q_{k+1} \subset Q^{\theta_0}_{\frac{\rho_k}{2},M_k}(t_0,x_0)$ and hence

$$\operatorname*{osc}_{Q_{k+1}} u \le \operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho_k}{2},M_k}(t_0,x_0)} u \le \omega_{k+1}.$$

□

Remarking that $M_j \ge M_{j+1}$ for $j \in \mathbb{N}$, we have

$$\operatorname*{osc}_{Q_{\rho_j,M_0}(t_0,x_0)} u \le \operatorname*{osc}_{Q_j} u \le \omega_j.$$

We choose $0 < \gamma < 1$ satisfying $r_0^\gamma \ge 1-\eta_0$. Then, we obtain

$$\operatorname*{osc}_{Q_{\rho_j,M_0}(t_0,x_0)} u \le (1-\eta_0)^j\omega_0 = \omega_0\left(\frac{\rho_j}{\rho_0}\right)^\gamma.$$

For $\rho \le \rho_0$, there exists $k \in \mathbb{N}$ such that $\rho_k \le \rho \le \rho_{k-1}$ and hence

$$\operatorname*{osc}_{Q_{\rho,M_0}(t_0,x_0)} u \le \omega_0\left(\frac{\rho_{k-1}}{\rho_0}\right)^\gamma = \omega_0 r_0^{-\gamma}\left(\frac{\rho_k}{\rho_0}\right)^\gamma \le M_0 r_0^{-\gamma}\left(\frac{\rho}{\rho_0}\right)^\gamma.$$

Taking $\rho_0 > 0$ as $\rho_0^{\gamma_0} = \delta_0\omega_0 M_0^{-\frac{1}{q}(1-\frac{1}{\alpha})}\Big(\|f\|^2_{L^q(0,\infty\,;L^p_{\mathrm{w}}(\mathbb{R}^n))} + \omega_0\|g\|_{L^{\frac{q}{2}}(0,\infty\,;L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))}\Big)^{-\frac{1}{2}}$, we find

$$\operatorname*{osc}_{Q_{\rho,M_0}(t_0,x_0)} u \le CM_0^{1-\frac{\gamma}{\gamma_0}} M_0^{\frac{\gamma}{q\gamma_0}(1-\frac{1}{\alpha})}\Big(\|f\|^2_{L^q(0,\infty\,;L^p_{\mathrm{w}}(\mathbb{R}^n))} + \omega_0\|g\|_{L^{\frac{q}{2}}(0,\infty\,;L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))}\Big)^{\frac{\gamma}{2\gamma_0}}\rho^{\gamma}$$

for $\rho \le \rho_0$ where the constant C depends only on n, α, p and q. Furthermore, if $\rho > \rho_0$, then

$$\begin{aligned}\operatorname*{osc}_{Q_{\rho,M_0}(t_0,x_0)} u &\le M_0 \le M_0\Big(\frac{\rho}{\rho_0}\Big)^{\gamma}\\ &\le CM_0^{1-\frac{\gamma}{\gamma_0}} M_0^{\frac{\gamma}{q\gamma_0}(1-\frac{1}{\alpha})}\Big(\|f\|^2_{L^q(0,\infty\,;L^p_{\mathrm{w}}(\mathbb{R}^n))} + \omega_0\|g\|_{L^{\frac{q}{2}}(0,\infty\,;L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))}\Big)^{\frac{\gamma}{2\gamma_0}}\rho^{\gamma}.\end{aligned}$$

Therefore, we find

$$\begin{aligned}\operatorname*{osc}_{Q_{\rho,M_0}(t_0,x_0)} u &\le C\Big(M_0 + M_0^{\frac{1}{q}(1-\frac{1}{\alpha})}\Big(\|f\|^2_{L^q(0,\infty\,;L^p_{\mathrm{w}}(\mathbb{R}^n))} + M_0\|g\|_{L^{\frac{q}{2}}(0,\infty\,;L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))}\Big)^{\frac{1}{2}}\Big)\rho^{\gamma}\\ &\le C\Big(M_0 + M_0^{\frac{1}{q}(1-\frac{1}{\alpha})}\|f\|_{L^q(0,\infty\,;L^p_{\mathrm{w}}(\mathbb{R}^n))} + M_0^{\frac{2}{q}(1-\frac{1}{\alpha})}\|g\|_{L^{\frac{q}{2}}(0,\infty\,;L^{\frac{p}{2}}_{\mathrm{w}}(\mathbb{R}^n))}\Big)\rho^{\gamma}\end{aligned}$$

and proof of Theorem 2.4 is complete. □

3. Proof of the lower bounds

Without loss of generality, we assume $t_0 = 0$ by using the parallel translation. And we omit the center of ball x_0. We hereafter write $\mu^+ = \sup_{Q_{\rho,M}} u$, $\mu^- = \inf_{Q_{\rho,M}} u$.

In this section, we show the lower bounds in Lemma 2.7. More precisely, we show the following proposition:

Proposition 2.10. *Let $\rho > 0$ satisfying $\rho^{\gamma_0} \le \omega M^{-\frac{1}{q}(1-\frac{1}{\alpha})}h(\rho, M, \omega)^{-\frac{1}{2}}$. Assume the inequality* (2.4) *and* (2.5). *Then there exists $0 < \theta_0 < 1$ depending only on n, α, p, q such that if*

$$\Big|Q_{\rho,M} \cap \Big\{u < \mu^- + \frac{\omega}{2}\Big\}\Big| \le \theta_0 |Q_{\rho,M}|,$$

where $\Big|Q_{\rho,M} \cap \big\{u < \mu^- + \frac{\omega}{2}\big\}\Big|$ denotes the Lebesgue measure on $\mathbb{R}^{n+1}$, then we have

$$u(t,x) \ge \mu^- + \frac{\omega}{4} \quad \textit{for } (t,x) \in Q_{\frac{\rho}{2},M}.$$

To show the lower bounds, the following Caccioppoli estimate plays the important role.

Lemma 2.11 (the Caccioppoli estimate for sub-level sets). *Let $\eta = \eta(t,x)$ be a cut-off function in $Q_{\rho,M}$. For $\mu^- < k < \mu^- + \frac{1}{2}\omega$, there exists a constant $C > 0$ depending only* *on α such that*

$$\begin{aligned}(2.9)\quad &\sup_{t\in I_{\rho,M}}\int_{B_\rho}(u(t)-k)_-^2\eta^2(t)\,dx+(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}|\nabla(u-k)_-|^2\eta^2\,dtdx\\ &\le C\Big\{\omega\iint_{Q_{\rho,M}}(u-k)_-\eta\partial_t\eta\,dtdx+(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}(u-k)_-^2|\nabla\eta|^2\,dtdx\\ &\qquad+(\mu^+)^{1-\frac{1}{\alpha}}h(\rho,M,\omega)\Big(\int_{I_{\rho,M}}|B_\rho\cap\{u(t)<k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,dt\Big)^{\frac{2}{q'}}\Big\},\end{aligned}$$

where $\frac{1}{2}=\frac{1}{q}+\frac{1}{q'}$ and $|B_\rho\cap\{u(t)<k\}|$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Proof of Lemma 2.11. Testing the function $-(u-k)_-\eta^2$ to the equation (2.3), we obtain

$$\begin{aligned}&\frac{1}{\alpha}\iint_{Q_{\rho,M}}\partial_t\Big(\int_0^{(u-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\Big)\eta^2\,dtdx+\iint_{Q_{\rho,M}}\nabla(u-k)_-\cdot\nabla\{(u-k)_-\eta^2\}\,dtdx\\ &=-\iint_{Q_{\rho,M}}f\cdot\nabla\{(u-k)_-\eta^2\}\,dtdx-\iint_{Q_{\rho,M}}g(u-k)_-\eta^2\,dtdx.\end{aligned}$$

By the integration by parts, we have

$$\begin{aligned}&\frac{1}{\alpha}\sup_{t\in I_{\rho,M}}\int_{B_\rho}\Big(\int_0^{(u(t)-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\Big)\eta^2(t)\,dx+\iint_{Q_{\rho,M}}|\nabla(u-k)_-|^2\eta^2\,dtdx\\ &\le\frac{1}{\alpha}\iint_{Q_{\rho,M}}\Big(\int_0^{(u-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\Big)\partial_t\eta^2\,dtdx-\iint_{Q_{\rho,M}}(\nabla(u-k)_-\cdot\nabla\eta^2)(u-k)_-\,dtdx\\ &-\iint_{Q_{\rho,M}}f\cdot\nabla(u-k)_-\eta^2\,dtdx-\iint_{Q_{\rho,M}}f\cdot\nabla\eta^2(u-k)_-\,dtdx-\iint_{Q_{\rho,M}}g(u-k)_-\eta^2\,dtdx.\end{aligned}$$

Using the Young inequality, we obtain

$$\begin{aligned}(2.10)\quad &\frac{1}{\alpha}\sup_{t\in I_{\rho,M}}\int_{B_\rho}\Big(\int_0^{(u(t)-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\Big)\eta^2(t)\,dx+\frac{1}{4}\iint_{Q_{\rho,M}}|\nabla(u-k)_-|^2\eta^2\,dtdx\\ &\le\frac{1}{\alpha}\iint_{Q_{\rho,M}}\Big(\int_0^{(u-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\Big)\partial_t\eta^2\,dtdx+3\iint_{Q_{\rho,M}}(u-k)_-^2|\nabla\eta|^2\,dtdx\\ &\qquad+2\iint_{Q_{\rho,M}\cap\{u<k\}}|f|^2\eta^2\,dtdx+\iint_{Q_{\rho,M}\cap\{u<k\}}|g|(u-k)_-\eta^2\,dtdx.\end{aligned}$$

We estimate the first term of the left-hand side of (2.10). Since (2.5) and $k\le\mu^-+\frac{\omega}{2}\le\mu^+-\operatorname{osc}_{Q_{\rho,M}}u+\frac{\omega}{2}\le\mu^+$, we have

$$(k-\xi)^{\frac{1}{\alpha}-1}\ge k^{\frac{1}{\alpha}-1}\ge(\mu^+)^{\frac{1}{\alpha}-1}\quad\text{for }\ \xi\ge 0$$

and hence

(2.11)
$$
\begin{aligned}
\frac{1}{2\alpha}\sup_{t\in I_{\rho,M}}\int_{B_\rho}(u(t)-k)_-^2\eta^2(t)\,dx+\frac{1}{4}(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}|\nabla(u-k)_-|^2\eta^2\,dtdx&\\
\le \frac{1}{\alpha}(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}\left(\int_0^{(u-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi\right)\partial_t\eta^2\,dtdx&\\
+3(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}}(u-k)_-^2|\nabla\eta|^2\,dtdx&\\
++2(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}\cap\{u<k\}}|f|^2\eta^2\,dtdx&\\
+(\mu^+)^{1-\frac{1}{\alpha}}\iint_{Q_{\rho,M}\cap\{u<k\}}|g|(u-k)_-\eta^2\,dtdx&\\
=: I_1+I_2+I_3+I_4.&
\end{aligned}
$$

We estimate I_3 and I_4. By the definition of the weak L^p space and by the Hölder inequality, we have

$$
\begin{aligned}
\iint_{Q_{\rho,M}\cap\{u<k\}}|f|^2\eta^2\,dtdx&=\int_{I_{\rho,M}}dt\int_{B_\rho\cap\{u(t)<k\}}|f|^2\,dx\\
&\le\int_{I_{\rho,M}}\big\||f(t)|^2\big\|_{L_{\mathrm{w}}^{\frac{p}{2}}(B_\rho)}|B_\rho\cap\{u(t)<k\}|^{1-\frac{2}{p}}\,dt\\
&\le\big\||f|^2\big\|_{L^{\frac{q}{2}}(L_{\mathrm{w}}^{\frac{p}{2}})(Q_{\rho,M})}\left(\int_{I_{\rho,M}}|B_\rho\cap\{u(t)<k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,dt\right)^{\frac{2}{q'}}
\end{aligned}
$$

and

$$
\begin{aligned}
\iint_{Q_{\rho,M}\cap\{u<k\}}|g|(u-k)_-\eta^2\,dtdx&=\frac{\omega}{2}\int_{I_{\rho,M}}dt\int_{B_\rho\cap\{u(t)<k\}}|g|\,dx\\
&\le\frac{\omega}{2}\int_{I_{\rho,M}}\|g(t)\|_{L_{\mathrm{w}}^{\frac{p}{2}}(B_\rho)}|B_\rho\cap\{u(t)<k\}|^{1-\frac{2}{p}}\,dt\\
&\le\frac{\omega}{2}\|g\|_{L^{\frac{q}{2}}(L_{\mathrm{w}}^{\frac{p}{2}})(Q_{\rho,M})}\left(\int_{I_{\rho,M}}|B_\rho\cap\{u(t)<k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,dt\right)^{\frac{2}{q'}}.
\end{aligned}
$$

Therefore

$$
I_3+I_4\le 2(\mu^+)^{1-\frac{1}{\alpha}}h(\rho,M,\omega)\left(\int_{I_{\rho,M}}|B_\rho\cap\{u(t)<k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,dt\right)^{\frac{2}{q'}}. \tag{2.12}
$$

Second, we estimate I_1. Since

$$
\begin{aligned}
\int_0^{(u-k)_-}(k-\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi&\le-\alpha(u-k)_-\int_0^{(u-k)_-}\frac{\partial}{\partial\xi}(k-\xi)^{\frac{1}{\alpha}}\,d\xi\\
&=\alpha(u-k)_-\Big[k^{\frac{1}{\alpha}}-\big(k-(u-k)_-\big)^{\frac{1}{\alpha}}\Big],
\end{aligned}
$$

we have

$$I_1 \le (\mu^+)^{1-\frac{1}{\alpha}} \iint_{Q_{\rho,M}} [k^{\frac{1}{\alpha}} - (k-(u-k)_-)^{\frac{1}{\alpha}}](u-k)_-\partial_t\eta^2\,dtdx$$

$$\le (\mu^+)^{1-\frac{1}{\alpha}} \iint_{Q_{\rho,M}} \left[\left(\mu^- + \frac{\omega}{2}\right)^{\frac{1}{\alpha}} - (\mu^-)^{\frac{1}{\alpha}}\right](u-k)_-\partial_t\eta^2\,dtdx.$$

Either if $\mu^- \le \frac{1}{2}\mu^+$, then $\mu^+ \le \omega + \mu^-$ and hence $\mu^+ \le 2\omega$. Therefore

$$(\mu^+)^{1-\frac{1}{\alpha}}\left[\left(\mu^- + \frac{\omega}{2}\right)^{\frac{1}{\alpha}} - (\mu^-)^{\frac{1}{\alpha}}\right] \le (2\omega)^{1-\frac{1}{\alpha}}\left(\frac{\omega}{2}\right)^{\frac{1}{\alpha}} \le 2^{1-\frac{2}{\alpha}}\omega$$

and hence

$$I_1 \le C(\alpha)\omega \iint_{Q_{\rho,M}} (u-k)_-\partial_t\eta^2\,dtdx.$$

Otherwise, if $\mu^- > \frac{1}{2}\mu^+$, then

$$\begin{aligned}\left(\mu^- + \frac{\omega}{2}\right)^{\frac{1}{\alpha}} - (\mu^-)^{\frac{1}{\alpha}} &= \int_0^1 \frac{d}{ds}\left(\mu^- + \frac{\omega}{2}s\right)^{\frac{1}{\alpha}} ds \\ &= \frac{\omega}{2\alpha}\int_0^1 \left(\mu^- + \frac{\omega}{2}s\right)^{\frac{1}{\alpha}-1} ds \\ &\le \frac{\omega}{2\alpha}(\mu^-)^{\frac{1}{\alpha}-1} \le \frac{\omega}{2\alpha}\left(\frac{1}{2}\mu^+\right)^{\frac{1}{\alpha}-1}\end{aligned}$$

and hence

$$(\mu^+)^{1-\frac{1}{\alpha}}\left[\left(\mu^- + \frac{\omega}{2}\right)^{\frac{1}{\alpha}} - (\mu^-)^{\frac{1}{\alpha}}\right] \le \frac{\omega}{2\alpha}(\mu^+)^{1-\frac{1}{\alpha}}\left(\frac{1}{2}\mu^+\right)^{\frac{1}{\alpha}-1} \le C(\alpha)\omega.$$

In either case, we obtain

$$I_1 \le C(\alpha)\omega \iint_{Q_{\rho,M}} (u-k)_-\partial_t\eta^2\,dtdx. \tag{2.13}$$

Substituting (2.12) and (2.13) for (2.11) we obtain (2.9). □

Proof of Proposition 2.10. We consider the scale transform

$$s = M^{1-\frac{1}{\alpha}}t,\ \ \tilde u(s,x) = u(t,x),\ \ \tilde\eta(s,x) = \eta(t,x),\ \ \tilde f(s,x) = f(t,x) \ \text{ and }\ \tilde g(s,x) = g(t,x)$$

and we put $\tilde h(\rho,\omega) := \|\tilde f\|^2_{L^q(L^p_{\mathrm{w}})(Q_\rho)} + \omega\|\tilde g\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_\rho)}$. We rewrite the Caccioppoli estimate (2.9) as follows:

$$\begin{aligned}&\sup_{s\in I_\rho}\int_{B_\rho}(\tilde u(s)-k)_-^2\tilde\eta^2(s)\,dx + \frac{(\mu^+)^{1-\frac{1}{\alpha}}}{M^{1-\frac{1}{\alpha}}}\iint_{Q_\rho}|\nabla(\tilde u-k)_-|^2\tilde\eta^2\,dsdx \\ &\le C(\alpha)\Bigg\{\omega\iint_{Q_\rho}(\tilde u-k)_-\partial_s\tilde\eta^2\,dsdx + \frac{(\mu^+)^{1-\frac{1}{\alpha}}}{M^{1-\frac{1}{\alpha}}}\iint_{Q_\rho}(\tilde u-k)_-^2|\nabla\tilde\eta|^2\,dsdx \\ &\qquad + \frac{(\mu^+)^{1-\frac{1}{\alpha}}}{M^{1-\frac{1}{\alpha}}}\tilde h(\rho,\omega)\left(\int_{I_\rho}|B_\rho\cap\{\tilde u(s)<k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,ds\right)^{\frac{2}{q'}}\Bigg\}.\end{aligned} \tag{2.14}$$

We take $p_*, q_* > 0$ as

$$\frac{2}{q'} = \frac{2}{q_*}\left(1 + \frac{2\gamma_0}{n}\right), \quad q'\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{q_*}{p_*}.$$

We remark that $\frac{2}{q_*} + \frac{n}{p_*} = \frac{n}{2}$. For $i \in \mathbb{N}$, we take $\rho = \rho_i$, $k = k_i$, $\tilde{\eta} = \tilde{\eta}_i$ satisfying $\tilde{\eta}_i \equiv 1$ on $Q_{\rho_{i+1}}$ and

$$\begin{aligned}
k_i &= \mu^- + \frac{1}{4}\omega + \frac{1}{2^{i+1}}\omega, \qquad \rho_i = \frac{1}{2}\rho + \frac{1}{2^{i+1}}\rho, \\
Y_i &:= \frac{|Q_{\rho_i} \cap \{\tilde{u} < k_i\}|}{|Q_\rho|}, \qquad Z_i = \frac{\rho^2}{|Q_\rho|}\left(\int_{I_{\rho_i}} |B_{\rho_i} \cap \{\tilde{u}(s) < k_i\}|^{\frac{q_*}{p_*}}\, ds\right)^{\frac{2}{q_*}}, \\
|\nabla \tilde{\eta}_i| &\le \frac{2}{\rho_i - \rho_{i+1}} \le \frac{8 \cdot 2^i}{\rho}, \quad \partial_s \tilde{\eta}_i \le \frac{2}{\rho_i^2 - \rho_{i+1}^2} \le \frac{16 \cdot 2^{2i}}{3\rho^2}.
\end{aligned}$$

Then, using (2.4) and $(\tilde{u} - k_i)_- \le \frac{\omega}{2}$, we rewrite (2.14) as

$$\begin{aligned}
&\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^\infty(L^2) \cap L^2(H^1)(Q_{\rho_i})} \\
&\quad \le C(\alpha)\Bigg\{\omega \iint_{Q_{\rho_i}} (\tilde{u} - k_i)_- \partial_s \tilde{\eta}_i{}^2 \, dsdx + \iint_{Q_{\rho_i}} (\tilde{u} - k_i)_-^2 |\nabla \tilde{\eta}_i|^2 \, dsdx \\
&\qquad + \tilde{h}(\rho, \omega)\left(\int_{I_{\rho_i}} |B_{\rho_i} \cap \{\tilde{u}(s) < k_i\}|^{\frac{q_*}{p_*}}\, ds\right)^{\frac{2}{q_*}(1 + \frac{2\gamma_0}{n})}\Bigg\} \\
&\le C(\alpha)\left\{\frac{2^{2i}\omega^2}{\rho^2}|Q_{\rho_i} \cap \{u < k_i\}| + \tilde{h}(\rho, \omega)\left(\int_{I_{\rho_i}} |B_{\rho_i} \cap \{\tilde{u}(s) < k_i\}|^{\frac{q_*}{p_*}}\, ds\right)^{\frac{2}{q_*}(1 + \frac{2\gamma_0}{n})}\right\} \\
&\qquad \le C(\alpha)\frac{\omega^2 |Q_\rho|}{\rho^2}\left\{2^{2i} Y_i + \tilde{h}(\rho, \omega)\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Z_i^{1 + \frac{2\gamma_0}{n}}\right\}.
\end{aligned}$$

Using the Ladyženskaja inequality (cf. Proposition B.2) and the Hölder inequality, we have

$$\begin{aligned}
\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^2(Q_{\rho_i})} &\le \|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^{2+\frac{4}{n}}(Q_{\rho_i})} \|\chi_{\{\tilde{u} < k_i\}}\|^2_{L^{n+2}(Q_{\rho_i})} \\
&\le C(\alpha, n)\omega^2 |Q_\rho| Y_i^{\frac{2}{n+2}}\left\{2^{2i} Y_i + \tilde{h}(\rho, \omega)\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Z_i^{1 + \frac{2\gamma_0}{n}}\right\}
\end{aligned}$$

and

$$\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^{q_*}(L^{p_*})(Q_{\rho_i})} \le C(\alpha, n)\frac{\omega^2 |Q_\rho|}{\rho^2}\left\{2^{2i} Y_i + \tilde{h}(\rho, \omega)\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}} Z_i^{1 + \frac{2\gamma_0}{n}}\right\}.$$

Since

$$\begin{aligned}
\|(\tilde{u} - k_i)_- \tilde{\eta}_i\|^2_{L^2(Q_{\rho_i})} &\ge \|(\tilde{u} - k_i)_-\|^2_{L^2(Q_{\rho_{i+1}} \cap \{\tilde{u} < k_{i+1}\})} \\
&\ge (k_i - k_{i+1})_-^2 |Q_{\rho_{i+1}} \cap \{\tilde{u} < k_{i+1}\}| = \frac{\omega^2}{64 \cdot 2^{2i}} |Q_\rho| Y_{i+1}
\end{aligned}$$

and

$$
\begin{aligned}
\|(\tilde{u}-k_i)_-\tilde{\eta}_i\|^2_{L^{q_*}(L^{p_*})(Q_{\rho_i})} &\geq \|(\tilde{u}-k_i)_-\|^2_{L^{q_*}(L^{p_*})(Q_{\rho_{i+1}}\cap\{\tilde{u}<k_{i+1}\})} \\
&\geq (k_i-k_{i+1})_-^2\left(\int_{I_{\rho_{i+1}}}|B_{\rho_{i+1}}\cap\{\tilde{u}(s)<k_{i+1}\}|^{\frac{q_*}{p_*}}\,ds\right)^{\frac{2}{q_*}} \\
&= \frac{\omega^2}{64\cdot 2^{2i}}\frac{|Q_\rho|}{\rho^2}Z_{i+1},
\end{aligned}
$$

we obtain

$$
Y_{i+1}\leq C(\alpha,n)\left\{2^{4i}Y_i^{1+\frac{2}{n+2}}+2^{2i}\tilde{h}(\rho,\omega)\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}}Y_i^{\frac{2}{n+2}}Z_i^{1+\varepsilon}\right\}
$$

and

$$
Z_{i+1}\leq C(\alpha,n)\left\{2^{4i}Y_i+2^{2i}\tilde{h}(\rho,\omega)\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2\gamma_0}{n}}Z_i^{1+\varepsilon}\right\}.
$$

Either if $q\geq p$, then $\frac{q_*}{p_*}\leq 1$ and we obtain

$$
\begin{aligned}
Z_0 &= \frac{\rho^2}{|Q_\rho|}\left(\int_{I_{\rho_0}}|B_{\rho_0}\cap\{\tilde{u}(s)<k_0\}|^{\frac{q_*}{p_*}}\,ds\right)^{\frac{2}{q_*}} \\
&\leq \frac{\rho^2}{|Q_\rho|}\left(\int_{I_{\rho_0}}|B_{\rho_0}\cap\{\tilde{u}(s)<k_0\}|\,ds\right)^{\frac{2}{p_*}}\rho^{\frac{4}{q_*}(1-\frac{q_*}{p_*})}\leq C(n,p,q)Y_0^{\frac{2}{p_*}},
\end{aligned}
$$

by the Hölder inequality. Otherwise, if $q<p$, then

$$
\begin{aligned}
Z_0 &= \frac{\rho^2}{|Q_\rho|}\left(\int_{I_{\rho_0}}|B_{\rho_0}\cap\{\tilde{u}(s)<k_0\}||B_{\rho_0}\cap\{\tilde{u}(s)<k_0\}|^{1-\frac{q_*}{p_*}}\,ds\right)^{\frac{2}{q_*}} \\
&\leq \frac{\rho^2}{|Q_\rho|}|B_{\rho_0}|^{\frac{2}{p_*}-\frac{2}{q_*}}\left(\int_{I_{\rho_0}}|B_{\rho_0}\cap\{\tilde{u}(s)<k_0\}|\,ds\right)^{\frac{2}{q_*}}\leq C(n,p,q)Y_0^{\frac{2}{q_*}}.
\end{aligned}
$$

Therefore, by using $\rho^{\gamma_0}\leq\omega\tilde{h}(\rho,\omega)^{-\frac{1}{2}}$ and Lemma B.13, there exists $0<\theta_0<1$ such that if $Y_0\leq\theta_0$, then $Y_i\to 0$ as $i\to\infty$, i.e.

$$
\tilde{u}(s,x)>\mu^-+\frac{\omega}{4}\quad\text{a.a. }(s,x)\in Q_{\frac{\rho}{2}}.
$$

□

4. Proof of the upper bounds

In this section, we show the upper bounds in Lemma 2.7. More precisely we show the following proposition:

PROPOSITION 2.12. *Let $0<\theta_0<1$. Assume the inequality* (2.4) *and* (2.5). *Then, there exist $\eta_1,\delta_1>0$ depending only on n,α,p,q and θ_0 such that if*

$$
\rho^{\gamma_0}\leq\delta_1\omega M^{-\frac{1}{q}(1-\frac{1}{\alpha})}h(\rho,M,\omega)^{-\frac{1}{2}}
$$

and
$$\left|Q_{\rho,M}\cap\left\{u<\mu^-+\frac{\omega}{2}\right\}\right|>\theta_0|Q_{\rho,M}|,$$
where $\left|Q_{\rho,M}\cap\left\{u<\mu^-+\frac{\omega}{2}\right\}\right|$ *and* $|Q_{\rho,M}|$ *denote the Lebesgue measure on* $\mathbb{R}^{n+1}$*, then we have*
$$u(t,x)\le\sup_{Q_{\rho,M}}u-\eta_1\omega\quad\text{for } (t,x)\in Q^{\theta_0}_{\frac{\rho}{2},M}.$$

We choose θ_0 as in Proposition 2.10 and δ_1, $\eta_1>0$ as in Proposition 2.12. Then taking
$$\delta_0=\min\{1,\delta_1\}\,,\ \eta_0=\min\Big\{\frac{1}{4},\eta_1\Big\},$$
we obtain Lemma 2.7.

Lemma 2.13. *Let* $0<\theta_0<1$. *If*
$$\left|Q_{\rho,M}\cap\left\{u<\mu^-+\frac{\omega}{2}\right\}\right|>\theta_0|Q_{\rho,M}|, \tag{2.15}$$
then for all $0<\theta<\theta_0$*, there exists* $-\frac{\rho^2}{M^{1-\frac{1}{\alpha}}}<\tau_0<-\theta\frac{\rho^2}{M^{1-\frac{1}{\alpha}}}$ *depending only on* θ *and* θ_0 *such that*
$$\left|B_\rho\cap\left\{u(\tau_0)>\mu^-+\frac{\omega}{2}\right\}\right|\le\frac{1-\theta_0}{1-\theta}|B_\rho|,$$
where $|B_\rho\cap\{u(\tau_0)>\mu^-+\frac{\omega}{2}\}|$ *and* $|B_\rho|$ *denote the Lebesgue measure on* $\mathbb{R}^n$.

Proof of Lemma 2.13. By the change of variable $t=\frac{\rho^2}{M^{1-\frac{1}{\alpha}}}s$, $\tilde u(s,x)=u(t,x)$ and the inequality (2.15), we obtain
$$\begin{aligned}\int_{-1}^0\left|B_\rho\cap\left\{\tilde u(s)>\mu^-+\frac{\omega}{2}\right\}\right|ds&=\frac{M^{1-\frac{1}{\alpha}}}{\rho^2}\left|Q_{\rho,M}\cap\left\{u>\mu^-+\frac{\omega}{2}\right\}\right|\\&\le\frac{M^{1-\frac{1}{\alpha}}}{\rho^2}\left(|Q_{\rho,M}|-\left|Q_{\rho,M}\cap\left\{u<\mu^-+\frac{\omega}{2}\right\}\right|\right)\\&<\frac{M^{1-\frac{1}{\alpha}}}{\rho^2}(1-\theta_0)|Q_{\rho,M}|=(1-\theta_0)|B_\rho|.\end{aligned}$$
If $|B_\rho\cap\{u(s)>\mu^-+\frac{\omega}{2}\}|>\frac{1-\theta_0}{1-\theta}|B_\rho|$ for all $-1<s<-\theta$, then
$$\begin{aligned}\int_{-1}^0\left|B_\rho\cap\left\{\tilde u(s)>\mu^-+\frac{\omega}{2}\right\}\right|ds&\ge\int_{-1}^{\theta_0}\left|B_\rho\cap\left\{\tilde u(s)>\mu^-+\frac{\omega}{2}\right\}\right|ds\\&\ge(1-\theta_0)|B_\rho|,\end{aligned}$$
which is contradiction. □

Lemma 2.14. *There exist* $r_0,\delta_2>0$ *depending only on* n,α,p,q *and* θ_0 *such that if* $\rho^{\gamma_0}\le\delta_2\omega M^{-\frac{1}{q}(1-\frac{1}{\alpha})}h(\rho,M,\omega)^{-\frac{1}{2}}$*, then*
$$\left|B_\rho\cap\left\{u(t)>\mu^+-\frac{\omega}{2^{r_0}}\right\}\right|\le\left(1-\left(\frac{\theta_0}{2}\right)^2\right)|B_\rho|$$
for $t\in I^{\theta_0}_{\rho,M}$*, where* $|B_\rho\cap\{u(t)>\mu^+-\frac{\omega}{2^{r_0}}\}|$ *and* $|B_\rho|$ *denote the Lebesgue measure on* $\mathbb{R}^n$.

Proof of Lemma 2.14. We rewrite (2.3) as

$$\partial_t u - \alpha u^{1-\frac{1}{\alpha}} \Delta u = -\alpha u^{1-\frac{1}{\alpha}} \operatorname{div} f + \alpha u^{1-\frac{1}{\alpha}} g.$$

Let

$$\psi(\xi) := \log_+ \left(\frac{H}{H - (\xi - k)_+ + c} \right),$$

where $k := \mu^- + \frac{\omega}{2}$, $H := \mu^+ - k = \operatorname{osc}_{Q_{\rho,M}} u - \frac{\omega}{2}$, $c := \frac{\omega}{2^{r_0}}$ and $r_0 > 2$ be chosen later. We remark that $\psi, \psi', \psi'' = (\psi')^2 \geq 0$, where $' = \frac{d}{d\xi}$. We take the cut-off function $\eta = \eta(x)$ as

$$\eta \in C_0^\infty(B_\rho), \ \eta \equiv 1 \text{ on } B_{(1-\sigma_0)\rho} \quad \text{and } |\nabla \eta| \leq \frac{2}{\sigma_0 \rho},$$

where $\sigma_0 > 0$ be chosen later. Putting $w = \psi(u)$ and taking the test function $(\psi^2)'(u)\eta^2$ in $(\tau_0, t) \times B_\rho$, where τ_0 be chosen later, we have

$$\begin{aligned}
\frac{1}{2} \int_{B_\rho} w^2 \eta^2 \, dx \Big|_{\tau_0}^{t} &+ \alpha \int_{\tau_0}^{t} \int_{B_\rho} \Big(\nabla u \cdot \nabla \big(u^{1-\frac{1}{\alpha}} (\psi^2)' \eta^2 \big) \Big) \, dt dx \\
&= \alpha \int_{\tau_0}^{t} \int_{B_\rho} \Big(f \cdot \nabla \big(u^{1-\frac{1}{\alpha}} (\psi^2)' \eta^2 \big) \Big) \, dt dx + \alpha \int_{\tau_0}^{t} \int_{B_\rho} u^{1-\frac{1}{\alpha}} g (\psi^2)' \eta^2 \, dt dx.
\end{aligned}$$

Since

$$\nabla \big(u^{1-\frac{1}{\alpha}} (\psi^2)' \eta^2 \big) = \Big(1 - \frac{1}{\alpha} \Big) u^{-\frac{1}{\alpha}} (\psi^2)' \eta^2 \nabla u + u^{1-\frac{1}{\alpha}} (\psi^2)'' \eta^2 \nabla u + u^{1-\frac{1}{\alpha}} (\psi^2)' \nabla \eta^2,$$

we obtain

$$\begin{aligned}
\frac{1}{2} \int_{B_\rho} w^2 \eta^2 \, dx \Big|_{\tau_0}^{t} &+ (\alpha - 1) \int_{\tau_0}^{t} \int_{B_\rho} u^{-\frac{1}{\alpha}} (\psi^2)' |\nabla u|^2 \eta^2 \, dt dx \\
&+ \alpha \int_{\tau_0}^{t} \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi^2)'' |\nabla u|^2 \eta^2 \, dt dx \\
&= -\alpha \int_{\tau_0}^{t} \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi^2)' (\nabla u \cdot \nabla \eta^2) \, dt dx \\
&+ (\alpha - 1) \int_{\tau_0}^{t} \int_{B_\rho} u^{-\frac{1}{\alpha}} (g^2)' (f \cdot \nabla u) \eta^2 \, dt dx \\
&+ \alpha \int_{\tau_0}^{t} \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi^2)'' (f \cdot \nabla u) \eta^2 \, dt dx \\
&+ \alpha \int_{\tau_0}^{t} \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi^2)' (f \cdot \nabla \eta^2) \, dt dx \\
&+ \alpha \int_{\tau_0}^{t} \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi^2)' g \eta^2 \, dt dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{2.16}$$

Using the property $(\psi^2)'\nabla u = 2w\nabla w$ and the Young inequality, we have

$$
\begin{aligned}
I_1 &\le \alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w|\nabla w|^2\eta^2 \, dtdx + 4\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w|\nabla \eta|^2 \, dtdx, \\
I_2 &\le \frac{\alpha-1}{2} \int_{\tau_0}^t \int_{B_\rho} u^{-\frac{1}{\alpha}} (\psi^2)'|\nabla u|^2\eta^2 \, dtdx + \frac{\alpha-1}{2} \int_{\tau_0}^t \int_{B_\rho} u^{-\frac{1}{\alpha}} (\psi^2)'|f|^2\eta^2 \, dtdx, \\
I_3 &\le \frac{\alpha}{4} \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi^2)''|\nabla u|^2\eta^2 \, dtdx + \alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi^2)''|f|^2\eta^2 \, dtdx, \\
I_4 &\le 4\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w\psi'|f||\nabla\eta|\eta \, dtdx \\
&\le 2\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w|\nabla \eta|^2 \, dtdx + 2\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} (g')^2 w|f|^2\eta^2 \, dtdx, \\
I_5 &\le 2\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w\psi'|g|\eta^2 \, dtdx.
\end{aligned}
\tag{2.17}
$$

Since $\psi'' = (\psi')^2$, $(\psi^2)'' = 2(\psi')^2(1+\psi)$, we have

$$
\begin{aligned}
\alpha \int_{\tau_0}^t \int_{B_\rho} & u^{1-\frac{1}{\alpha}} (\psi^2)''|\nabla u|^2\eta^2 \, dtdx \\
&= 2\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} |\nabla w|^2\eta^2 \, dtdx + 2\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w|\nabla w|^2\eta^2 \, dtdx.
\end{aligned}
$$

Combining estimates (2.17), we have from (2.16) that

$$
\begin{aligned}
\frac{1}{2}\int_{B_\rho} & w^2(t)\eta^2(t) \, dx + \frac{\alpha-1}{2} \int_{\tau_0}^t \int_{B_\rho} u^{-\frac{1}{\alpha}} (\psi^2)'\eta^2|\nabla u|^2 \, dtdx \\
&\quad + \frac{3}{2}\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} |\nabla w|^2\eta^2 \, dtdx + \frac{\alpha}{2} \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w|\nabla w|^2\eta^2 \, dtdx \\
&\le \frac{1}{2}\int_{B_\rho} w^2(\tau_0)\eta^2(\tau_0) \, dx + 6\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w|\nabla \eta|^2 \, dtdx \\
&\quad + \frac{\alpha-1}{2} \int_{\tau_0}^t \int_{B_\rho} u^{-\frac{1}{\alpha}} (\psi^2)'|f|^2\eta^2 \, dtdx \\
&\quad + 2\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} (\psi')^2(1+2w)|f|^2\eta^2 \, dtdx \\
&\quad + 2\alpha \int_{\tau_0}^t \int_{B_\rho} u^{1-\frac{1}{\alpha}} w\psi'|g|\eta^2 \, dtdx \\
&=: I_6 + I_7 + I_8 + I_9 + I_{10}.
\end{aligned}
\tag{2.18}
$$

For simplicity, we put $k' = \mu^+ - c = \mu^+ - \frac{\omega}{2^{r_0}}$. First, we estimate the left-hand side of (2.18). Since $k' > k$, we have

$$
\begin{aligned}
\frac{1}{2}\int_{B_\rho} w^2(t)\eta^2(t)\,dx &\geq \frac{1}{2}\int_{B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}} w^2(t)\,dx \\
&\geq \frac{1}{2}\int_{B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}} \log^2\Big(\frac{H}{H-(k'-k)+c}\Big)\,dx \\
&\geq \frac{1}{2}\log^2\Big(\frac{\frac{\omega}{4}}{\frac{\omega}{2^{r_0-1}}}\Big)|B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}| \\
&= \frac{1}{2}(r_0-3)^2\log^2 2|B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}|.
\end{aligned}
\tag{2.19}
$$

Second, we estimate I_6. Taking τ_0 as in Lemma 2.13 with $\theta = \frac{\theta_0}{2}$, we obtain

$$
w = \log_+\Big(\frac{H}{H-(u-k)_+ + c}\Big) \leq \log\Big(\frac{\frac{1}{2}\omega}{\frac{1}{2^{r_0}}\omega}\Big) = (r_0-1)\log 2
$$

and hence

$$
\begin{aligned}
I_6 &\leq \frac{1}{2}\int_{B_\rho\cap\{u(\tau_0)>k\}} w^2(\tau_0)\,dx \\
&\leq \frac{1}{2}(r_0-1)^2\log^2 2|B_\rho\cap\{u(\tau_0)>k\}| \leq \frac{1}{2}\cdot\frac{1-\theta_0}{1-\frac{\theta_0}{2}}(r_0-1)^2\log^2 2|B_\rho|.
\end{aligned}
\tag{2.20}
$$

We estimate I_7. From $t - \tau_0 \leq \frac{\rho^2}{M^{1-\frac{1}{\alpha}}}$ and the inequality (2.4), we have

$$
I_7 \leq 6\alpha(\mu^+)^{1-\frac{1}{\alpha}}(t-\tau_0)(r_0-1)\log 2\Big(\frac{2}{\sigma_0\rho}\Big)^2|B_\rho| \leq C(\alpha)\Big(\frac{r_0-1}{\sigma_0^2}\Big)|B_\rho|.
\tag{2.21}
$$

We estimate I_8. Since

$$
\psi' \leq \frac{1}{H-(u-k)_+ + c} \leq \frac{1}{c} = \frac{2^{r_0}}{\omega},
$$

$$
(\psi^2)' = 2\psi\psi' \leq \frac{2^{r_0+1}}{\omega}(r_0-1)\log 2
$$

and

$$
u^{-\frac{1}{\alpha}} \leq k^{-\frac{1}{\alpha}} \leq \Big(\frac{\omega}{2}\Big)^{-\frac{1}{\alpha}} \quad \text{for } u \geq k,
$$

we have

$$
I_8 \leq C(\alpha)(r_0-1)2^{r_0}\omega^{-1-\frac{1}{\alpha}}\int_{\tau_0}^t\int_{B_\rho\cap\{u(s)>k\}} |f|^2\,dtdx.
$$

By the definition of the weak L^p space and by the Hölder inequality, we have

$$
\begin{aligned}
\int_{\tau_0}^t\int_{B_\rho\cap\{u(s)>k\}} |f|^2\,dtdx &\leq \int_{\tau_0}^t \big\||f(s)|^2\big\|_{L^{\frac{p}{2}}_{\mathrm{w}}(B_\rho)}|B_\rho\cap\{u(s)>k\}|^{1-\frac{2}{p}}\,ds \\
&\leq C(n,p)M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\||f|^2\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\rho^{2\gamma_0}\frac{|B_\rho|}{M^{1-\frac{1}{\alpha}}}.
\end{aligned}
$$

Using the inequality (2.5), we obtain

$$
\begin{aligned}
I_8 &\le C(n,\alpha,p)\Big(\frac{\rho^{2\gamma_0}}{\omega^2}M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\||f|^2\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\Big)\Big(\frac{\omega}{M}\Big)^{1-\frac{1}{\alpha}}(r_0-1)2^{r_0}|B_\rho| \\
&\le C(n,\alpha,p)\Big(\frac{\rho^{2\gamma_0}}{\omega^2}M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\||f|^2\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\Big)(r_0-1)2^{r_0}|B_\rho|.
\end{aligned}
\tag{2.22}
$$

We estimate I_9 and I_{10}. As the estimate of I_8 (more easy to estimate since $u^{1-\frac{1}{\alpha}} \le M^{1-\frac{1}{\alpha}}$), we have

$$
I_9 \le C(n,\alpha,p)\Big(\frac{\rho^{2\gamma_0}}{\omega^2}M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\||f|^2\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\Big)2^{2r_0}(1+2(r_0-1)\log 2)|B_\rho| \tag{2.23}
$$

and

$$
I_{10} \le C(n,\alpha,p)\Big(\frac{\rho^{2\gamma_0}}{\omega}M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\|g\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\Big)2^{r_0}(r_0-1)|B_\rho|. \tag{2.24}
$$

Combining those estimates (2.19)–(2.24), we have

$$
\begin{aligned}
|B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}| \le \Bigg\{&\frac{1-\theta_0}{1-\frac{\theta_0}{2}}\Big(\frac{r_0-1}{r_0-3}\Big)^2+\frac{C_1(\alpha)}{\sigma_0^2}\frac{r_0-1}{(r_0-3)^2} \\
&+C_2(n,\alpha,p)\Big(\frac{\rho^{2\gamma_0}}{\omega^2}M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\||f|^2\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\Big)\frac{2^{r_0}(r_0-1)}{(r_0-3)^2} \\
&+C_3(n,\alpha,p)\Big(\frac{\rho^{2\gamma_0}}{\omega^2}M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\||f|^2\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\Big)\frac{2^{2r_0}(1+2(r_0-1)\log 2)}{(r_0-3)^2} \\
&+C_4(n,\alpha,p)\Big(\frac{\rho^{2\gamma_0}}{\omega}M^{\frac{2}{q}(1-\frac{1}{\alpha})}\big\|g\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q_{\rho,M})}\Big)\frac{2^{r_0}(r_0-1)}{(r_0-3)^2}\Bigg\}|B_\rho|.
\end{aligned}
$$

Since

$$
\begin{aligned}
|B_\rho\cap\{u(t)>k'\}| &= |(B_\rho\setminus B_{(1-\sigma_0)\rho})\cap\{u(t)>k'\}|+|B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}| \\
&\le (1-(1-\sigma_0)^n)|B_\rho|+|B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}|,
\end{aligned}
$$

we have

$$
\begin{aligned}
|B_{(1-\sigma_0)\rho}\cap\{u(t)>k'\}| \le \Bigg\{&\frac{1-\theta_0}{1-\frac{\theta_0}{2}}\Big(\frac{r_0-1}{r_0-3}\Big)^2+\frac{C_1(\alpha)}{\sigma_0^2}\frac{r_0-1}{(r_0-3)^2}+(1-(1-\sigma_0)^n) \\
&+\max\{C_2,C_3,C_4\}\frac{\rho^{2\gamma_0}}{\omega^2}M^{\frac{2}{q}(1-\frac{1}{\alpha})}h(\rho,M,\omega)C_5(r_0)\Bigg\}|B_\rho|,
\end{aligned}
$$

where

$$
C_5(r_0)=\max\Big\{\frac{2^{r_0}(r_0-1)}{(r_0-3)^2},\frac{2^{2r_0}(1+2(r_0-1)\log 2)}{(r_0-3)^2}\Big\}.
$$

We choose parameters r_0, γ and δ_2. First we choose $\sigma_0=\sigma_0(n,\theta_0)$ satisfying $1-(1-\sigma_0)^n \le \frac{1}{8}\theta_0^2$. Second, we choose $r_0=r_0(n,\alpha,\theta_0)$ satisfying

$$
\Big(\frac{r_0-1}{r_0-3}\Big)^2 \le \Big(1-\frac{\theta_0}{2}\Big)(1+\theta_0) \ \text{ and } \ \frac{C_1(\alpha)}{\sigma_0^2}\frac{r_0-1}{(r_0-3)^2} \le \frac{1}{8}\theta_0^2.
$$

Finally, we choose $\delta_2 = \delta_2(n, \alpha, p, \theta_0) > 0$ sufficiently small such that

$$\max\{C_2, C_3, C_4\} C_5(r_0)\delta_2 \leq \frac{1}{2}\theta_0^2.$$

Then, if $\rho^{2\gamma_0} \leq \delta_2 \omega^2 M^{-\frac{2}{q}(1-\frac{1}{\alpha})} h(\rho, M, \omega)^{-1}$, we have

$$|B_\rho \cap \{u(t) > k'\}| \leq \left(1 - \left(\frac{\theta_0}{2}\right)^2\right)|B_\rho|.$$

$\square$

LEMMA 2.15 (the Caccioppoli estimate for super-level sets). *Let $\eta = \eta(t, x)$ be a cut-off function in $Q^{\theta_0}_{\rho,M}$. For $k \geq \mu^+ - \frac{\omega}{2}$, there exists a constant $C > 0$ depending only on α such that*

$$\begin{aligned}
(2.25)\quad &\sup_{t \in I^{\theta_0}_{\rho,M}} \int_{B_\rho} (u(t) - k)_+^2 \eta^2(t)\, dx + M^{1-\frac{1}{\alpha}} \iint_{Q^{\theta_0}_{\rho,M}} |\nabla(u-k)_+|^2 \eta^2\, dtdx \\
&\leq C\Bigg\{ \left(\frac{M}{\mu^+}\right)^{1-\frac{1}{\alpha}} \iint_{Q^{\theta_0}_{\rho,M}} (u-k)_+^2 \partial_t \eta^2\, dtdx + M^{1-\frac{1}{\alpha}} \iint_{Q^{\theta_0}_{\rho,M}} (u-k)_+^2 |\nabla \eta|^2\, dtdx \\
&\qquad + M^{1-\frac{1}{\alpha}} h(\rho, M, \omega) \left(\int_{t \in I^{\theta_0}_{\rho,M}} |B_\rho \cap \{u(t) > k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\, dt \right)^{\frac{2}{q'}} \Bigg\},
\end{aligned}$$

where $\frac{1}{2} = \frac{1}{q} + \frac{1}{q'}$ and $|B_\rho \cap \{u(t) > k\}|$ denotes the Lebesgue measure on $\mathbb{R}^n$.

PROOF OF LEMMA 2.15. Testing the function $(u-k)_+\eta^2$ to (2.3), we have

$$\begin{aligned}
&\frac{1}{\alpha} \iint_{Q^{\theta_0}_{\rho,M}} \partial_t \left(\int_0^{(u-k)_+} (k+\xi)^{\frac{1}{\alpha}-1} \xi\, d\xi \right) \eta^2\, dtdx + \iint_{Q^{\theta_0}_{\rho,M}} \nabla(u-k)_+ \cdot \nabla\{(u-k)_+\eta^2\}\, dtdx \\
&\qquad = \iint_{Q^{\theta_0}_{\rho,M}} f \cdot \nabla\{(u-k)_+\eta^2\}\, dtdx + \iint_{Q^{\theta_0}_{\rho,M}} g(u-k)_+\eta^2\, dtdx.
\end{aligned}$$

By the integration by parts, we obtain

$$\begin{aligned}
(2.26)\quad &\frac{1}{\alpha} \sup_{t \in I^{\theta_0}_{\rho,M}} \int_{B_\rho} \left(\int_0^{(u(t)-k)_+} (k+\xi)^{\frac{1}{\alpha}-1} \xi\, d\xi \right) \eta^2(t)\, dx + \iint_{Q^{\theta_0}_{\rho,M}} |\nabla(u-k)_+|^2 \eta^2\, dtdx \\
&\leq \frac{1}{\alpha} \iint_{Q^{\theta_0}_{\rho,M}} \left(\int_0^{(u-k)_+} (k+\xi)^{\frac{1}{\alpha}-1} \xi\, d\xi \right) \partial_t \eta^2\, dtdx - \iint_{Q^{\theta_0}_{\rho,M}} (\nabla(u-k)_+ \cdot \nabla \eta^2)(u-k)_+\, dtdx \\
&\quad + \iint_{Q^{\theta_0}_{\rho,M}} f \cdot \nabla(u-k)_+ \eta^2\, dtdx + \iint_{Q^{\theta_0}_{\rho,M}} (f \cdot \nabla \eta^2)(u-k)_+\, dtdx \\
&\quad + \iint_{Q^{\theta_0}_{\rho,M}} g(u-k)_+\eta^2\, dtdx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By the Young inequality and $k > \mu^+ - \frac{\omega}{2}$, we have

$$
\begin{aligned}
I_2 &\le \frac{1}{2}\iint_{Q^{\theta_0}_{\rho,M}} |\nabla(u-k)_+|^2\eta^2\,dtdx + 2\iint_{Q^{\theta_0}_{\rho,M}} (u-k)_+^2|\nabla\eta|^2\,dtdx,\\
I_3 &\le \frac{1}{4}\iint_{Q^{\theta_0}_{\rho,M}} |\nabla(u-k)_+|^2\eta^2\,dtdx + \iint_{Q^{\theta_0}_{\rho,M}\cap\{u>k\}} |f|^2\eta^2\,dtdx,\\
I_4 &\le \iint_{Q^{\theta_0}_{\rho,M}} (u-k)_+^2|\nabla\eta|^2\,dtdx + \iint_{Q^{\theta_0}_{\rho,M}\cap\{u>k\}} |f|^2\eta^2\,dtdx,\\
I_5 &\le \frac{\omega}{2}\iint_{Q^{\theta_0}_{\rho,M}\cap\{u>k\}} |g|\eta^2\,dtdx.
\end{aligned}
\tag{2.27}
$$

We estimate the first term of the left-hand side in (2.26). Since

$$
(k+\xi)^{\frac{1}{\alpha}-1} \ge u^{\frac{1}{\alpha}-1} \ge (\mu^+)^{\frac{1}{\alpha}-1} \ge M^{\frac{1}{\alpha}-1} \quad \text{for} \quad 0 \le \xi \le (u-k)_+,
$$

we have

$$
\int_0^{(u(t)-k)_+} (k+\xi)^{\frac{1}{\alpha}-1}\xi\,d\xi \ge \frac{1}{2}M^{\frac{1}{\alpha}-1}(u(t)-k)_+^2. \tag{2.28}
$$

Finally, we estimate I_1. By (2.5), we have

$$
(k+\xi)^{\frac{1}{\alpha}-1} \le k^{\frac{1}{\alpha}-1} \le \Big(\mu^+ - \frac{\omega}{2}\Big)^{\frac{1}{\alpha}-1} \le \Big(\frac{1}{3}\mu^+\Big)^{\frac{1}{\alpha}-1}
$$

and hence

$$
I_1 \le \frac{1}{2}\Big(\frac{1}{3}\mu^+\Big)^{\frac{1}{\alpha}-1}\iint_{Q^{\theta_0}_{\rho,M}} (u-k)_+^2\partial_t\eta^2\,dtdx. \tag{2.29}
$$

Combining of those estimates (2.27), (2.28) and (2.29), we obtain

$$
\begin{aligned}
&M^{\frac{1}{\alpha}-1}\sup_{t\in I^{\theta_0}_{\rho,M}}\int_{B_\rho}(u(t)-k)_+^2\eta^2(t)\,dx + \iint_{Q^{\theta_0}_{\rho,\omega}} |\nabla(u-k)_+|^2\eta^2\,dtdx\\
&\quad\le C(\alpha)\Big\{(\mu^+)^{\frac{1}{\alpha}-1}\iint_{Q^{\theta_0}_{\rho,M}}(u-k)_+^2\partial_t\eta^2\,dtdx + \iint_{Q^{\theta_0}_{\rho,M}}(u-k)_+^2|\nabla\eta|^2\,dtdx\\
&\qquad\qquad + \iint_{Q^{\theta_0}_{\rho,M}\cap\{u>k\}}|f|^2\eta^2\,dtdx + \frac{\omega}{2}\iint_{Q^{\theta_0}_{\rho,M}\cap\{u>k\}}|g|\eta^2\,dtdx\Big\}.
\end{aligned}
$$

As the same argument of the proof of Lemma 2.11, we have

$$
\begin{aligned}
\iint_{Q^{\theta_0}_{\rho,M}\cap\{u>k\}}|f|^2\eta^2\,dtdx &\le \big\||f|^2\big\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q^{\theta_0}_{\rho,M})}\left(\int_{I^{\theta_0}_{\rho,M}}|B_\rho\cap\{u(t)>k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,dt\right)^{\frac{2}{q'}},\\
\iint_{Q^{\theta_0}_{\rho,M}\cap\{u>k\}}|g|\eta^2\,dtdx &\le \|g\|_{L^{\frac{q}{2}}(L^{\frac{p}{2}}_{\mathrm{w}})(Q^{\theta_0}_{\rho,M})}\left(\int_{I^{\theta_0}_{\rho,M}}|B_\rho\cap\{u(t)>k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\,dt\right)^{\frac{2}{q'}}.
\end{aligned}
$$

Substituting these estimates, we obtain (2.25). □

LEMMA 2.16. *Let $\rho_0 = \frac{3}{4}\rho$. For $0 < \nu < 1$, there exist q_0, $\delta_1 > 0$ depending only on $n, \alpha, p, q, \theta_0$ and ν such that if $\rho^{\gamma_0} \le \delta_2 \omega M^{-\frac{1}{q}(1-\frac{1}{\alpha})} h(\rho, M, \omega)$, then*

$$\left| Q^{\theta_0}_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{q_0+1}} \right\} \right| \le \nu |Q^{\theta_0}_{\rho_0,M}|,$$

where $|Q^{\theta_0}_{\rho_0,M} \cap \{u > \mu^+ - \frac{\omega}{2^{q_0+1}}\}|$ and $|Q^{\theta_0}_{\rho_0,M}|$ denote the Lebesgue measure on $\mathbb{R}^{n+1}$.

REMARK 2.17. We obtain the estimate of δ_1 as

$$\delta_1 \le \theta_0^{\frac{1}{q}-\frac{1}{2}} 2^{-q_0}.$$

PROOF OF LEMMA 2.16. We fix $t \in I^{\theta_0}_{\rho,M}$ and set

$$l := \mu^+ - \frac{\omega}{2^{j+1}}, \quad k := \mu^+ - \frac{\omega}{2^j},$$

where $j \ge r_0$ and the constant r_0 is given by Lemma 2.14. By the Poincaré type inequality (cf. Proposition B.4), we have

$$\frac{\omega}{2^{j+1}} |B_{\rho_0} \cap \{u(t) > l\}| \le \frac{C(n)\rho_0^{n+1}}{|B_{\rho_0} \cap \{u(t) \le k\}|} \int_{B_{\rho_0} \cap \{k < u(t) \le l\}} |\nabla u(t)|\, dx.$$

Since $k > \mu^+ - \frac{\omega}{2^{r_0}}$ and Lemma 2.14, we have

$$|B_{\rho_0} \cap \{u(t) \le k\}| = |B_{\rho_0}| - |B_{\rho_0} \cap \{u(t) > k\}| \ge \left(\frac{\theta_0}{2}\right)^2 |B_{\rho_0}|$$

and hence

$$\frac{\omega}{2^{j+1}} |B_{\rho_0} \cap \{u(t) > l\}| \le \frac{C(n)\rho_0}{\theta_0^2} \int_{B_{\rho_0} \cap \{k < u(t) \le l\}} |\nabla u(t)|\, dx. \tag{2.30}$$

Integrating over $I^{\theta_0}_{\rho_0,M}$ for (2.30), we obtain

$$\begin{aligned} \frac{\omega}{2^{j+1}} |Q^{\theta_0}_{\rho_0,M} \cap \{u > l\}| &\le \frac{C(n)\rho_0}{\theta_0^2} \int_{I^{\theta_0}_{\rho_0,M}} \int_{B_{\rho_0} \cap \{k < u(t) \le l\}} |\nabla u(t)|\, dtdx \\ &\le \frac{C(n)\rho_0}{\theta_0^2} \|\nabla (u-k)_+\|_{L^2(Q^{\theta_0}_{\rho_0,M})} |Q^{\theta_0}_{\rho_0,M} \cap \{k < u \le l\}|^{\frac{1}{2}}. \end{aligned}$$

We estimate $\|\nabla(u-k)_+\|_{L^2(Q^{\theta_0}_{\rho_0,M})}$. Let $\eta = \eta(t,x)$ be a cut-off function in $Q^{\theta_0}_{\rho,M}$ satisfying

$$\eta \equiv 1 \text{ on } Q^{\theta_0}_{\rho_0,M}, \quad |\nabla \eta| \le \frac{8}{\rho} \quad \text{and} \quad \partial_t \eta \le \frac{10 M^{1-\frac{1}{\alpha}}}{\theta_0 \rho^2}.$$

Then, by the Caccioppoli estimate (Lemma 2.15), we have

$$\begin{aligned} \|\nabla(u-k)_+\|^2_{L^2(Q^{\theta_0}_{\rho_0,M})} &\le \|\nabla(u-k)_+ \eta\|^2_{L^2(Q^{\theta_0}_{\rho,M})} \\ &\le C(\alpha) \Bigg\{ \iint_{Q^{\theta_0}_{\rho,M}} (u-k)_+^2 (|\nabla \eta|^2 + (\mu^+)^{\frac{1}{\alpha}-1} \partial_t \eta^2)\, dtdx \\ &\qquad + h(\rho, M, \omega) \left(\int_{I^{\theta_0}_{\rho,M}} |B_\rho \cap \{u(t) > k\}|^{q'(\frac{1}{2}-\frac{1}{p})}\, dt \right)^{\frac{2}{q'}} \Bigg\} \\ &=: I_1 + I_2. \end{aligned} \tag{2.31}$$

We estimate I_1. By the inequality (2.4), we have

$$
\begin{aligned}
I_1 &\le C(\alpha)(\mu^+ - k)_+^2 \left(\frac{1}{\rho^2} + \frac{M^{1-\frac{1}{\alpha}}}{\theta_0 \rho^2} (\mu^+)^{\frac{1}{\alpha}-1} \right) |Q^{\theta_0}_{\rho_0,M}| \\
&\le C(\alpha) \left(\frac{\omega}{2^j} \right)^2 \frac{1}{\theta_0 \rho^2} \left(\frac{M}{\mu^+} \right)^{1-\frac{1}{\alpha}} |Q^{\theta_0}_{\rho_0,M}| \le C(\alpha) \left(\frac{\omega}{2^j} \right)^2 \frac{1}{\theta_0 \rho^2} |Q^{\theta_0}_{\rho_0,M}|.
\end{aligned}
\tag{2.32}
$$

We estimate I_2. Since

$$
\begin{aligned}
&\left(\int_{I^{\theta_0}_{\rho,M}} |B_\rho \cap \{u(t) > k\}|^{q'(\frac{1}{2}-\frac{1}{p})} \, dt \right)^{\frac{2}{q'}} \\
&\le |B_\rho|^{1-\frac{2}{p}} \left(\frac{\theta_0}{2} \frac{\rho^2}{M^{1-\frac{1}{\alpha}}} \right)^{\frac{2}{q'}} \\
&\le C(q) |B_\rho|^{-\frac{2}{p}} \left(\frac{\theta_0}{2} \frac{\rho^2}{M^{1-\frac{1}{\alpha}}} \right)^{\frac{2}{q'}-1} |Q^{\theta_0}_{\rho_0,M}| \\
&\le C(q) |B_\rho|^{-\frac{2}{p}} \left(\frac{\theta_0}{2} \frac{\rho^2}{M^{1-\frac{1}{\alpha}}} \right)^{\frac{2}{q'}-1} |Q^{\theta_0}_{\rho_0,M}| \\
&\le C(n,p,q) \left(\rho^{2\gamma_0} M^{\frac{2}{q}(1-\frac{1}{\alpha})} \left(\frac{2^j}{\omega} \right)^2 \theta_0^{\frac{2}{q'}} \right) \frac{1}{\theta_0 \rho^2} \left(\frac{\omega}{2^j} \right)^2 |Q^{\theta_0}_{\rho_0,M}|,
\end{aligned}
$$

we obtain

$$
I_2 \le C(n,\alpha,p,q) \left(\rho^{2\gamma_0} M^{\frac{2}{q}(1-\frac{1}{\alpha})} \left(\frac{2^j}{\omega} \right)^2 \theta_0^{\frac{2}{q'}} h(\rho, M, \omega) \right) \frac{1}{\theta_0 \rho^2} \left(\frac{\omega}{2^j} \right)^2 |Q^{\theta_0}_{\rho_0,M}|.
\tag{2.33}
$$

Substituting those estimates (2.32) and (2.33) for (2.31), we obtain

$$
\begin{aligned}
&\|\nabla (u-k)_+\|^2_{L^2(Q^{\theta_0}_{\rho_0,M})} \\
&\qquad \le C(n,\alpha,p,q) \left(1 + \rho^{2\gamma_0} M^{\frac{2}{q}(1-\frac{1}{\alpha})} \left(\frac{2^j}{\omega} \right)^2 \theta_0^{\frac{2}{q'}} h(\rho, M, \omega) \right) \frac{1}{\theta_0 \rho^2} \left(\frac{\omega}{2^j} \right)^2 |Q^{\theta_0}_{\rho_0,M}|
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\left(\frac{\omega}{2^{j+1}} \right)^2 |Q^{\theta_0}_{\rho_0,M} \cap \{u > l\}|^2 \\
&\qquad \le \frac{C(n,\alpha,p,q)}{\theta_0^5} \left(\frac{\omega}{2^j} \right)^2 \left(1 + \rho^{2\gamma_0} M^{\frac{2}{q}(1-\frac{1}{\alpha})} \left(\frac{2^j}{\omega} \right)^2 \theta_0^{\frac{2}{q'}} h(\rho, M, \omega) \right) \\
&\qquad\qquad \times |Q^{\theta_0}_{\rho_0,M}| \cdot |Q^{\theta_0}_{\rho_0,M} \cap \{k < u \le l\}|.
\end{aligned}
$$

Summing over $i = r_0 + 1, \dots, q_0$, we have

$$
\begin{aligned}
&\sum_{i=r_0+1}^{q_0} \left| Q^{\theta_0}_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right|^2 \\
&\qquad \le \frac{C(n,\alpha,p,q)}{\theta_0^5} |Q^{\theta_0}_{\rho_0,M}| \sum_{i=r_0+1}^{q_0} \left(1 + \rho^{2\gamma_0} M^{\frac{2}{q}(1-\frac{1}{\alpha})} \left(\frac{2^j}{\omega} \right)^2 \theta_0^{\frac{2}{q'}} h(\rho, M, \omega) \right) \\
&\qquad\qquad \times \left| Q^{\theta_0}_{\rho_0,M} \cap \left\{ \mu^+ - \frac{\omega}{2^i} < u \le \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right| \\
&\qquad \le \frac{C(n,\alpha,p,q)}{\theta_0^5} |Q^{\theta_0}_{\rho_0,M}| \left(1 + \rho^{2\gamma_0} M^{\frac{2}{q}(1-\frac{1}{\alpha})} \left(\frac{2^{q_0}}{\omega} \right)^2 \theta_0^{\frac{2}{q'}} h(\rho, M, \omega) \right) \\
&\qquad\qquad \times \sum_{i=r_0+1}^{\infty} \left| Q^{\theta_0}_{\rho_0,M} \cap \left\{ \mu^+ - \frac{\omega}{2^i} < u \le \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right| \\
&\qquad\qquad \le \frac{C(n,\alpha,p,q)}{\theta_0^5} |Q^{\theta_0}_{\rho_0,M}|^2 . \left(1 + \rho^{2\gamma_0} M^{\frac{2}{q}(1-\frac{1}{\alpha})} \left(\frac{2^{q_0}}{\omega} \right)^2 \theta_0^{\frac{2}{q'}} h(\rho, M, \omega) \right).
\end{aligned}
$$

We take $q_0 > 0$ enough large such that

$$
\frac{2C(n,\alpha,p,q)}{\theta_0^5 (q_0 - r_0)} \le \nu^2.
$$

Since

$$
\sum_{i=r_0+1}^{q_0} \left| Q^{\theta_0}_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{i+1}} \right\} \right|^2 \ge (q_0 - r_0) \left| Q^{\theta_0}_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{q_0+1}} \right\} \right|^2,
$$

we have

$$
\begin{aligned}
\left| Q^{\theta_0}_{\rho_0,M} \cap \left\{ u > \mu^+ - \frac{\omega}{2^{q_0+1}} \right\} \right|^2 &\le \frac{2C(n,\alpha,p,q)}{\theta_0^5 (q_0 - r_0)} |Q^{\theta_0}_{\rho_0,M}|^2 \\
&\le \nu^2 |Q^{\theta_0}_{\rho_0,M}|^2
\end{aligned}
$$

provided $\rho^{2\gamma_0} \le \min\{\theta_0^{-\frac{2}{q'}} 2^{-2q_0}, \delta_2^2\} \omega^2 M^{-\frac{2}{q}(1-\frac{1}{\alpha})} h(\rho, M, \omega)^{-1}$, where $\delta_2 > 0$ is given by Lemma 2.14. Taking $\delta_1^2 := \min\{\theta_0^{-\frac{2}{q'}} 2^{-2q_0}, \delta_2^2\}$, we obtain Lemma 2.16. $\square$

Proof of Proposition 2.12. This argument is same as the proof of Proposition 2.10. Let $0 < \nu < 1$ be chosen later. We take $\delta_1 > 0$ and q_0 as in Lemma 2.16. We introduce the following scale transform

$$
s = M^{1-\frac{1}{\alpha}} t, \;\; \tilde{u}(s,x) = u(t,x), \;\; \tilde{\eta}(s,x) = \eta(t,x), \;\; \tilde{f}(s,x) = f(t,x) \;\text{ and }\; \tilde{g}(s,x) = g(t,x).
$$

Then, using (2.4), we can rewrite the Caccioppoli estimate (2.25) as follows:

(2.34)

$$
\begin{aligned}
&\sup_{s\in I_\rho^{\theta_0}}\int_{B_\rho}(\tilde u(s)-k)_+^2\tilde\eta^2(s)\,dx+\iint_{Q_\rho^{\theta_0}}|\nabla(\tilde u-k)_+|^2\tilde\eta^2\,dsdx\\
&\qquad\le C(\alpha)\Big\{\iint_{Q_\rho^{\theta_0}}(\tilde u-k)_+^2\Big\{\Big(\frac{M}{\mu^+}\Big)^{1-\frac1\alpha}\partial_s\tilde\eta^2+|\nabla\tilde\eta|^2\Big\}\,dsdx\\
&\qquad\quad+\tilde h(\rho,\omega)\Big(\int_{I_\rho^{\theta_0}}|B_\rho\cap\{\tilde u(s)>k\}|^{q'(\frac12-\frac1p)}\,ds\Big)^{\frac{2}{q'}}\Big\}\\
&\qquad\le C(\alpha)\Big\{\iint_{Q_\rho^{\theta_0}}(\tilde u-k)_+^2\Big\{\partial_s\tilde\eta^2+|\nabla\tilde\eta|^2\Big\}\,dsdx\\
&\qquad\quad+\tilde h(\rho,\omega)\Big(\int_{I_\rho^{\theta_0}}|B_\rho\cap\{\tilde u(s)>k\}|^{q'(\frac12-\frac1p)}\,ds\Big)^{\frac{2}{q'}}\Big\}
\end{aligned}
$$

where $\tilde h(\rho,\omega):=\|\tilde f\|^2_{L^q(L^p_{\mathrm w})(Q_\rho)}+\omega\|\tilde g\|_{L^{\frac q2}(L^{\frac p2}_{\mathrm w})(Q_\rho)}$.

We take $p_*,q_*>0$ as in the proof of Proposition 2.10 and for $i\in\mathbb N$ we take $\rho=\rho_i$, $k=k_i$, $\tilde\eta=\tilde\eta_i$ satisfying $\tilde\eta_i\equiv1$ on $Q^{\theta_0}_{\rho_{i+1}}$ and

$$
\begin{aligned}
&k_i=\mu^+-\frac{\omega}{2^{q_0}}+\frac{1}{2^{q_0+i+2}}\omega,\qquad \rho_i=\frac12\rho+\frac{1}{2^{i+2}}\rho,\\
&Y_i:=\frac{|Q^{\theta_0}_{\rho_i}\cap\{\tilde u>k_i\}|}{|Q^{\theta_0}_{\rho_0}|},\qquad Z_i=\frac{\rho_0^2}{|Q^{\theta_0}_{\rho_0}|}\Big(\int_{I^{\theta_0}_{\rho_i}}|B_{\rho_i}\cap\{\tilde u(s)>k_i\}|^{\frac{q_*}{p_*}}\,ds\Big)^{\frac{2}{q_*}},\\
&|\nabla\tilde\eta_i|\le\frac{2}{\rho_i-\rho_{i+1}}\le\frac{12\cdot2^i}{\rho_0},\qquad \partial_s\tilde\eta_i\le\frac{4}{\theta_0}\frac{1}{\rho_i^2-\rho_{i+1}^2}\le\frac{48\cdot2^{2i}}{\theta_0\rho^2}.
\end{aligned}
$$

From (2.34) and $(\tilde u-k_i)_+\le\frac{\omega}{2^{q_0+1}}$, we obtain

$$
\begin{aligned}
&\|(\tilde u-k_i)_+\tilde\eta_i\|^2_{L^\infty(L^2)\cap L^2(\dot H^1)(Q^{\theta_0}_{\rho_i})}\\
&\qquad\le C(\alpha)\Big\{\iint_{Q^{\theta_0}_{\rho_i}}(\tilde u-k_i)_+^2\Big\{\partial_s\tilde\eta_i{}^2+|\nabla\tilde\eta_i|^2\Big\}\,dsdx\\
&\qquad\quad+\tilde h(\rho,\omega)\Big(\int_{I^{\theta_0}_{\rho_i}}|B_\rho\cap\{\tilde u(s)>k_i\}|^{q'(\frac12-\frac1p)}\,ds\Big)^{\frac{2}{q'}}\Big\}\\
&\qquad\le C(\alpha)\Big\{\Big(\frac{\omega}{2^{q_0}}\Big)^2\Big(\frac{1}{\theta_0}+1\Big)\frac{2^{2i}}{\rho^2}|Q^{\theta_0}_{\rho_i}\cap\{\tilde u>k_i\}|\\
&\qquad\quad+\tilde h(\rho,\omega)\Big(\int_{I^{\theta_0}_{\rho_i}}|B_\rho\cap\{\tilde u(s)>k_i\}|^{\frac{q_*}{p_*}}\,ds\Big)^{\frac{2}{q_*}(1+\frac{2\gamma_0}{n})}\Big\}\\
&\qquad\le C(\alpha,\theta_0)\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0^2}\Big(\frac{\omega}{2^{q_0}}\Big)^2\Big\{2^{2i}Y_i+\tilde h(\rho,\omega)\Big(\frac{2^{q_0}}{\omega}\Big)^2\Big(\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0^2}\Big)^{\frac{2\gamma_0}{n}}Z_i^{1+\frac{2\gamma_0}{n}}\Big\}.
\end{aligned}
$$

Since $\delta_1 \le \theta_0^{-\frac{1}{q'}} 2^{-q_0}$, we have

$$\tilde{h}(\rho,\omega)\Big(\frac{2^{q_0}}{\omega}\Big)^2\Big(\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0^2}\Big)^{\frac{2\gamma_0}{n}} \le C(n,p,q,\theta_0)$$

and hence

$$\|(\tilde{u}-k_i)_+\tilde{\eta}_i\|^2_{L^\infty(L^2)\cap L^2(\dot{H}^1)(Q^{\theta_0}_{\rho_i})} \le C(n,\alpha,p,q,\theta_0)\Big(\frac{\omega}{2^{q_0}}\Big)^2\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0}\Big\{2^{2i}Y_i + Z_i^{1+\frac{2\gamma_0}{n}}\Big\}.$$

By the Ladyženskaja inequality (cf. Proposition B.2) and the Hölder inequality, we have

$$\begin{aligned}\|(\tilde{u}-k_i)_+\tilde{\eta}_i\|^2_{L^2(Q^{\theta_0}_{\rho_i})} &\le \|(\tilde{u}-k_i)_+\tilde{\eta}_i\|^2_{L^{2+\frac{4}{n}}(Q^{\theta_0}_{\rho_i})}\|\chi_{\{\tilde{u}>k_i\}}\|^2_{L^{n+2}(Q^{\theta_0}_{\rho_i})}\\ &\le C(n,\alpha,p,q,\theta_0)\Big(\frac{\omega}{2^{q_0}}\Big)^2\frac{|Q^{\theta_0}_{\rho_0}|^{1+\frac{2}{n+2}}}{\rho_0^2}Y_i^{\frac{2}{n+2}}\Big\{2^{2i}Y_i + Z_i^{1+\frac{2\gamma_0}{n}}\Big\}\end{aligned}$$

and

$$\|(\tilde{u}-k_i)_+\tilde{\eta}_i\|^2_{L^{q*}(L^{p*})(Q^{\theta_0}_{\rho_i})} \le C(n,\alpha,p,q,\theta_0)\Big(\frac{\omega}{2^{q_0}}\Big)^2\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0^2}\Big\{2^{2i}Y_i + Z_i^{1+\frac{2\gamma_0}{n}}\Big\}.$$

Since

$$\begin{aligned}\|(\tilde{u}-k_i)_+\tilde{\eta}_i\|^2_{L^2(Q^{\theta_0}_{\rho_i})} &\ge \|(\tilde{u}-k_i)_+\|^2_{L^2(Q^{\theta_0}_{\rho_{i+1}}\cap\{\tilde{u}>k_{i+1}\})}\\ &\ge (k_{i+1}-k_i)^2|Q^{\theta_0}_{\rho_{i+1}}\cap\{\tilde{u}>k_{i+1}\}| = \left(\frac{\omega}{2^{q_0+i+3}}\right)^2|Q^{\theta_0}_{\rho_0}|Y_{i+1}\end{aligned}$$

and

$$\begin{aligned}\|(\tilde{u}-k_i)_+\tilde{\eta}_i\|^2_{L^{q*}(L^{p*})(Q^{\theta_0}_{\rho_i})} &\ge \|(\tilde{u}-k_i)_+\|^2_{L^{q*}L^{p*}(Q^{\theta_0}_{\rho_{i+1}}\cap\{\tilde{u}>k_{i+1}\})}\\ &\ge \left(\frac{\omega}{2^{q_0+i+3}}\right)^2\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0^2}Z_{i+1},\end{aligned}$$

we obtain

$$Y_{i+1} \le C(n,\alpha,p,q,\theta_0)\Big\{2^{4i}Y_i^{1+\frac{2}{n+2}} + 2^{2i}Y_i^{\frac{2}{n+2}}Z_i^{1+\frac{2\gamma_0}{n}}\Big\}$$

and

$$Z_{i+1} \le C(n,\alpha,p,q,\theta_0)\Big\{2^{4i}Y_i + 2^{2i}Z_i^{1+\frac{2\gamma_0}{n}}\Big\}.$$

As the similar calculus in the proof of Proposition 2.10, we obtain

$$Z_0 \le \begin{cases} C(n,p,q,\theta_0)Y_0^{\frac{2}{p*}} & \text{if} \quad q \ge p,\\ C(n,p,q,\theta_0)Y_0^{\frac{2}{q*}} & \text{if} \quad q < p.\end{cases}$$

Therefore, by Lemma B.13, there exists $0 < \nu = \nu(n,\alpha,p,q,\theta_0) < 1$ such that if $Y_0 \le \nu$, then $Y_i \to 0$ as $i \to \infty$, i.e.

$$\tilde{u}(s,x) < \mu^+ - \frac{\omega}{2^{q_0+2}} \quad \text{a.a. } (s,x) \in Q^{\theta_0}_{\frac{\rho}{2}}.$$

By Lemma 2.16, we obtain the upper bounds of u. □

CHAPTER 3

Regularity and asymptotic behavior for the Keller-Segel system of degenerate type with critical nonlinearity

1. The Keller-Segel system of degenerate type

We consider the large time behavior of the global solution of the degenerate parabolic elliptic system:

$$
\begin{cases} \partial_t u - \Delta u^\alpha + \operatorname{div}(u\nabla\psi) = 0, & t > 0\,, x \in \mathbb{R}^n, \\ -\Delta\psi + \psi = u, & t > 0\,, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{3.1}
$$

where $\alpha > 1$. This system is described as the dynamics of the chemical attracted mold. The equation originally consists of two reaction diffusion equations. By taking the zero relaxation time limit, one can obtain the above form as the result. For the case of $\alpha = 1$, it is semi-linear problem and the system (3.1) is analyzed by many authors. For $\alpha > 1$, the problem (3.1) is degenerate parabolic elliptic system and there are some work on it (Biler-Nadzieja-Stańczy [**7**], Díaz-Galiano-Jüngel [**16, 17**], Luckhaus-Sugiyama [**32**], Ogawa [**39, 40**], Sugiyama [**46**], Sugiyama-Kunii [**48**]). On the other hand, the system has a strong relation with the variational structure and the large time behavior of the solution is really depending on the variational functional reduced from the entropy-energy inequality.

$$
W[u](t) \equiv \frac{1}{\alpha - 1}\|u(t)\|_\alpha^\alpha - \frac{1}{2}\int_{\mathbb{R}^n} u(t)\psi(t)\,dx \leq W[u_0].
$$

Then it appears that there exists a critical exponent $\alpha = 2 - \frac{2}{n}$ that the global behavior of the solution is changed. This exponent is considered as a threshold exponent to separate the global stability of the weak solution. Roughly speaking, the small solution with small initial data decays as $t \to \infty$. Then the main concern for this case is its asymptotic profile. By the self-similar rescaling, one may find that there appears some particular profile in its rescaled form. On the other hand, the equation is degenerated and it has some hyperbolic like feature in its weak solution when the solution meets zero. In this case, the regularity breaks down and the behavior is governed by the hyperbolic like structure. The most possible regularity for the weak solution is generally known as the Hölder continuity. Indeed, to show the asymptotic profile of the decaying solution, the regularity of the weak solution plays an important role.

In this chapter, we consider the regularity problem of the system (3.1) and apply it for the asymptotic stability of the decaying solution in the critical case and show its convergence rate for the asymptotic profile if it is rescaled in the self-similar way. Since

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This chapter is taken from the paper [**41**] which is the joint work with Professor Ogawa.

the equation is degenerated, the smoothness of the solution is not generally obtained and we necessarily consider the weak solution.

Definition 3.1. Let $\alpha > 1$. For non-negative initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$, we call (u, ψ) a weak solution of the system (3.1) if there exists $T > 0$ such that
(1) $u(t,x) \ge 0$ for almost all $(t,x) \in [0,T) \times \mathbb{R}^n$;
(2) $u \in L^\infty(0,T\,; L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n))$ with $\nabla u^\alpha \in L^2((0,T) \times \mathbb{R}^n)$;
(3) u satisfies (3.1) in the sense of distribution, that is for any $\phi \in C^\infty([0,T]\,; C_0^\infty(\mathbb{R}^n))$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} u(t)\phi(t)\,dx - \int_{\mathbb{R}^n} u_0\phi(0)\,dx \\ = \int_0^t d\tau \int_{\mathbb{R}^n} \{u(\tau)\partial_t\phi(\tau) - \nabla u^\alpha(\tau)\cdot\nabla\phi(\tau) + u(\tau)\nabla\psi(\tau)\cdot\nabla\phi(\tau)\}\,dx\end{aligned} \tag{3.2}$$

for almost all $0 < t < T$, where $\psi = (-\Delta + 1)^{-1}u$ is given by the Bessel potential.

We may obtain the time local weak solution of (3.1) by some approximating procedure. Then the existence of time global weak solution is classified by a threshold exponent $\alpha = 2 - \frac{2}{n}$. We summarize the known results for the existence and non-existence of time global weak solutions.

Proposition 3.2 (Biler-Nadzieja-Stańczy [**7**], Sugiyama [**47**], Sugiyama-Kunii [**48**]). *Let $n \ge 3$, $\alpha > 1$ and assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$. Then there exists a weak solution of (u, ψ) of (3.1) that satisfies for $0 < t < T$,*

$$\begin{aligned}&\|u(t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)},\\ &W(t) + \int_0^t \int_{\mathbb{R}^n} u(\tau)\left|\frac{\alpha}{\alpha-1}\nabla u^{\alpha-1} - \nabla\psi\right|^2 dx d\tau \le W(0),\end{aligned} \tag{3.3}$$

where

$$W(t) = \frac{1}{\alpha-1}\|u\|^\alpha_{L^\alpha(\mathbb{R}^n)} - \frac{1}{2}\|(-\Delta+1)^{-\frac{1}{2}}u(t)\|^2_{L^2(\mathbb{R}^n)}.$$

In addition:
(1) If $\alpha > 2 - \frac{2}{n}$, then for any initial data u_0 the solution exists globally in time and the solution is uniformly bounded.
(2) For $2 - \frac{4}{n+2} < \alpha \le 2 - \frac{2}{n}$ and the initial data satisfying $W(0) > 0$ with

$$\|u_0\|^{1-\gamma}_{L^1(\mathbb{R}^n)} W(0)^{\frac{\gamma-\alpha+1}{\alpha}} < C\|E_n\|^{-1}_{L_{\mathrm{w}}^{\frac{n}{n-2}}(\mathbb{R}^n)}, \tag{3.4}$$

then the weak solution exists globally in time where E_n is the fundamental solution of $-\Delta + 1$ in $\mathbb{R}^n$, $L^q_{\mathrm{w}}(\mathbb{R}^n)$ is the weak Lebesgue space and $\gamma + 1 = \frac{\alpha}{\alpha-2}\frac{n-2}{n}$.
(3) In particular, if $\alpha = 2 - \frac{2}{n}$, then the above condition (3.4) is given by

$$\|u_0\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)} < \frac{2n}{n-2}\|E_n\|^{-1}_{L_{\mathrm{w}}^{\frac{n}{n-2}}(\mathbb{R}^n)}. \tag{3.5}$$

(4) If $1 < \alpha \le 2 - \frac{2}{n}$, and the initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ with $|x|^2 u_0 \in L^1(\mathbb{R}^n)$ satisfies $W(0) < 0$, then the weak solution blows up in a finite time T in the following sense:

$$\limsup_{t\to T} \|u(t)\|_{L^q(\mathbb{R}^n)} = \infty \quad \textit{for all } \alpha \le q \le \infty.$$

By Proposition 3.2, the weak solution to (3.1) exists globally in time when $n \geq 3$, $2 - \frac{4}{n+2} < \alpha \leq 2 - \frac{2}{n}$ and the initial data is sufficiently small. When we consider the small data problem, then system can be regarded as the perturbed problem from the porous medium equation:

$$\tag{3.6} \begin{cases} \partial_t w - \Delta w^\alpha = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \qquad w(0,x) = w_0(x), & \qquad x \in \mathbb{R}^n. \end{cases}$$

For the porous medium equation, there exists an explicit solution called the Barenblatt-Pattle solution

DEFINITION 3.3 (the Barenblatt-Pattle solution). For $\alpha > 1$, we set $\sigma = n(\alpha - 1) + 2$. For some $A > 0$, the function $\mathscr{U}(t)$ defined by

$$\tag{3.7} \mathscr{U}(t,x) = (1+\sigma t)^{-\frac{n}{\sigma}} \left(A - \frac{\alpha-1}{2\alpha} \frac{|x|^2}{(1+\sigma t)^{\frac{2}{\sigma}}} \right)_+^{\frac{1}{\alpha-1}}$$

is called as the Barenblatt-Pattle solution, where $(f(t,x))_+ = \max\{f(t,x), 0\}$.

It is well-known that the Barenblatt-Pattle solution solves the porous medium equation (3.6) with the initial data $w_0(x) = \left(A - \frac{\alpha-1}{2\alpha}|x|^2\right)_+^{\frac{1}{\alpha-1}}$.

In the case $\alpha \leq 2 - \frac{2}{n}$, and the initial data u_0 is small, then we may regard the nonlinear term $\operatorname{div}(u\nabla\psi)$ in (3.1) as a small perturbation and we speculate that the solution (3.1) asymptotically converges to the solution of the porous medium equation. In fact, Luckhaus-Sugiyama [**32**] showed the asymptotic behavior of the solution in L^p spaces for $1 < \alpha \leq 2 - \frac{2}{n}$, $n \geq 3$ and $1 \leq p \leq \infty$. Ogawa [**40**] showed that if $1 < \alpha < 2 - \frac{2}{n}$, then we obtain the algebraic convergence rate of the solution in L^1 space via the argument due to Carrillo-Toscani [**15**] and the critical Sobolev type inequality (cf. Ogawa-Taniuchi [**42**]). Namely, for $1 < \alpha < 2 - \frac{2}{n}$ and $W(0) > 0$ with (3.4) then there exist $\nu > 0$ and $C > 0$ such that

$$\tag{3.8} \|u(t) - \mathscr{U}(t)\|_{L^1(\mathbb{R}^n)} \leq C(1+t)^{-\nu}, \quad t > 0,$$

where $\mathscr{U}$ is the Barenblatt-Pattle solution with $\|\mathscr{U}(0)\|_1 = \|u_0\|_1$.

In this chapter, we show the same asymptotic convergence in $L^1(\mathbb{R}^n)$ for the critical case $\alpha = 2 - \frac{2}{n}$. Our main theorem is the following:

THEOREM 3.4. *Let $\alpha = 2 - \frac{2}{n}$ and $n \geq 3$. Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ satisfying $W(0) > 0$, (3.5), $\|u_0\|_1 < 2$ and $|x|^a u_0 \in L^1(\mathbb{R}^n)$ for some $a > n$. Then, there exist $C > 0$ and $\nu > 0$ such that the corresponding global weak solution u of (3.1) satisfies*

$$\tag{3.9} \|u(t) - \mathscr{U}(t)\|_{L^1(\mathbb{R}^n)} \leq C(1+\sigma t)^{-\nu}, \quad t > 0,$$

where $\mathscr{U}$ is the Barenblatt-Pattle solution with $\|\mathscr{U}(0)\|_1 = \|u_0\|_1$.

To show the asymptotic stability of decaying solution, we necessarily consider the regularity of the solution. Indeed, the weak solution to the degenerate problem (3.1) has a hyperbolic feature in it when the solution meets zero. In this case, the equation lost the parabolic behavior and the solution behaves as if it is a solution of the hyperbolic equation. In the proof of Ogawa [**40**], the Hölder regularity is used essential way to show the asymptotic stability estimate. To see this, we firstly introduce the forward self-similar transform, which plays an important role in studying the asymptotic behavior of the solution. We introduce the forward self-similar scaling (t', x') as

$$t' = \frac{1}{\sigma}\log(1+\sigma t), \quad x' = \frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}},$$

where $\sigma = n(\alpha-1)+2$ and the forward self-similar transform $(v(t', x'), \phi(t', x'))$ as

$$v(t', x') = (1+\sigma t)^{\frac{n}{\sigma}} u(t, x), \quad \phi(t', x') = (1+\sigma t)^{\frac{n}{\sigma}} \psi(t, x).$$

Then, the forward self-similar transform (v, ϕ) satisfies the following degenerate parabolic elliptic system:

$$\begin{cases} \partial_{t'} v - \operatorname{div}_{x'}(\nabla_{x'} v^\alpha + x'v - e^{-\kappa t'} v \nabla_{x'}\phi) = 0, & t' > 0\,, x' \in \mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t' > 0\,, x' \in \mathbb{R}^n, \\ v(0, x') = u_0(x') \geq 0, & x' \in \mathbb{R}^n, \end{cases} \tag{3.10}$$

where $\kappa = n+2-\sigma = n(2-\alpha)$. The weak solution of the system (3.10) is similarly defined as in the case for (3.1).

For $1 \leq p \leq \infty$, we obtain

$$(1+\sigma t)^{\frac{n}{\sigma}(1-\frac{1}{p})} \|u(t) - \mathscr{U}(t)\|_p = \|v(t') - \mathscr{V}\|_p \tag{3.11}$$

where

$$\mathscr{V}(x') := \left(A - \frac{\alpha-1}{2\alpha}|x'|^2\right)_+^{\frac{1}{\alpha-1}} = (1+\sigma t)^{\frac{n}{\sigma}} \mathscr{U}(t, x)$$

is a self-similar profile of the Barenblatt-Pattle solution. If $p = 1$, then equation (3.11) is rewritten by

$$\|u(t) - \mathscr{U}(t)\|_1 = \|v(t') - \mathscr{V}\|_1.$$

For the sake of obtaining the convergence rate of the solution in L^1, we show the convergence rate of the forward self-similar transform in L^1 space. Ogawa [**40**] showed that if the self-similar transformed solution v is the uniformly Hölder continuous, then we obtain the exponential convergence rate of the self-similar transform v. More precisely,

Proposition 3.5 (Ogawa [**40**]). *Let $\alpha = 2 - \frac{2}{n}$. Assume that an initial data u_0 satisfies $W(0) > 0$ and (3.5). If the corresponding forward self-similar transform v is uniformly Hölder continuous, then there exist $\nu > 0$ and $C > 0$ such that*

$$\|v(t') - \mathscr{V}\|_{L^1(\mathbb{R}^n_y)} \leq Ce^{-\nu t'}, \quad t' > 0, \tag{3.12}$$

where $\mathscr{V}$ is the self-similar profile of the Barenblatt-Pattle solution with $\|\mathscr{V}\|_1 = \|u_0\|_1$.

Our main concern is to obtain the algebraic convergence rate of the solution in L^1 space for the case of critical exponent $\alpha = 2 - \frac{2}{n}$. The reason why the critical case is excluded in Ogawa [**40**] is because the uniform Hölder continuity of the rescaled weak solution $v(t', x')$ is required for proving algebraic convergence rate. The Hölder continuity was obtained in Ogawa [**40**] for $v(t', x')$ via the rescaled weak solution $u(t, x)$ and hence it was not the uniform estimate for (t', x'). By this argument, the critical case has to be necessarily excluded since the decaying factor $e^{-(\kappa-2)t'}$ disappears in the crucial estimates. To cover the critical case, we necessarily derive the uniform Hölder regularity of the rescaled solution $v(t', x')$ directly by assuming that the moment of the solution is uniformly bounded in time. Using Theorem 2.4 in Chapter 2, we obtain the following regularity result:

Theorem 3.6. *Let (v, ϕ) be a weak solution of the rescaled Keller-Segel system* (3.10) *in $(u, \psi) \in L^\infty(0, T\,; L^1 \cap L^\alpha) \times L^\infty(0, T\,; W^{2,\alpha})$. Assume that $|x|^a u_0 \in L^1(\mathbb{R}^n)$ for some $a > n$. Then $v(t', x')$ is uniformly Hölder continuous. Namely, there exist constants $C > 0$ and $0 < \gamma < 1$ such that for any (t', x') and $(s', y') \in (1, \infty) \times \mathbb{R}^n$, we obtain*

$$|u(t', x') - u(s', y')| \le C(|t' - s'|^{\frac{\gamma}{2}} + |x' - y'|^\gamma).$$

To show Theorem 3.6, we put $f = x'v - e^{-\kappa t'} v \nabla_{x'} \phi$ and apply Theorem 2.4. The integrability of $x'v$ is essential. Thanks to the uniform moment bound for the weak solution, we may apply Theorem 2.4 with the external term $\mathrm{div}(x'v - e^{-\kappa t'} v \nabla \phi)$. From the uniform Hölder continuity of v, we may derive the convergence rate of the solution.

This chapter is organized as follows. In section 2, we study some properties of the forward self-similar transform. Using these properties, we show uniform Hölder continuity of the rescaled solution. In section 3, we consider the asymptotic convergence of the weak solution of (3.1) by using the uniform Hölder continuity of the rescaled solution. We compute the time derivative of the free energy functional.

2. Forward self-similar transform

In this section, we show the time decay of the global weak solution of the degenerated Keller-Segel system. This is originally shown in Sugiyama [**46**] however, we present the method of rescaling which is shown by Ogawa [**39**].

2.1. Rescaled equation. We introduce the new scaled variables (t', x') as

$$t' = \frac{1}{\sigma} \log(1 + \sigma t), \quad x' = \frac{x}{(1 + \sigma t)^{\frac{1}{\sigma}}}, \tag{3.13}$$

where $\sigma = n(\alpha - 1) + 2$ and introduce the new scaled unknown $v(t', x')$, $\phi(t', x')$ as

$$\begin{aligned} u(t, x) &= (1 + \sigma t)^{-\frac{n}{\sigma}} v\biggl(\frac{1}{\sigma}\log(1 + \sigma t), \frac{x}{(1 + \sigma t)^{\frac{1}{\sigma}}}\biggr), \\ \psi(t, x) &= (1 + \sigma t)^{-\frac{n}{\sigma}} \phi\biggl(\frac{1}{\sigma}\log(1 + \sigma t), \frac{x}{(1 + \sigma t)^{\frac{1}{\sigma}}}\biggr), \end{aligned}$$

or one may write as

$$\begin{aligned} v(t', x') &= e^{nt'} u\biggl(\frac{1}{\sigma}(e^{\sigma t'} - 1), x' e^{t'}\biggr), \\ \phi(t', x') &= e^{nt'} \psi\biggl(\frac{1}{\sigma}(e^{\sigma t'} - 1), x' e^{t'}\biggr). \end{aligned}$$

The resulting scaling equation of (v, ϕ) follows by setting $\kappa = n + 2 - \sigma = n(2 - \alpha)$,

$$\begin{cases} \partial_{t'} v - \mathrm{div}_{x'}(\nabla_{x'} v^\alpha + x'v - e^{-\kappa t'} v \nabla_{x'} \phi) = 0, & t' > 0\,, x' \in \mathbb{R}^n, \\ -e^{-2t'} \Delta_{x'} \phi + \phi = v, & t' > 0\,, x' \in \mathbb{R}^n, \\ v(0, x') = u_0(x') \ge 0, & x' \in \mathbb{R}^n. \end{cases} \tag{3.14}$$

In this case, the vanishing exponent as before can be found as $\alpha = 2$ by

$$0 = \sigma - n - 2 = n(\alpha - 2)$$

and thus the sub-critical case is corresponding to $\alpha < 2$. Hereafter we analyze the above rescaled equation (3.14) to see the asymptotic behavior of the solution. We slightly change the outlook of the solution as follows:

The existence of the weak solution of (3.14) may be proven by a similar way to the original equation. Indeed, the scaling does not change any analytical feature of (3.1) except the weighted restriction such as $v \in C((0,T)\,; L^\alpha \cap L^1_a(\mathbb{R}^n))$ for $a \geq 2$. Similar to the original system, we consider the approximated system by the parabolic regularization:

$$\begin{cases} \partial_{t'} v - \mathrm{div}_{x'}(\nabla_{x'}(v+\varepsilon)^\alpha + x'v - e^{-\kappa t'} v \nabla_{x'}\phi) = 0, & t' > 0\,, x' \in \mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t' > 0\,, x' \in \mathbb{R}^n, \\ v(0,x') = u_0(x') \geq 0, & x' \in \mathbb{R}^n. \end{cases} \tag{3.15}$$

Namely, we again consider the nonnegative weak solution $v(t',x')$ as before. Note that for the construction of the weak solution, we need to use the diagonal argument obtaining the weak solution (u,ψ) and (v,ϕ) simultaneously, since we do not know the uniqueness of the weak solution.

2.2. Rescaled uniform bounds. The following estimate is a direct consequence of the above a priori bound of the rescaled solution.

PROPOSITION 3.7. *Let $1 < \alpha \leq 2 - \frac{2}{n}$ and $(v(t), \phi(t))$ be a weak solution of* (3.14) *for the initial data $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Assume that*

$$\|u_0\|_1^{1-\gamma} W(0)^{\frac{\gamma-\alpha+1}{\alpha}} < C\|E_n\|^{-1}_{L_{\mathrm{w}}^{\frac{n}{n-2}}} \tag{3.16}$$

for $1 < \alpha \leq 2 - \frac{2}{n}$ and $\gamma + 1 = \frac{\alpha}{\alpha-1}\frac{n-2}{n}$, where E_n is the fundamental solution to $-\Delta + 1$ in $\mathbb{R}^n$, and $C > 0$ is the constant in Proposition 3.2. Then

(1) we have

$$\|v(t)\|_q \leq C$$

for all $1 \leq q \leq \infty$.

(2) for all $\frac{n}{n-1} < r \leq \infty$,

$$\|\nabla\phi(t)\|_r \leq Ce^t.$$

Once we obtain the above uniform bound for the rescaled solution, we can immediately obtain the time decay estimate for the solution of the original equation.

$$\int_{\mathbb{R}^n} v^q(t',x')\,dx' = \int_{\mathbb{R}^n} e^{n(q-1)t'} u^q(t,x)\,dx = (1+\sigma t)^{\frac{n}{\sigma}(q-1)} \int_{\mathbb{R}^n} u^q(t,x)\,dx \tag{3.17}$$

in the original variables (t,x). Hence we obtain the following decay estimate for the original solution as the corollary of Proposition 3.7.

PROPOSITION 3.8 (Ogawa [**39**], Sugiyama [**48**]). *Let $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and let $(u(t), \psi(t))$ be a weak solution of* (3.1). *If $1 < \alpha \leq 2 - \frac{2}{n}$ with small initial data* (3.16), *we have*

$$\|u(t)\|_q \leq C(1+\sigma t)^{-\frac{n}{\sigma}(1-\frac{1}{q})}$$

for all $1 \leq q \leq \infty$.

2.3. The moment bounds. The last part of this section, we show the second moment of the weak solution remains bounded for $0 \le t \le T$.

PROPOSITION 3.9. *Let $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ with $|x|^2 u_0 \in L^1(\mathbb{R}^n)$. Then the weak solution (v, ϕ) of (3.14) satisfies*

$$\int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' \le e^{-nt'} \int_{\mathbb{R}^n} |x'|^2 u_0\, dx' + \frac{2(n-2)}{n} W_s(0), \tag{3.18}$$

where

$$W_s(t') = \frac{1}{\alpha - 1} \int_{\mathbb{R}^n} v^\alpha(t')\, dx' + \frac{1}{2} \int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' - \frac{1}{2} \int_{\mathbb{R}^n} e^{-\kappa t'} v(t') \phi(t')\, dx'.$$

Namely, $|x'|^2 v(t') \in L^1(\mathbb{R}^n)$ for almost all t'. In addition, if we assume that $u_0 \in L^1_a(\mathbb{R}^n)$ with $a \ge 2$, then we have

$$\int_{\mathbb{R}^n} |x'|^a v(t')\, dx' \le e^{-at'} \int_{\mathbb{R}^n} |x'|^a u_0\, dx' + C. \tag{3.19}$$

PROOF OF PROPOSITION 3.9. We only give the formal proof. It can be justified some appropriate cut-off and approximation procedure. To show (3.18) we test $|x'|^2$ to the equation and we see

$$\frac{d}{dt'} \int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' = 2n \|v(t')\|_\alpha^\alpha - 2 \int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' + 2e^{-\kappa t'} \int_{\mathbb{R}^n} \big(x' v(t') \cdot \nabla \phi(t')\big)\, dx'. \tag{3.20}$$

We invoke the Pokhozaev identity for the second equation. We multiply the elliptic part of the system by the generator of the dilation $x \cdot \nabla \psi$ and integrate it by parts. Then it follows

$$\begin{aligned} \int_{\mathbb{R}^n} \big(x' v(t') \cdot \nabla \phi(t')\big)\, dx' &= e^{-2t'} \Big(1 - \frac{n}{2}\Big) \int_{\mathbb{R}^n} |\nabla \phi(t')|^2\, dx' - \frac{n}{2} \int_{\mathbb{R}^n} |\phi(t')|^2\, dx' \\ &= \Big(1 - \frac{n}{2}\Big) \int_{\mathbb{R}^n} v(t') \phi(t')\, dx' - \|\phi(t')\|_2^2. \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21), we obtain

$$\begin{aligned} &\frac{d}{dt'} \int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' + n \int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' \\ &= 2n \|v(t')\|_\alpha^\alpha + (n-2) \int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' + (2-n) e^{-\kappa t'} \int_{\mathbb{R}^n} v(t') \phi(t')\, dx' - 2e^{-\kappa t'} \|\phi(t')\|_2^2 \\ &\le 2(n-2) W_s(t') + 2n \bigg(\frac{\alpha - 2 + \frac{2}{n}}{\alpha - 1}\bigg) \|v(t')\|_\alpha^\alpha - 2e^{-\kappa t'} \|\phi(t')\|_2^2. \end{aligned} \tag{3.22}$$

Thus under the condition $\alpha \le 2 - \frac{2}{n}$, we see that

$$\int_{\mathbb{R}^n} |x'|^2 v(t')\, dx' \le e^{-nt'} \int_{\mathbb{R}^n} |x'|^2 u_0\, dx' + \frac{2(n-2)}{n} W_s(0)(1 - e^{-nt'}).$$

For further weighted estimate, we modify (3.20) to have

$$\begin{aligned}(3.23)\quad &\frac{d}{dt'}\int_{\mathbb{R}^n}|x'|^a v(t')\,dx' + a\int_{\mathbb{R}^n}|x'|^a v(t')\,dx' \\ &\qquad = a(n-2+a)\int_{\mathbb{R}^n}|x'|^{a-2}v^\alpha(t')\,dx' + ae^{-\kappa t'}\int_{\mathbb{R}^n}|x'|^{a-2}x'v(t')\cdot\nabla\phi(t')\,dx'.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{d}{dt'}\left[e^{at'}\int_{\mathbb{R}^n}|x'|^a v(t')\,dx'\right] \\ &\quad \le a(n-2+a)e^{at'}\|v(t')\|_\infty^{\alpha-1}\int_{\mathbb{R}^n}|x'|^{a-2}v(t')\,dx' + ae^{-(\kappa-a)t'}\|\nabla\phi\|_\infty\int_{\mathbb{R}^n}|x'|^{a-1}v\,dx'.\end{aligned}$$

By the uniform boundedness for $\|v(t')\|_\infty$, $e^{t'}\|\nabla\phi\|_\infty$ (Proposition 3.7) and the lower moment bounds implies that

$$\int_{\mathbb{R}^n}|x'|^a v(t')\,dx' \le e^{-at'}\int_{\mathbb{R}^n}|x'|^a v(0)\,dx' + C. \tag{3.24}$$

□

Proof of Theorem 3.6. By Proposition 3.9, we obtain $yv \in L^\infty(0,\infty\,;L^a(\mathbb{R}^n))$ and by Proposition 3.7, we have $e^{-\kappa t'}v\nabla\phi \in L^\infty((0,\infty)\times\mathbb{R}^n)$. Since $a>n$, we can apply Theorem 2.4 and we obtain the uniform Hölder continuity of v in $(1,\infty)\times\mathbb{R}^n$. □

3. The asymptotic profile

In this section, we show the asymptotic convergence of the weak solution $u(t,x)$ of (3.1) to the Barenblatt-Pattle solution by using the uniform Hölder estimate.

To show the convergence rate, we consider as we mentioned in the introduction that the self-similar transform of the system and the weak solution of the rescaled system:

$$\begin{cases}\partial_{t'}v - \mathrm{div}_{x'}(\nabla_{x'}v^\alpha + x'v - e^{-\kappa t'}v\nabla_{x'}\phi) = 0, & t'>0\,, x'\in\mathbb{R}^n,\\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t'>0\,, x'\in\mathbb{R}^n,\\ v(0,x') = u_0(x') \ge 0, & x'\in\mathbb{R}^n,\end{cases} \tag{3.25}$$

where $\kappa = n+2-\sigma = n(2-\alpha)$.

In what follows, we only treat the scaled system (3.25) and hence we use a simpler notations as $t'\to t$ and $x'\to x$ if it does not cause any confusion.

Applying the method of the Fokker-Planck equation due to Carrillo-Toscani [**15**], we compute the time derivative of the free energy functional: For a weak solution (v,ϕ) of (3.25), we let

$$\begin{aligned}H(v(t)) &\equiv \frac{1}{\alpha-1}\int_{\mathbb{R}^n}v^\alpha(t)\,dx + \frac{1}{2}\int_{\mathbb{R}^n}|x|^2 v(t)\,dx,\\ J(v(t)) &\equiv \int_{\mathbb{R}^n}v(t)\left|\nabla\left(\frac{\alpha}{\alpha-1}v^{\alpha-1}(t) + \frac{|x|^2}{2}\right)\right|^2 dx,\\ I(v(t)) &\equiv \int_{\mathbb{R}^n}v(t)\left|\nabla\left(\frac{\alpha}{\alpha-1}v^{\alpha-1}(t) + \frac{|x|^2}{2} - e^{-\kappa t}\phi\right)\right|^2 dx.\end{aligned}$$

The key idea to show the asymptotic behavior is to consider the decay of the dissipative flux term $I(v)$ in t. We firstly observe that the entropy functional has a certain relation:

PROPOSITION 3.10. *For a weak solution v and ϕ of (3.14), we have*

$$\begin{aligned} H(v(t)) &+ \frac{1}{2}(e^{-(\kappa+2)t}\|\nabla\phi(t)\|_2^2 + e^{-\kappa t}\|\phi(t)\|_2^2) \\ &+ \int_s^t \left(\frac{\kappa-2}{2} e^{-(\kappa+2)\tau}\|\nabla\phi(\tau)\|_2^2 + \frac{\kappa}{2} e^{-\kappa\tau}\|\phi(\tau)\|_2^2 + J(v(\tau)) \right) d\tau \\ &\le H(v(s)) + \frac{1}{2}(e^{-(\kappa+2)s}\|\nabla\phi(s)\|_2^2 + e^{-\kappa s}\|\phi(s)\|_2^2) + \int_s^t e^{-2\kappa\tau}\, d\tau \int_{\mathbb{R}^n} v|\nabla\phi(\tau)|^2\, dx. \end{aligned} \tag{3.26}$$

In particular, for $1 < \alpha \le 2 - \frac{2}{n}$, we have that $H(v(t))$ is uniformly bounded in t under the smallness condition (3.16) and

$$H(v(t)) \le H(u_0) + \frac{1}{2}\int_{\mathbb{R}^n} \phi(0)v(0)\, dx + C \sup_{\tau>0}\Big[e^{-2\tau}\|v(\tau)\|_\infty \|\nabla\phi(\tau)\|_2^2\Big] \tag{3.27}$$

for any $t > 0$.

REMARK 3.11. The restriction of the exponent $\alpha \le 2 - \frac{2}{n}$ follows from the restriction on $\kappa \ge 2$ in view of the integrability of the third term of the right-hand side of the above inequality.

PROOF OF PROPOSITION 3.10. The equation (3.14) can be rewritten as the following form:

$$\partial_t v - \operatorname{div}\left(v\nabla \left(\frac{\alpha}{\alpha-1} v^{\alpha-1}(t) + \frac{|x|^2}{2} - e^{-\kappa t}\phi \right)\right) = 0.$$

Testing $\frac{\alpha}{\alpha-1}v^{\alpha-1}(t) + \frac{|x|^2}{2} + e^{-\kappa t}\phi$, we see that

$$\partial_t H(v(t)) + e^{-\kappa t}\int_{\mathbb{R}^n} \phi\partial_t v\, dx + J(v(t)) = e^{-2\kappa t}\int_{\mathbb{R}^n} v|\nabla\phi|^2\, dx.$$

Since

$$\begin{aligned} e^{-\kappa t}\int_{\mathbb{R}^n} \phi\partial_t v\, dx &= e^{-\kappa t}\int_{\mathbb{R}^n} \phi\partial_t(-e^{-2t}\Delta\phi + \phi)\, dx \\ &= e^{-\kappa t}\int_{\mathbb{R}^n} \nabla\phi\cdot\nabla(\partial_t(e^{-2t}\phi))\, dx + e^{-\kappa t}\int_{\mathbb{R}^n} \phi\partial_t\phi\, dx \\ &= \frac{1}{2}\partial_t(e^{-(\kappa+2)t}\|\nabla\phi\|_2^2 + e^{-\kappa t}\|\phi\|_2^2) + \frac{\kappa-2}{2}e^{-(\kappa+2)t}\|\nabla\phi\|_2^2 + \frac{\kappa}{2}e^{-\kappa t}\|\phi\|_2^2, \end{aligned}$$

we obtain

$$\begin{aligned} \partial_t &\left(H(v(t)) + \frac{1}{2}(e^{-(\kappa+2)t}\|\nabla\phi\|_2^2 + e^{-\kappa t}\|\phi\|_2^2)\right) \\ &+ \frac{\kappa-2}{2}e^{-(\kappa+2)t}\|\nabla\phi\|_2^2 + \frac{\kappa}{2}e^{-\kappa t}\|\phi\|_2^2 + J(v(t)) = e^{-2\kappa t}\int_{\mathbb{R}^n} v|\nabla\phi|^2\, dx. \end{aligned} \tag{3.28}$$

Integrating (3.28) over $[s,t]$ we obtain (3.26). Under the condition $1 < \alpha \le 2 - \frac{2}{n}$, we have $\kappa \ge 2$ and by Proposition 3.7 $e^{-2t}\|v(t)\|_\infty \|\nabla\phi(t)\|_2^2 \le C$. Therefore it follows

$$\begin{aligned} H(v(t)) &\le H(v(0)) + \frac{1}{2}\big[\|\nabla\phi(0)\|_2^2 + \|\phi(0)\|_2^2\big] + \frac{1}{\kappa-1}\sup_{t>0}\big(e^{-2t}\|v(t)\|_\infty\|\nabla\phi(t)\|_2^2\big) \\ &\le H(v(0)) + \frac{1}{2}\int_{\mathbb{R}^n} v(0)\phi(0)\,dx + C\sup_{\tau>0}\big(e^{-2\tau}\|v(\tau)\|_\infty\|\nabla\phi(\tau)\|_2^2\big) \end{aligned}$$

for all $t > 0$. $\square$

For a solution v and ϕ of (3.14), we let

$$K(x, v(t), \phi(t)) := \nabla\left(\frac{\alpha}{\alpha-1}v^{\alpha-1} + \frac{|x|^2}{2} - e^{-\kappa t}\phi\right).$$

It is not so difficult to see that the asymptotic profile is given by $J(v(t)) \to 0$ from the above inequality. However to obtain the convergence rate for a weak solution in the weighted class $L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we derive that $I(v(t))$ is exponentially decaying. To this end, we observe the time derivative of the functional $I(v(t))$. We assume that $\kappa > 0$ namely $\alpha < 2$.

Following Carrillo-Toscani [**15**], we formally have

(3.29)
$$\begin{aligned} \frac{d}{dt}I(v(t)) = &-2\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx - 2(\alpha-1)\int_{\mathbb{R}^n} v^\alpha|\operatorname{div}K(x,v,\phi)|^2\,dx \\ &- 2\int_{\mathbb{R}^n} v^\alpha|\nabla K(x,v,\phi)|^2\,dx + 2e^{-\kappa t}\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x,v,\phi)(\partial_i\partial_j\phi)\,dx \\ &+ 2e^{-\kappa t}\int_{\mathbb{R}^n}\operatorname{div}(vK(x,v,\phi))\partial_t\phi\,dx - 2\kappa e^{-\kappa t}\int_{\mathbb{R}^n} vK(x,v,\phi)\nabla\phi\,dx. \end{aligned}$$

Since the weak solution does not have enough regularity, the above identity is not necessarily valid and the actual estimate should be obtained in the form of the integral inequality. This is justified by an appropriate approximation: Let (v,ϕ) be a solution of the regularized system:

$$\begin{cases} \partial_t v - \operatorname{div}((v+\varepsilon)K_\varepsilon(x,v,\phi)) = -\varepsilon(e^{-\kappa-2t}(v-\phi)+n), & t>0,\ x\in\mathbb{R}^n, \\ -e^{-2t}\Delta\phi + \phi = v, & t>0,\ x\in\mathbb{R}^n, \\ v(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{3.30}$$

where

$$K_\varepsilon(x, v(t), \phi(t)) := \nabla\left(\frac{\alpha}{\alpha-1}(v+\varepsilon)^{\alpha-1} + \frac{|x|^2}{2} - e^{-\kappa t}\phi\right).$$

Note that the above system (3.30) is equivalent to (3.15). The existence of the smooth and sufficiently fast decaying solution at $|x| \to \infty$ of (3.30) is obtained in a similar manner in Sugiyama [**46**].

PROPOSITION 3.12. *Let ζ be a smooth cut-off function such that $\zeta = 1$ in B_R and whose derivatives are supported in $B_{2R} \setminus B_R$. For a solution v and ϕ of* (3.30) *belonging to L^1, we let*

$$I_\varepsilon(v(t)) := \int_{\mathbb{R}^n} v(t)\,|K_\varepsilon(x,v,\phi)|^2\,\zeta^2\,dx.$$

Then we have

$$
\begin{aligned}
\frac{d}{dt}I_\varepsilon(v(t)) \le -2\int_{\mathbb{R}^n}&(v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2\zeta^2\,dx\\
&-2(\alpha-1)\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|\operatorname{div}K_\varepsilon(x,v,\phi)|^2\zeta^2\,dx\\
&-2\int_{\mathbb{R}^n}(v+\varepsilon)^\alpha|\nabla K_\varepsilon(x,v,\phi)|^2\zeta^2\,dx\\
&+2e^{-\kappa t}\int_{\mathbb{R}^n}v(D^2\phi K_\varepsilon(x,t,\phi)\cdot K_\varepsilon(x,t,\phi))\zeta^2\,dx\\
&+2e^{-(\kappa-2)t}\int_{\mathbb{R}^n}|vK_\varepsilon(x,v,\phi)|^2\zeta^2\,dx-2\kappa e^{-\kappa t}\int_{\mathbb{R}^n}vK_\varepsilon(x,v,\phi)\nabla\phi\zeta^2\,dx\\
&\quad+E_I(x,v,\phi,\varepsilon,\nabla\zeta),
\end{aligned}
\tag{3.31}
$$

where $E_I(x,v,\phi,\varepsilon,\nabla\zeta)$ denotes the error term and it will be vanishing when we take the limit $R\to\infty$ and $\varepsilon\to 0$.

The derivation and rigorous treatment of (3.31) is given in Appendix of [**40**]. We proceed to the following.

Proposition 3.13. *Let (v,ϕ) be a weak solution of* (3.14)*. Then under the condition $1<\alpha\le 2-\frac{2}{n}$ and the solution v has uniform estimate $\sup_{t>0}\|v(t)\|_\infty\le C_0$ for some constant, there exists $T_0>0$ such that for any $T_0<t$,*

$$
I(v(t))+\int_{T_0}^t I(v(\tau))\,d\tau\le I(v(T_0)),
\tag{3.32}
$$

in particular, there exist constants $\nu, C>0$ such that

$$
I(v(t))\le Ce^{-\nu t}
$$

for $t\ge T_0$.

To obtain the above proposition, we need the following two ingredients. First one is the Sobolev type inequality in the critical type originally due to Brezis-Gallouet [**8**]. This is the generalized version obtained in Ogawa-Taniuchi [**42**] and Kozono-Ogawa-Taniuchi [**28**].

Proposition 3.14 (Kozono-Ogawa-Taniuchi [**28**], Ogawa-Taniuchi [**42**]). *There exists a constant C depending only on n such that for $f\in L^2(\mathbb{R}^n)\cap C^\gamma(\mathbb{R}^n)$, the following inequality holds:*

$$
\|f\|_\infty\le C(1+\|f\|_{BMO}\log(e+\|f\|_2+\|f\|_{C^\gamma})),
\tag{3.33}
$$

where

$$
BMO:=\left\{f\in L^1_{\mathrm{loc}}(\mathbb{R}^n):\|f\|_{BMO}:=\sup_{x\in\mathbb{R}^n,\,R>0}\frac{1}{|B_R(x)|}\int_{B_R(x)}|f-(f)_{B_R(x)}|\,dx<\infty\right\}.
$$

Proof of Proposition 3.13. To avoid the complexity of the notation, we treat the estimate only for the essential parts in rather formal way, namely dropping the parameter ε and cut-off function ζ. The rigorous procedure requires that all those estimates are proceeded before passing to the limit $R\to\infty$ and $\varepsilon\to 0$ and the rigorous treatment can be found in [**40**]. Observing the estimate (3.31), we need to estimate the last four terms in the right-hand side. The fourth error term $E_I(x,v,\phi,\varepsilon,\nabla\zeta)$ is handled in Appendix A in [**40**] since it does not give any effect for the estimation of the other terms. Firstly, the sixth term of the right-hand side of (3.31) can be estimated as follows:

$$
\begin{aligned}
-2\kappa e^{-\kappa t}\int_{\mathbb{R}^n} vK(x,v,\phi)\nabla\phi\,dx &\le 2\kappa e^{-\kappa t}\|\nabla\phi\|_\infty\Big(\int_{\mathbb{R}^n} v\,dx\Big)^{\frac12}\Big(\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx\Big)^{\frac12}\\
&\le 2\kappa\delta^{-1}e^{-2\kappa t}\|\nabla\phi\|_\infty^2\|v\|_1+\frac{\delta}{2}\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx,
\end{aligned}
$$

where $\delta>0$ is a small parameter. Hence from Proposition 3.12, we obtain that

$$
\begin{aligned}
\frac{d}{dt}I(v(t)) \le& -(2-\delta)\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx-2(\alpha-1)\int_{\mathbb{R}^n} v^\alpha|\operatorname{div}K(x,v,\phi)|^2\,dx\\
&-2\int_{\mathbb{R}^n} v^\alpha|\nabla K(x,v,\phi)|^2\,dx+2e^{-(\kappa-2)t}\int_{\mathbb{R}^n}|vK(x,v,\phi)|^2\,dx\\
&+2e^{-\kappa t}\int_{\mathbb{R}^n} v(D^2\phi K(x,t,\phi)\cdot K(x,t,\phi))\,dx\\
&+C\delta^{-1}e^{-2(\kappa-1)t}\|v(t)\|_1\sup_t(e^{-2t}\|\nabla\phi(t)\|_2^2).
\end{aligned}
\tag{3.34}
$$

We now tern into how to treat the following term:

$$
e^{-\kappa t}\int_{\mathbb{R}^n} v(D^2\phi K(x,t,\phi)\cdot K(x,t,\phi))\,dx.
$$

Applying the logarithmic interpolation inequality of Brezis-Gallouet type (3.33), we see

$$
\|D^2\phi(t)\|_\infty \le C\Big(1+\|D^2\phi(t)\|_{BMO}\log\big(e+\|D^2\phi(t)\|_2+\|D^2\phi(t)\|_{C^\gamma}\big)\Big).
$$

By the Calderon-Zygmund inequality, we have

$$
\|D^2\phi\|_2\le C\|\Delta\phi\|_2\le Ce^{2t}(\|v\|_2+\|\phi\|_2)\le Ce^{2t}\|v\|_2.
$$

By the Schauder estimate, we obtain

$$
\|D^2\phi\|_{C^\gamma}\le Ce^{2t}(\|v\|_{C^\gamma}+\|\phi\|_{C^\gamma})\le Ce^{2t}.
$$

Finally, by the Calderon-Zygmund inequality again, we have

$$
\|D^2\phi\|_{BMO}\le C\|\Delta\phi\|_{BMO}\le C\|\Delta(-e^{-2t}\Delta+1)^{-1}v\|_{BMO}.
\tag{3.35}
$$

We notice that the corresponding Fourier multiplier of the operator appearing the right-hand side of (3.35) is given by

$$
\frac{|\xi|^2}{e^{-2t}|\xi|^2+1}=\frac{e^{2t}|\xi|^{2-\gamma}}{|\xi|^2+e^{2t}}|\xi|^\gamma=e^{(2-\gamma)t}\frac{e^{\gamma t}|\xi|^{2-\gamma}}{|\xi|^2+e^{2t}}|\xi|^\gamma
$$

and the multiplier satisfies the condition so that the operator $e^{(\gamma-2)t}|\nabla|^{2-\gamma}(-e^{-2t}\Delta+1)^{-1}$ is bounded in BMO. Therefore

$$
\|D^2\phi\|_{BMO}\le Ce^{(2-\gamma)t}\||\nabla|^\gamma v\|_{BMO}.
$$

From the uniform Hölder estimate Proposition 3.6, $\||\nabla|^{\gamma} v\|_{BMO}$ is bounded uniformly in t. This enable us to proceed the estimate as

$$
\begin{aligned}
2e^{-\kappa t}\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x,v,\phi)\partial_i\partial_j\phi\,dx &\le 2e^{-\kappa t}\|D^2\phi(t)\|_\infty \int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx \\
&\le Ce^{-(\kappa-2+\gamma')t}\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx
\end{aligned} \tag{3.36}
$$

for some $\gamma' > 0$. Combining (3.34) and (3.36), we obtain that if $\kappa = 2$ i.e. $\alpha = 2 - \frac{2}{n}$,

$$
\begin{aligned}
\frac{d}{dt}I(v(t)) \le& -(2-\varepsilon)I(v(t)) \\
&+ C(1+t)\sup_t \|v(t)\|_\infty I(v(t)) + C\varepsilon^{-1}e^{-2t}\|v(t)\|_\infty^2 \sup_t(e^{-2t}\|\nabla\phi(t)\|_2^2).
\end{aligned} \tag{3.37}
$$

Note that at this stage, the inequality (3.37) does not include the higher order terms so that it is possible to justify it for the weak solution. Since $2(\kappa-2) = 0$ when $\alpha = 2 - \frac{2}{n}$, we can choose $\nu, \eta > 0$ such that for some large $T_0 > 0$, which depends on C, for any $t \ge T_0$,

$$
\frac{d}{dt}(e^{\nu t}I(v(t)) \le Ce^{-\eta t}. \tag{3.38}
$$

Immediately we obtain that

$$
I(v(t)) \le e^{-\nu t}\left(I(v(T_0)) + C\int_{T_0}^{\infty} e^{\eta\tau}\,d\tau\right).
$$

Since T_0 is only depending on C we may conclude that $I(v(t)) \le C(T_0)$ for $0 \le t \le T_0$ and this concludes the desired estimate. □

The proof of the asymptotic profile in Theorem 3.4 completes after proving the convergence of the rescaled solution and rescaling.

Proposition 3.15. *Let $1 < \alpha \le 2 - \frac{2}{n}$ and let (v, ϕ) be a weak solution to (3.14). If the initial data satisfies the condition (3.5), then we have for some $\nu > 0$ that*

$$
\|v(t) - \mathscr{V}\|_1 \le Ce^{-\nu t}
$$

where

$$
\mathscr{V}(x) = \left(A - \frac{\alpha-1}{2\alpha}|x|^2\right)_+^{\frac{1}{\alpha-1}}
$$

and the constant A is chosen as $\|\mathscr{V}\|_1 = \|u_0\|_1$.

Proof of Proposition 3.15. Due to the result from Proposition 3.13, we immediately obtain that

$$
\lim_{t\to\infty} I(v(t)) = 0 \tag{3.39}
$$

On the other hand, since by Proposition 3.7,

$$
\begin{aligned}
J(v(t)) &\le 2I(v(t)) + 2e^{-2\kappa t}\int_{\mathbb{R}^n} v(t)|\nabla\phi(t)|^2\,dx \\
&\le 2I(v(t)) + 2e^{-2\kappa t}\|v(t)\|_\infty\|\nabla\phi(t)\|_2^2 \\
&\le 2I(u_0)e^{-\nu t} + 2Ce^{-2(\kappa-1)t}.
\end{aligned}
$$

We conclude from (3.26) and Proposition 3.7 that for any $s < t$,

$$
\begin{aligned}
&\Big|H(v(t)) - H(v(s))\Big| \\
&\le |e^{-\kappa t}(e^{-2t}\|\nabla\phi(t)\|_2^2 + \|\phi(t)\|_2^2) - e^{-\kappa s}(e^{-2s}\|\nabla\phi(s)\|_2^2 + \|\phi(s)\|_2^2)| \\
&\quad + \int_s^t \left(\frac{\kappa-2}{2}e^{-(\kappa+2)\tau}\|\nabla\phi(\tau)\|_2^2 + \frac{\kappa}{2}e^{-\kappa\tau}\|\phi(\tau)\|_2^2 + J(v(\tau))\right) d\tau \\
&\quad + \int_s^t e^{-2\kappa\tau}\left(\int_{\mathbb{R}^n} v(\tau)|\nabla\phi(\tau)|^2\, dx\right) d\tau \\
&\le C(\kappa)e^{-\kappa s}\sup_{\tau>0}\big(e^{-2\tau}\|\nabla\phi(\tau)\|_2^2 + \|\phi(\tau)\|_2^2\big) + 2I(v(u_0))e^{-\nu s} + 2C(\kappa)e^{-2(\kappa-1)s} \\
&\quad + e^{-2(\kappa-1)s}\sup_{\tau>0}\big(e^{-2\tau}\|v(\tau)\|_\infty\|\nabla\phi(\tau)\|_2^2\big) \le Ce^{-\nu s} \to 0, \quad \text{as } s,t\to\infty
\end{aligned}
\tag{3.40}
$$

and this shows that $\{H(v(t_n))\}_n$ is the Cauchy sequence in $t_n \to \infty$. Besides the moment bound (3.19) in Proposition 3.9, $|x|^a v \in L^1(\mathbb{R}^n)$ for some $a > 2$. Therefore by the compactness $W^{1,1}(\mathbb{R}^n) \cap L^1_a(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$, we have a subsequence $v(t_n)$ such that it converges strongly to $\mathscr{V} \in L^\alpha(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$. The similar argument found in Carrillo-Toscani [**15**, Theorem 3.1] works for our case and we see that there exists a limit function $\mathscr{V}$ in $L^1_2(\mathbb{R}^n)$ such that

$$
v(t_n) \to \mathscr{V}, \quad t_n \to \infty
$$

in $L^1(\mathbb{R}^n)$. It turns out that the limit function is also non-negative and bounded. While by (3.39), the moment bound Proposition 3.9 and the natural regularity of the weak solution, we see that

$$
J(v(t)) \to J(\mathscr{V}) = \int_{\mathbb{R}^n} \mathscr{V}\left|\frac{\alpha}{\alpha-1}\nabla\mathscr{V}^{\alpha-1} + x\right|^2 dx = 0
$$

and we obtain either $\mathscr{V} = 0$ or $\nabla\mathscr{V}^{\alpha-1} = -\frac{\alpha-1}{\alpha}$ almost everywhere. This concludes by recalling $M = \|u_0\|_1$,

$$
\mathscr{V}(x) = \left[A - \frac{\alpha-1}{\alpha}|x|^2\right]_+^{\frac{1}{\alpha-1}},
$$

where A is chosen such that the L^1 norm of $\mathscr{V}(x)$ is normalized as 1. Again the estimate (3.26) in Proposition 3.10 and (3.40) gives

$$
|H(v(t)) - H(\mathscr{V})| \le Ce^{-\nu t} \tag{3.41}
$$

and that desired estimate follows from the argument in Carrillo-Toscani [**15**]. Namely we see firstly that

$$
\int_{\{v<\mathscr{V}\}} |v(t) - \mathscr{V}|\, dx \le \left(\frac{1}{\alpha}|H(\chi_B v(t)) - H(\mathscr{V})|\right)^{\frac{1}{2}}\left(\int_B \mathscr{V}(x)^{\frac{2}{n}}\, dx\right)^{\frac{1}{2}} \tag{3.42}
$$

by the special structure of the Barenblatt-Pattle solution, where $B = \operatorname{supp}\mathscr{V} = \{|x| \le \frac{2\alpha A}{\alpha-1}\}$ and χ_B is the characteristic function on B. While by $M = \|\mathscr{V}\|_1 = \|v(t)\|_1$ and $\mathscr{V} \ge 0$, we see

$$
\int_{\{v\ge\mathscr{V}\}} |v(t) - \mathscr{V}|\, dx = \int_{\{v<\mathscr{V}\}} |\mathscr{V} - v(t)|\, dx = \int_{\{v<\mathscr{V}\}} |v(t) - \mathscr{V}|\, dx. \tag{3.43}
$$

We note that over B^c, $\mathscr{V}$ is vanishing and by Carrillo-Toscani [**15**, Lemma 4.4]

$$\begin{aligned}&\frac{1}{\alpha-1}\int_{|x|^2>C} v^\alpha(t)\,dx + \frac{1}{2}\int_{|x|^2>C}(|x|^2-D)v(t)\,dx \leq |H(v(t)) - H(\mathscr{V})|,\\ &D\int_{|x|^2>C} v(t)\,dx \leq Ce^{-\gamma t}.\end{aligned} \tag{3.44}$$

Combining (3.42), (3.43) and (3.44) with (3.40) we conclude that

$$\|v(t) - \mathscr{V}\|_1 \leq C^{-\nu' t}$$

for some $\nu' > 0$. □

CHAPTER 4

Hölder continuity for solutions of the p-harmonic heat flow

1. The p-harmonic heat flow

We consider the following initial value problem of the p-harmonic heat flow:

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $p > 2$ is a constant, $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is unknown, $f : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $u_0 : \mathbb{R}^n \to \mathbb{R}$ are given external and initial data.

It is well-known that a classical solution of (4.1) does not generally exist. In fact, when $f \equiv 0$, for $\sigma = n(p-2) + p$ the Barenblatt solution

$$U(t,x) := (1+\sigma t)^{-\frac{n}{\sigma}} \left\{ 1 - \frac{p-2}{p} \left(\frac{|x|}{(1+\sigma t)^{\frac{1}{\sigma}}} \right)^{\frac{p}{p-1}} \right\}_+^{1+\frac{1}{p-2}}$$

satisfies (4.1) in the sense of distribution. Since the Barenblatt solution is not twice differentiable, a classical solution of (4.1) does not generally exist. Hence we introduce the notion of weak solutions.

Definition 4.1. For $u_0 \in L^1(\mathbb{R}^n)$ and $f \in L^1((0,\infty) \times \mathbb{R}^n)$, we call u a weak solution of (4.1) if there exists $T > 0$ such that

(1) $u \in L^\infty(0,T\,; L^2(\mathbb{R}^n))$ with $\nabla u \in L^p(0,T\,; L^p(\mathbb{R}^n))$; and
(2) u satisfies (4.1) in the sense of distribution, namely, for all $\varphi \in C^1(0,T\,; C_0^1(\mathbb{R}^n))$ and for almost all $0 < t < T$,

$$\begin{aligned} \int_{\mathbb{R}^n} u(t)\varphi(t)\,dx - \int_0^T \int_{\mathbb{R}^n} u\partial_t\varphi\,dtdx + \int_0^T \int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dtdx \\ = \int_{\mathbb{R}^n} u_0\varphi(0)\,dx - \int_0^T \int_{\mathbb{R}^n} f \cdot \nabla\varphi\,dtdx. \end{aligned}$$

For the existence of a weak solution is shown by Browder [**9**], Ladyženskaja-Solonnikov-Ural'ceva [**29**] (cf. Ôtani [**44**]). In this chapter, we study the Hölder continuity of ∇u, particularly, we show a relationship between the Hölder continuity of ∇u and the regularity of the external force f.

The Hölder continuity of ∇u was firstly shown by DiBenedetto-Friedman [**19, 20**] and Wiegner [**51**] when $f \equiv 0$. Misawa [**33**] showed the Hölder continuity of ∇u if f is locally Hölder continuous with respect to t and x. In this chapter, we give a weaker condition of the external force f for the Hölder continuity of ∇u than the condition given by Misawa.

This chapter is based on the paper [**36**].

Theorem 4.2. *Let u be a weak solution of* (4.1) *satisfying $\nabla u \in L^\infty((0,\infty)\times\mathbb{R}^n)$. Assume that for some constants $K>0$ and $\gamma_0 > n+2-\frac{p}{p-1}$, the external force f satisfies*

$$\iint_{Q_R(t_0,x_0)} |\nabla f|^{\frac{p}{p-1}}\, dtdx \le KR^{\gamma_0} \tag{4.2}$$

for all $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n$ and $0<R<1$ satisfying $Q_R(t_0,x_0)\subset(0,\infty)\times\mathbb{R}^n$. Then ∇u is Hölder continuous with exponent $\gamma>0$ depending only on n,p,γ_0. And for $\varepsilon>0$, there exists a constant $C>0$ depending only on $n,p,\gamma_0,K,\varepsilon$ such that for all $(t,x),(s,y)\in(\varepsilon,\infty)\times\mathbb{R}^n$,

$$|\nabla u(t,x)-\nabla u(s,y)| \le C(|t-s|^{\frac{\gamma}{2}}+|x-y|^\gamma).$$

Remark 4.3. The assumption (4.2) holds for $\nabla f\in L^r_{\mathrm{loc}}(0,\infty\,;L^q_{\mathrm{loc}}(\mathbb{R}^n))$ when $\frac{2}{r}+\frac{n}{q}<1$, $1\le r,q\le\infty$ and $n\ge 2$. In particular, if f is Hölder continuous in $(0,\infty)\times\mathbb{R}^n$, then $\nabla f\in L^r_{\mathrm{loc}}(0,\infty\,;L^q_{\mathrm{loc}}(\mathbb{R}^n))$ for some $r,q\ge1$ with $\frac{2}{r}+\frac{n}{q}<1$.

To prove Theorem 4.2, we show the decay estimate of the mean oscillation of ∇u by the perturbation argument. It is well-known that we obtain the Hölder continuity by the decay estimate of the mean oscillation (cf. Campanato [**14**]). Considering the time dependent mean oscillation of f and using the Poincare inequality, we treat f as a perturbation under the condition (4.2). If the external force f is locally Hölder continuous with respect to t and x, then we have (4.2) hence our results cover the results of Misawa.

2. Proof of the Hölder continuity

We show the decay estimate of the mean oscillation of ∇u at (t_0,x_0). By the scaling argument, we only consider $(t_0,x_0)\in(1,\infty)\times\mathbb{R}^n$ and we omit to denote the center of parabolic cylinders (t_0,x_0).

For $R<1$, we consider the following reference equation:

$$\begin{cases} \partial_t v - \mathrm{div}(|\nabla v|^{p-2}\nabla v) = 0, & (t,x)\in Q_R, \\ v = u, & (t,x)\in\partial_p Q_R. \end{cases} \tag{4.3}$$

For the existence of a solution of (4.3), we refer to Ladyženskaja-Solonnikov-Ural'ceva [**29**, Theorem 6.7 in p.466]

Lemma 4.4. *Let $\lambda=\lambda(t): I_R\to\mathbb{R}^n$. Then there exists a constant $C>0$ depending only on n,p such that*

$$\int_{B_R}(v(t_0)-u(t_0))^2\,dx + \iint_{Q_R}|\nabla v-\nabla u|^p\,dtdx \le C\iint_{Q_R}|f-\lambda(t)|^{\frac{p}{p-1}}\,dtdx.$$

Proof of Lemma 4.4. Subtracting (4.1) from (4.3), multiplying $(v-u)$ and integrating in Q_R, we obtain

$$\begin{aligned}\frac{1}{2}\iint_{Q_R}\partial_t(v-u)^2\,dtdx + \iint_{Q_R}\Big((|\nabla v|^{p-2}\nabla v-|\nabla u|^{p-2}\nabla u)\cdot(\nabla v-\nabla u)\Big)\,dtdx \\ = \iint_{Q_R}\Big((f-\lambda(t))\cdot(\nabla v-\nabla u)\Big)\,dtdx.\end{aligned}$$

Using Lemma B.1 and the Hölder inequality, we have

$$
\begin{aligned}
&\frac{1}{2}\iint_{Q_R} \partial_t (v-u)^2\, dtdx + C_0(n,p)\iint_{Q_R} |\nabla v - \nabla u|^p\, dtdx \\
&\le \iint_{Q_R} \Big((f-\lambda(t))\cdot(\nabla v - \nabla u)\Big)\, dtdx \\
&\le \left(\iint_{Q_R} |f-\lambda(t)|^{\frac{p}{p-1}}\, dtdx\right)^{1-\frac{1}{p}} \left(\iint_{Q_R} |\nabla v - \nabla u|^p\, dtdx\right)^{\frac{1}{p}} \\
&\le \frac{C_0(n,p)}{2}\iint_{Q_R} |\nabla v - \nabla u|^p\, dtdx + C_1(n,p)\iint_{Q_R} |f-\lambda(t)|^{\frac{p}{p-1}}\, dtdx.
\end{aligned}
$$

□

The following local maximum principle for (4.3) is given by DiBenedetto [**18**]

Lemma 4.5 (DiBenedetto [**18**, Theorem 5.1 in p.238]). *There exists a constant $C > 0$ depending only on n, p such that*

$$
\sup_{Q_{\frac{R}{2}}} |\nabla v| \le C\left(\frac{1}{|Q_R|}\iint_{Q_R} |\nabla v|^p\, dtdx\right)^{\frac{1}{2}}.
$$

Using Lemma 4.5, we obtain the following lemma:

Lemma 4.6. *Let $\lambda = \lambda(t) : I_R \to \mathbb{R}^n$. Then there exists a constant $C > 0$ depending only on n, p such that*

$$
\sup_{Q_{\frac{R}{2}}} |\nabla v| \le C\left\{\left(\frac{1}{|Q_R|}\iint_{Q_R} |f-\lambda(t)|^{\frac{p}{p-1}}\, dtdx\right)^{\frac{1}{2}} + M^{\frac{p}{2}}\right\}. \tag{4.4}
$$

In particular, $|\nabla v|$ is bounded on $Q_{\frac{R}{2}}$.

Proof of Lemma 4.6. By Lemma 4.4 and $|\nabla u| \le M$, we have

$$
\begin{aligned}
\iint_{Q_R} |\nabla v|^p\, dtdx &\le C(p)\left(\iint_{Q_R} |\nabla v - \nabla u|^p\, dtdx + \iint_{Q_R} |\nabla u|^p\, dtdx\right) \\
&\le C(p)\left(\iint_{Q_R} |f-\lambda(t)|^{\frac{p}{p-1}}\, dtdx + |Q_R| M^p\right).
\end{aligned}
$$

Using Lemma 4.5, we obtain (4.4). We let $\lambda(t) := (f(t))_{B_R}$, then by the Poincare inequality and (4.2), we have

$$
\begin{aligned}
\frac{1}{|Q_R|}\iint_{Q_R} |f-\lambda(t)|^{\frac{p}{p-1}}\, dtdx &\le C(n,p) R^{-n-2+\frac{p}{p-1}} \iint_{Q_R} |\nabla f|^{\frac{p}{p-1}}\, dtdx \\
&\le C(n,p) K R^{\gamma_0 - n - 2 + \frac{p}{p-1}}.
\end{aligned}
$$

Since $\gamma_0 > n + 2 - \frac{p}{p-1}$, we find that $|\nabla v|$ is bounded on $Q_{\frac{R}{2}}$. □

The following Hölder estimates of ∇v is shown by DiBenedetto [**18**].

LEMMA 4.7 (DiBenedetto [**18**, Theorem 1.1' in p.256]). *For $0 < \delta < 1$ and $R < 1$, there exist constants $\alpha_0 > 0$ and $C > 0$ depending only on n, p such that*

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \le C\|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} \left(\frac{R + R\max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}}{\operatorname{dist}_p(Q_{\frac{R}{2}}, \partial_p Q_{\frac{R^{1-\delta}}{2}})} \right)^{\alpha_0}.$$

Using Lemma 4.7 we show the following lemma:

LEMMA 4.8. *For $0 < \delta < 1$, there exists a constant $C > 0$ depending only on n, p such that,*

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \le 8R^\delta \|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} + C\|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} \left(1 + \max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\right)^{\alpha_0} R^{\delta\alpha_0}.$$

PROOF OF LEMMA 4.8. Either if $R^\delta \ge \frac{1}{4}$, then

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \le 2\|\nabla v\|_{L^\infty(Q_{\frac{R}{2}})} \le 8R^\delta \|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}.$$

Otherwise, if $0 < R^\delta < \frac{1}{4}$, then

$$\begin{aligned}
\operatorname{dist}(Q_{\frac{R}{2}}, \partial_p Q_{\frac{R^{1-\delta}}{2}}) &= \min\left\{ \left(\left(\frac{R^{1-\delta}}{2}\right)^2 - \left(\frac{R}{2}\right)^2 \right)^{\frac{1}{2}}, \left(\frac{R^{1-\delta}}{2} - \frac{R}{2} \right) \right\} \\
&= \frac{R^{1-\delta}}{2}(1 - R^\delta) \ge \frac{3}{8} R^{1-\delta}.
\end{aligned}$$

Therefore, by Lemma 4.7, we obtain

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \le C(p,n)\|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} \left(1 + \max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\right)^{\alpha_0} R^{\delta\alpha_0}.$$

□

PROOF OF THEOREM 4.2. Fix $0 < \rho < R < 1$. Then

(4.5)
$$\iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx \le C(p) \left(\iint_{Q_\rho} |\nabla u - \nabla v|^p \, dtdx + \iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx \right).$$

First, we estimate $\iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx$. Either if $0 < \rho \le \frac{R}{2}$, then by Lemma 4.8,

$$\begin{aligned}
&\iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx \\
&\le \iint_{Q_\rho} \operatorname{osc}(\nabla v)^p \, dtdx \\
&\le C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} R^{n+2+p\delta} \\
&\quad + C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} \left(1 + \max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\right)^{p\alpha_0} R^{n+2+p\delta\alpha_0}.
\end{aligned}$$

Otherwise, if $\frac{R}{2} \le \rho \le R$, then

$$\iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx \le 2^{n+2+p} \Big(\frac{\rho}{R}\Big)^{n+2+p} \iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx.$$

Since

$$\begin{aligned}
&\iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx \\
&\le C(p) \iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_R}|^p \, dtdx + C(p)|Q_\rho||(\nabla v)_{Q_\rho} - (\nabla v)_{Q_R}|^p \\
&\le C(p) \iint_{Q_R} |\nabla v - (\nabla v)_{Q_R}|^p \, dtdx + C(p) \iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_R}|^p \, dtdx \\
&\le C(p) \iint_{Q_R} |\nabla v - (\nabla v)_{Q_R}|^p \, dtdx,
\end{aligned}$$

we obtain

$$\iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx \le C(n,p) \Big(\frac{\rho}{R}\Big)^{n+2+p} \iint_{Q_R} |\nabla v - (\nabla v)_{Q_R}|^p \, dtdx.$$

Using the inequality

$$\begin{aligned}
&\iint_{Q_R} |\nabla v - (\nabla v)_{Q_R}|^p \, dtdx \\
&\qquad \le C(n,p) \iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx + C(n,p) \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dtdx,
\end{aligned}$$

we have

$$\begin{aligned}
&\iint_{Q_\rho} |\nabla v - (\nabla v)_{Q_\rho}|^p \, dtdx \\
&\le C(n,p) \left(\frac{\rho}{R}\right)^{n+2+p} \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dtdx + \|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} R^{n+2+p\delta} \\
&+ C(n,p) \|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} \Big(1 + \max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\Big)^{p\alpha_0} R^{n+2+p\delta\alpha_0} \\
&+ C(n,p) \iint_{Q_R} |\nabla u - \nabla v|^p \, dtdx.
\end{aligned}$$

We estimate $\iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx$. By Lemma 4.4, we have for $\lambda = \lambda(t) : I_R \to \mathbb{R}^n$,

$$\iint_{Q_R} |\nabla u - \nabla v|^p \, dtdx \le C(n,p) \iint_{Q_R} |f - \lambda(t)|^{\frac{p}{p-1}} \, dtdx.$$

Therefore, by (4.5), we obtain

$$\begin{aligned}\iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx &\le C(n,p)\left(\frac{\rho}{R}\right)^{n+2+p} \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dtdx \\ &\quad + C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} R^{n+2+p\delta} \\ + C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}&\Big(1+\max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\Big)^{p\alpha_0} R^{n+2+p\delta\alpha_0} \\ &\quad + C(n,p)\left(\left(\frac{\rho}{R}\right)^{n+2+p} + 1\right) \iint_{Q_R} |f-\lambda(t)|^{\frac{p}{p-1}} \, dtdx.\end{aligned}$$

We let $\lambda(t) := (f(t))_{B_R}$, then by the Poincare inequality and (4.2), we have

$$\iint_{Q_R} |f-\lambda(t)|^{\frac{p}{p-1}} \, dtdx \le C(n,p) R^{\frac{p}{p-1}} \iint_{Q_R} |\nabla f|^{\frac{p}{p-1}} \, dtdx \le C(n,p) K R^{\gamma_0 + \frac{p}{p-1}}.$$

Applying Lemma B.14 for $\phi(\rho) = \iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx$, we obtain

$$\iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx \le C\rho^{n+2+pn\gamma}$$

for some $\gamma > 0$. Since $\gamma_0 + \frac{p}{p-1} > n+2$, ∇u is Hölder continuous with exponent γ. $\square$

APPENDIX A

Harnack estimates for some nonlinear parabolic equation

1. Introduction and main result

We consider the following nonlinear parabolic equation:

$$
\begin{cases} \partial_t u - \Delta u + \dfrac{u}{\varepsilon}(|\nabla u|^2 - 1) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ \qquad\qquad\qquad\qquad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{A.1}
$$

where $u(t,x)$ is the unknown function, $u_0(x)$ is a given initial data and $\varepsilon > 0$ is a small parameter.

To compute the motion by mean curvature, Bence-Merriman-Osher [**5**] proposed a numerical algorithm which is called B-M-O algorithm, based on a simple procedure using a solution of heat equations. There are some mathematical justifications and extensions of the B-M-O algorithm given by Evans [**21**], Barles-Georgelin [**4**], H. Ishii [**25**] and H. Ishii-K. Ishii [**26**]. Considering the B-M-O algorithm, Goto-K. Ishii-Ogawa [**24**] introduced the singular limiting problem (A.1) of the nonlinear parabolic equation. Moreover, Goto-K. Ishii-Ogawa gave another proof of the convergence of the B-M-O algorithm and a solution u of the limiting problem (A.1) satisfies the level set equation of the motion by mean curvature:

$$
\partial_t u - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0. \tag{A.2}
$$

This problem (A.1) is similar to a singular limiting problem of the Allen-Cahn equation and the behavior of the solution of limiting problem (A.1) might be singular as $\varepsilon \to 0$. In general, it is difficult to obtain the regularity of the solution of the limiting problem (A.2). Besides, the regularity of the limiting problem (A.1) is related to a convergence of the B-M-O algorithm. Hence, it is important to study the regularity of the solution of (A.1) depending on the parameter $\varepsilon > 0$.

We note the existence of a solution of (A.1). Let $A_\varepsilon = \Delta + \frac{1}{\varepsilon}$ with a domain $D(A_\varepsilon) = H^2(\mathbb{R}^n)$ and e^{tA_ε} is a semigroup generated by A_ε on $\mathbb{R}^n$.

Definition A.1. We call $u = u(t,x)$ a mild solution of (A.1) if there exists $T > 0$ such that u satisfies the integral equation:

$$
u(t,x) = e^{tA_\varepsilon} u_0(x) - \frac{1}{\varepsilon} \int_0^t e^{(t-\tau)A_\varepsilon} u(\tau,x) |\nabla u(\tau,x)|^2 \, d\tau \tag{A.3}
$$

for all $0 < t < T$.

The existence of the mild solution of (A.1) is as follows.

This chapter is taken from the paper [**34**].

Proposition A.2. *Let $1 < p, r \le \infty$ be satisfying*

$$\frac{1}{p} + \frac{1}{r} < \frac{1}{n}, \quad \frac{1}{p} + \frac{2}{r} \le 1.$$

For any initial data $u_0 \in L^p(\mathbb{R}^n)$ with $\nabla u_0 \in L^r(\mathbb{R}^n)$, we take $T > 0$ enough small such that

$$0 < T^{1-\gamma}(\|u_0\|_{L^p(\mathbb{R}^n)} + \|\nabla u_0\|^2_{L^r(\mathbb{R}^n)}) \ll 1, \quad e^{\frac{3T}{\varepsilon}} < \frac{3}{2},$$

where $\gamma = \frac{n}{2}\left(\frac{1}{p} + \frac{1}{r}\right) + \frac{1}{2}$. Then, there exists a unique mild solution of (1.3) such that $u \in L^\infty(0, T\,; L^p(\mathbb{R}^n))$ and $\nabla u \in L^\infty(0, T\,; L^r(\mathbb{R}^n))$.

We will show the proof of Proposition A.2 in Section 3.

In Proposition A.2, we can obtain that the solution u is Hölder continuous in the spacial variable by the Sobolev embedding. Moreover, using the maximal regularity of heat equations, we find that the solution u is smooth in $(0, T)$. However it is not clear how the regularity of the solution depends on the parameter $\varepsilon > 0$.

To study the regularity, we consider the Hölder estimate of the solution of (A.1). It is well known that the Harnack inequality gives the interior Hölder continuity for solutions of parabolic equations. The Harnack constant, the constant in the Harnack inequality, is related to the Hölder exponent of the solution, hence we can regard that the Harnack constant has some information of regularity of solutions of (A.1). Now, we study explicit dependence on the parameter $\varepsilon > 0$ of the Harnack constant for nonnegative solutions of (A.1) and state our main theorem.

Theorem A.3 (The Harnack inequality). *Let u_ε be a nonnegative mild solution of (A.1) on $(0, 8T) \times B_{4R}$ and $0 < \varepsilon < 1$. Suppose that $0 \le u_\varepsilon \le M$ for some $M \ge 0$. Then we have*

$$\sup_{(T,2T)\times B_R} u_\varepsilon \le CM \exp\left(\frac{\theta}{\varepsilon}\right) \inf_{(7T,8T)\times B_R} u_\varepsilon,$$

where the constant C depends on n, T, R and the constant θ depends on n, M.

The basic strategy to prove theorem is to use the De Giorgi-Nash-Moser method. For linear parabolic equations, Moser [**37**] showed the Harnack inequality and it is well-known that his method may be extended to a nonlinear case. However we can not apply Moser's method directly since our equation has the strong nonlinearity and it is generally difficult to treat the equation by a perturbation method, whenever the parameter $\varepsilon > 0$ is small. To overcome this difficulty, we employ the Cole-Hopf transform. Formally by using the Cole-Hopf transform, the nonlinear equation (A.1) is transformed into some linear heat equation and hence Moser's method is applicable. Since we consider the mild solution, we need to justify the Cole-Hopf transform in the weak formulation. For this purpose, we modify Trudinger's argument [**49**] and we investigate the explicit dependence of the constant on ε.

Once we obtain the theorem, we obtain the Hölder continuity of solutions of (1.1) and the estimate of the Hölder exponent of solutions. Furthermore, our main theorem may be developed a finer analysis of the singular limiting problem (A.1) as $\varepsilon \to 0$. For instance, our theorem is connected with the regularity of the derivative of the solution of singular limiting problem (1.1). Moreover, by the regularity of the gradient of the solution, the interface of (A.1) make sense and we study the mean curvature flow and B-M-O algorithm more clear.

This chapter is organized as follows. In section 2, we show the local maximum principle, the weak Harnack inequality and we prove Theorem A.3. In section 3, we give the existence theorem of the initial value problem of (A.1).

2. Proof of the Harnack inequality

In this section, we consider the Harnack estimate of the solution of the problem (A.1) and investigate the dependence on the parameter $\varepsilon > 0$ of the Harnack constant.

To prove Theorem A.3, we show the local maximum principle, estimating the supremum of u by the L^p-norm of u, and show the weak Harnack inequality, estimating the L^p-norm of u by the infimum of u.

First, we give the local maximum principle:

PROPOSITION A.4 (the local maximum principle). *Let u_ε be a nonnegative mild solution of* (A.1) *on* $(0,T)\times B_R$. *Then, for all $p>1$, $0\le\tau<\tau'<T$, $0<R'<R$ and $0<\varepsilon<1$, we have*

$$\sup_{(\tau',T)\times B_{R'}} u_\varepsilon \le C\varepsilon^{-\frac{n+2}{2p}}\|u_\varepsilon\|_{L^p((\tau,T)\times B_R)},$$

where the constant C depends on n,p,τ',τ,R,R'.

REMARK A.5. We consider the following problem:

$$\partial_t v - \Delta v - v = 0, \quad (t,x)\in(0,T)\times B_R. \tag{A.4}$$

For a nonnegative subsolution v of (A.4) and for all $p>1$, $0\le\tau<\tau'<T$, $0<R'<R$, we may obtain

$$\sup_{(\tau',T)\times B_{R'}} v \le C\|v\|_{L^p((\tau,T)\times B_R)},$$

where the constant C depends on n,p,τ,τ',R,R'. We put

$$v_\varepsilon(t,x) := v\left(\frac{t}{\varepsilon},\frac{x}{\sqrt{\varepsilon}}\right),$$

then we have

$$\partial_t v_\varepsilon - \Delta v_\varepsilon - \frac{1}{\varepsilon}v_\varepsilon = 0\,, \quad (t,x)\in(0,\varepsilon T)\times B_{\sqrt{\varepsilon}R}.$$

By change of variable, we find

$$\sup_{(\varepsilon\tau',\varepsilon T)\times B_{\sqrt{\varepsilon}R'}} v_\varepsilon \le C\varepsilon^{-\frac{n+2}{2p}}\|v_\varepsilon\|_{L^p((\varepsilon\tau,\epsilon T)\times B_{\sqrt{\varepsilon}R})}.$$

Therefore the power of ε in Proposition A.4 naturally arises.

Second, we give the weak Harnack inequality:

PROPOSITION A.6 (the weak Harnack inequality). *Let u_ε be a nonnegative mild solution of* (A.1) *on* $(0,T)\times B_R$. *Suppose that $0\le u_\varepsilon\le M$ for some $M\ge 0$. Then, for all $p\ge 1$, $0<\tau\le\frac{T}{4}$ and $0<R'<R$, we have*

$$\|u_\varepsilon\|_{L^p((0,\tau)\times B_{R'})} \le CM\exp\left(\frac{\theta}{\varepsilon}\right)\inf_{(3\tau,4\tau)\times B_{R'}} u_\varepsilon,$$

where the constant C depends on n,p,τ,R',R and the constant θ depends on n,M.

Using the local maximum principle and the weak Harnack inequality, we obtain Theorem A.3.

2.1. Proof of the local maximum principle. Hereafter, we abbreviate the solution u_ε of (A.1) as u. Before proving Proposition A.4, we show the reverse Hölder inequality.

LEMMA A.7. *Let u be a nonnegative mild solution of* (A.1)*, Then for all $\beta > 0$, $0 < s < s' < T$, $0 < r' < r$ and $\varepsilon < 1$, we have the following reverse Hölder inequality:*

$$\text{(A.5)} \quad \|u\|^{\beta+1}_{L^{(1+\frac{2}{n})(\beta+1)}((s',T)\times B_{r'})} \\ \leq C\left(1+\frac{1}{\beta}\right)^2\left(\frac{1}{\varepsilon}(\beta+1)+\frac{1}{(r-r')^2}+\frac{1}{(s'-s)}\right)\|u\|^{\beta+1}_{L^{\beta+1}((s,T)\times B_r)},$$

where the constant C depends on n only.

PROOF OF LEMMA A.7. For simplicity, we treat the estimate for the classical solution. Set a cut-off function η satisfying

$$0 \leq \eta \leq 1, \quad \eta(t,x) = 1 \text{ on } (s',T)\times B_{r'}, \quad |\partial_t \eta| \leq \frac{4}{s'-s}, \quad |\nabla \eta| \leq \frac{4}{r-r'}.$$

Taking the test function $\eta^2 u^\beta$ in the equation of (A.1), integrating over $(s,t)\times B_r$ and neglecting the term $\frac{u}{\varepsilon}|\nabla u|^2$, we obtain

$$\frac{1}{\beta+1}\int_s^t\int_{B_r}\eta^2\partial_t(u^{\beta+1})\,d\tau dx + \beta\int_s^t\int_{B_r}\eta^2 u^{\beta-1}|\nabla u|^2\,d\tau dx \\ \leq -2\int_s^t\int_{B_r}u^\beta\eta\nabla\eta\cdot\nabla u\,d\tau dx + \frac{1}{\varepsilon}\int_s^t\int_{B_r}\eta^2u^{\beta+1}\,d\tau dx.$$

Using the Young inequality by the first integral of right-hand side, we have

$$\frac{1}{\beta+1}\int_{B_r}\eta^2(t)u^{\beta+1}(t)\,dx + \frac{2\beta}{(\beta+1)^2}\int_s^t\int_{B_r}\eta^2\left|\nabla\left(u^{\frac{\beta+1}{2}}\right)\right|^2 d\tau dx \\ \leq \frac{1}{\varepsilon}\int_s^T\int_{B_r}\eta^2u^{\beta+1}\,d\tau dx + \frac{2}{\beta+1}\int_s^T\int_{B_r}\eta|\partial_t\eta|u^{\beta+1}\,d\tau dx + \frac{2}{\beta}\int_s^T\int_{B_r}|\nabla\eta|^2u^{\beta+1}\,d\tau dx.$$

From this inequality, we obtain

$$\left\|\eta u^{\frac{\beta+1}{2}}\right\|^2_{L^\infty(s,T;L^2(B_r))} \\ \leq C\left\{\frac{1}{\varepsilon}(\beta+1)+\left(1+\frac{1}{\beta}\right)\frac{1}{(r-r')^2}+\frac{1}{s'-s}\right\}\left\|u^{\frac{\beta+1}{2}}\right\|^2_{L^2(s,T;L^2(B_r))}$$

and

$$\left\|\eta u^{\frac{\beta+1}{2}}\right\|^2_{L^2(s,T;H^1_0(B_r))} \\ \leq C\left\{\frac{\beta+1}{\varepsilon}\left(1+\frac{1}{\beta}\right)+\left(1+\frac{1}{\beta}\right)^2\frac{1}{(r-r')^2}+\left(1+\frac{1}{\beta}\right)\frac{1}{s'-s}\right\}\left\|u^{\frac{\beta+1}{2}}\right\|^2_{L^2(s,T;L^2(B_r))},$$

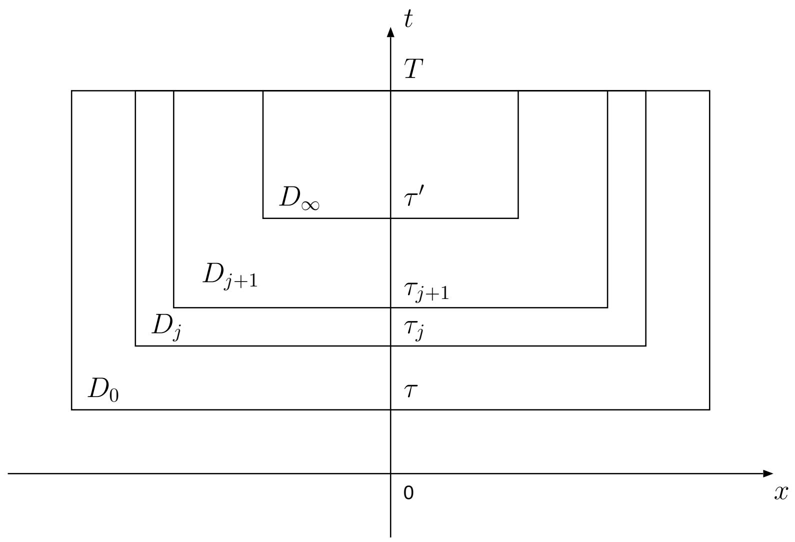

FIGURE A.1. Figure of D_j (We let $D_\infty := (\tau', T) \times B_{R'}$)

where C is the universal constant. Using the Ladyženskaja inequality (B.2), we have

$$\begin{aligned}\|u^{\frac{\beta+1}{2}}\|^2_{L^{2(1+\frac{2}{n})}((s',T)\times B_{r'})} &\le \|\eta u^{\frac{\beta+1}{2}}\|^2_{L^{2(1+\frac{2}{n})}((s',T)\times B_r)} \\ &\le C(n)\left(1+\frac{1}{\beta}\right)^2\left(\frac{1}{\varepsilon}(\beta+1)+\frac{1}{(r-r')^2}+\frac{1}{(s'-s)}\right)\|u^{\frac{\beta+1}{2}}\|^2_{L^2((s,T)\times B_r)}.\end{aligned}$$

This implies the inequality (A.5). $\square$

PROOF OF PROPOSITION A.4. For $j \in \mathbb{N}_0$, we put

$$\tau_j := \tau' - 2^{-j}(\tau'-\tau), \quad R_j := R' + 2^{-j}(R-R'),$$
$$\alpha_j := \left(1+\frac{2}{n}\right)^j, \qquad D_j := (\tau_j, T) \times B_{R_j}.$$

In the inequality (A.5), we set

$$\beta + 1 = p\alpha_j, \quad s' = \tau_{j+1}, \ s = \tau_j, \quad r' = R_{j+1}, \quad r = R_j,$$

then we obtain

$$\|u\|_{L^{p\alpha_{j+1}}(D_{j+1})} \le C(p,n)^{\frac{j}{\alpha_j}}\left(\frac{1}{\varepsilon}+\frac{1}{(\tau'-\tau)}+\frac{1}{(R-R')^2}\right)^{\frac{1}{p\alpha_j}}\|u\|_{L^{p\alpha_j}(D_j)}. \tag{A.6}$$

This inequality (A.6) asserts that if $\|u\|_{L^{p\alpha_j}(D_j)}$ is finite, then $\|u\|_{L^{p\alpha_{j+1}}(D_{j+1})}$ is also finite. Iterating this inequality (A.6), we find

$$\begin{aligned}\|u\|_{L^{p\alpha_{j+1}}((\tau',T)\times B_{R'})} &\le \|u\|_{L^{p\alpha_{j+1}}(D_{j+1})} \\ &\le \prod_{j=0}^{\infty}\left(C(p,n)^{\frac{j}{\alpha_j}}\left(\frac{1}{\varepsilon}+\frac{1}{(\tau'-\tau)}+\frac{1}{(R-R')^2}\right)^{\frac{1}{p\alpha_j}}\right)\|u\|_{L^p(D_0)} \\ &= C(p,n)^{\sum_{i=1}^{\infty}\frac{i}{\alpha_i}}\left(\frac{1}{\varepsilon}+\frac{1}{(\tau'-\tau)}+\frac{1}{(R-R')^2}\right)^{\frac{n+2}{2p}}\|u\|_{L^p((\tau,T)\times B_R)}.\end{aligned}$$

We remark that $\sum_{i=1}^{\infty} \frac{i}{\alpha_i}$ is finite. Taking $j \to \infty$, we have

$$\sup_{(\tau',T)\times B_{R'}} u \leq C(n,p)\varepsilon^{-\frac{n+2}{2p}} \left(1 + \frac{1}{\tau'-\tau} + \frac{1}{(R-R')^2}\right)^{\frac{n+2}{2p}} \|u\|_{L^p((\tau,T)\times B_R)}.$$

$\square$

REMARK A.8. In the proof of Proposition A.4 and Lemma A.7, we only consider the classical solution of (A.1). However using the Steklov average, we may extend our results for weak solutions of (A.1).

2.2. Proof of the weak Harnack inequality. First, as Lemma A.7, we show the reverse Hölder inequality.

LEMMA A.9. *Let u be a nonnegative mild solution of* (A.1). *Suppose that $0 \leq u \leq M$ for some $M \geq 0$. Then, for all $\beta < -1\,, 0 < s < s' < T$ and $0 < r' < r$, we have the following reverse Hölder inequality:*

$$\|u^{\beta+1}\|_{L^{(1+\frac{2}{n})}((s',T)\times B_{r'})} \leq Ce^{\frac{\theta}{\varepsilon}} \left(\frac{1}{s'-s} + \frac{1}{(r-r')^2}\right) \|u^{\beta+1}\|_{L^1((s,T)\times B_r)}, \tag{A.7}$$

where the constant C depends on n and the constant θ depends on M, β only.

LEMMA A.10. *Let u be a nonnegative mild solution of* (A.1). *Suppose that $0 \leq u \leq M$ for some $M \geq 0$. Then, for all $-1 < \beta < 0\,, 0 < s' < s < T$ and $0 < r' < r$, we have the following reverse Hölder inequality:*

$$\begin{aligned} &\|u^{\beta+1}\|_{L^{(1+\frac{2}{n})}((0,s')\times B_{r'})} \\ &\quad \leq Ce^{\frac{\theta}{\varepsilon}} \max\left\{1, \left|1+\frac{1}{\beta}\right|, \left|1+\frac{1}{\beta}\right|^2\right\} \left(\frac{1}{s-s'} + \frac{1}{(r-r')^2}\right) \|u^{\beta+1}\|_{L^1((0,s)\times B_r)}, \end{aligned} \tag{A.8}$$

where the constant C depends on n and the constant θ depends on M, β only.

Since their proofs are similar, we show these lemmas at the same time.

PROOF OF LEMMA A.9 AND LEMMA A.10. Set a cut-off function η satisfying $0 \leq \eta \leq 1$, and we require more condition for η later. We put $b_0 = \frac{M}{\varepsilon}$ for our convenience. Taking a test function $\eta^2 e^{-b_0 u} u^\beta$ in the equation of (A.1), integrating over $(t_0, t) \times B_r$ and neglecting the term $\frac{u}{\varepsilon}$, we obtain

$$\begin{aligned} &-\int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^\beta \partial_t u \, d\tau dx - \int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} (\beta u^{\beta-1} - b_0 u^\beta) |\nabla u|^2 \, d\tau dx \\ &\qquad \leq 2\int_{t_0}^t \int_{B_r} \eta e^{-b_0 u} u^\beta \nabla\eta \cdot \nabla u \, d\tau dx + b_0 \int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^\beta |\nabla u|^2 \, d\tau dx. \end{aligned}$$

Using the Young inequality, we have

$$\begin{aligned} &-\int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^\beta \partial_t u \, d\tau dx - \frac{\beta}{2} \int_{t_0}^t \int_{B_r} \eta^2 e^{-b_0 u} u^{\beta-1} |\nabla u|^2 \, d\tau dx \\ &\qquad \leq -\frac{2}{\beta} \int_{t_0}^t \int_{B_r} e^{-b_0 u} u^{\beta+1} |\nabla\eta|^2 \, d\tau dx. \end{aligned} \tag{A.9}$$

For $\beta \neq -1$, we set

$$f(u) := \begin{cases} (\beta+1)\displaystyle\int_0^u e^{-b_0 s} s^\beta \, ds, & \text{if } \beta > -1, \\ -(\beta+1)\displaystyle\int_u^\infty e^{-b_0 s} s^\beta \, ds, & \text{if } \beta < -1. \end{cases}$$

We remark that $\partial_t f(u) = (\beta+1)e^{-b_0 u} u^\beta \partial_t u$. Either if $\beta < -1$, by the integral by part, we have

$$\begin{aligned} f(u) &= -(\beta+1)u^{\beta+1} \int_1^\infty e^{-b_0 ur} r^\beta \, dr \qquad (s = ur) \\ &= b_0 u^{\beta+2} \int_1^\infty e^{-b_0 ur}(1 - r^{\beta+1}) \, dr \\ &\geq b_0 u^{\beta+2} \int_{2^{-\frac{1}{\beta+1}}}^\infty e^{-b_0 ur}(1 - r^{\beta+1}) \, dr \geq \frac{1}{2} u^{\beta+1} e^{-b_0 M 2^{-\frac{1}{\beta+1}}}. \end{aligned}$$

Otherwise, if $-1 < \beta < 0$, we have

$$\begin{aligned} f(u) &= (\beta+1)u^{\beta+1} \int_0^1 e^{-b_0 ur} r^\beta \, dr \qquad (s = ur) \\ &\geq e^{-b_0 M} u^{\beta+1} (\beta+1) \int_0^1 r^\beta \, dr = e^{-b_0 M} u^{\beta+1}. \end{aligned}$$

On the other hand, since $f(u) \leq u^{\beta+1}$, there exists $0 < \theta = \theta(M, \beta) \leq 1$ such that

$$\frac{1}{2} e^{-b_0 \theta(M,\beta)} u^{\beta+1} \leq f(u) \leq u^{\beta+1}. \tag{A.10}$$

We remark that

$$\begin{aligned} &\theta(M, \beta) \to \infty \quad \text{as } \beta \to -1, \\ &\theta(M, \beta) \to \theta(M, -\infty) < \infty \quad \text{as } \beta \to -\infty. \end{aligned}$$

From (A.9) we obtain

$$\begin{aligned} -\frac{1}{\beta+1} \int_{t_0}^t \int_{B_r} \partial_t(\eta^2 f(u)) \, d\tau dx - \frac{2\beta}{(\beta+1)^2} e^{-b_0 M} \int_{t_0}^t \int_{B_r} \eta^2 |\nabla u^{\frac{\beta+1}{2}}|^2 \, d\tau dx \\ \leq \frac{2}{|\beta|} \int_{t_0}^t \int_{B_r} u^{\beta+1} |\nabla \eta|^2 \, d\tau dx + \frac{2}{|\beta+1|} \int_{t_0}^t \int_{B_r} \eta |\partial_t \eta| u^{\beta+1} \, d\tau dx. \end{aligned} \tag{A.11}$$

We show the inequality (A.7) under the following additional condition

$$\eta(t, x) = 1 \text{ on } (s', T) \times B_{r'}, \quad |\partial_t \eta| \leq \frac{4}{s' - s}, \quad |\nabla \eta| \leq \frac{4}{r - r'}, \quad t_0 = s \tag{A.12}$$

to the cut-off function η. Applying the estimates (A.11) and (A.12) to (A.10), and noting that $-\frac{2\beta}{(\beta+1)^2} > 0$, we have

$$\|\eta u^{\frac{\beta+1}{2}}\|^2_{L^\infty(s,T;L^2(B_r))} \leq C e^{b_0 \theta} \left(\frac{1}{s' - s} + \frac{1}{(r - r')^2} \right) \|u^{\frac{\beta+1}{2}}\|^2_{L^2((s,T) \times B_r)}$$

and

$$\|\eta u^{\frac{\beta+1}{2}}\|^2_{L^2(s,T;H^1_0(B_r))} \leq C e^{b_0 \theta} \left(\frac{1}{s' - s} + \frac{1}{(r - r')^2} \right) \|u^{\frac{\beta+1}{2}}\|^2_{L^2((s,T) \times B_r)}.$$

Using the Ladyženskaja inequality (B.2), we obtain

$$
\begin{aligned}
\|u^{\frac{\beta+1}{2}}\|^2_{L^{2(1+\frac{2}{n})}((s',T)\times B_{r'})} &\le \|\eta u^{\frac{\beta+1}{2}}\|^2_{L^{2(1+\frac{2}{n})}((s,T)\times B_r)} \\
&\le C(n)e^{b_0\theta}\left(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\right)\|u^{\frac{\beta+1}{2}}\|^2_{L^2((s,T)\times B_r)},
\end{aligned}
$$

and this implies (A.7).

Next, we show the inequality (A.8). We assume further condition on the test function η as

$$
\eta(t,x)=1 \text{ on } (0,s')\times B_{r'}, \quad |\partial_t\eta|\le\frac{4}{s-s'}, \quad |\nabla\eta|\le\frac{4}{r-r'}, \quad t_0=0.
$$

Then it follows from (A.11) that

$$
\|\eta u^{\frac{\beta+1}{2}}\|^2_{L^\infty(0,s\,;L^2(B_r))}\le Ce^{b_0\theta}\max\left\{1,\left|1+\frac{1}{\beta}\right|\right\}\left(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\right)\|u^{\frac{\beta+1}{2}}\|^2_{L^2((0,s)\times B_r)}
$$

and

$$
\begin{aligned}
&\|\eta u^{\frac{\beta+1}{2}}\|^2_{L^2(0,s\,;H^1_0(B_r))} \\
&\qquad\le Ce^{b_0\theta}\max\left\{1,\left|1+\frac{1}{\beta}\right|,\left|1+\frac{1}{\beta}\right|^2\right\}\left(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\right)\|u^{\frac{\beta+1}{2}}\|^2_{L^2((0,s)\times B_r)}.
\end{aligned}
$$

Using the Ladyženskaja inequality (B.2), we obtain

$$
\begin{aligned}
\|u^{\frac{\beta+1}{2}}\|^2_{L^{2(1+\frac{2}{n})}((0,s')\times B_{r'})} &\le \|\eta u^{\frac{\beta+1}{2}}\|^2_{L^{2(1+\frac{2}{n})}((0,s)\times B_r)} \\
&\le Ce^{b_0\theta}\max\left\{1,\left|1+\frac{1}{\beta}\right|,\left|1+\frac{1}{\beta}\right|^2\right\}\left(\frac{1}{s'-s}+\frac{1}{(r-r')^2}\right)\|u^{\frac{\beta+1}{2}}\|^2_{L^2((0,s)\times B_r)},
\end{aligned}
$$

and this implies (A.8). □

Remark A.11. Introducing the Cole-Hopf transform $v=e^{-\frac{M}{\varepsilon}u}$, we find that v is a subsolution of the linear heat equation under the assumption that u is the classical solution of (A.1). We may regard that the test function $\phi=\eta^2e^{-b_0u}u^\beta$ as the justification of the Cole-Hopf transform for weak formulations. The original idea to cancel out the nonlinear term may be go-back to Aronson-Serrin [**1**] and Trudinger [**49**].

Lemma A.12. *Let u be a nonnegative mild solution in $(0,T)\times B_R$ with $0\le u\le M$. Then, for all $q>0$, $0\le\tau<\tau'<T$ and $0<R'<R$, we have*

$$
\inf_{(\tau',T)\times B_{R'}} u\ge C\exp\left(\frac{-M\theta(n+2)}{2q\varepsilon}\right)\left(\int_\tau^T\int_{B_R}u^{-q}\,dtdx\right)^{-\frac{1}{q}} \tag{A.13}
$$

where the constant C depends on $n,q,\tau'-\tau,R-R'$ and the constant θ depends on M,q.

Proof of Lemma A.12. For $j\in\mathbb{N}_0$, we put

$$
\begin{aligned}
\tau_j&=(1-2^{-j})(\tau'-\tau)\,, &\quad r_j&=R'+2^{-j}(R-R')\,,\\
\alpha_j&=\left(1+\frac{2}{n}\right)^j, &\quad D_j&=(0,\tau_j)\times B_{R_j}.
\end{aligned}
$$

In the inequality (A.7), we set

$$\beta+1=p\alpha_j,\quad s=\tau_j,\quad s'=\tau_{j+1},\quad r'=R_{j+1},\quad r=R_j.$$

Then we obtain

$$\|u^{-q}\|_{L^{\alpha_{j+1}}(D_{j+1})}\le\left\{C(n,q)e^{\frac{M}{\varepsilon}\theta}\left(\frac{1}{\tau'-\tau}+\frac{1}{(R-R')^2}\right)\right\}^{\frac{1}{\alpha_j}}2^{\frac{2j+2}{\alpha_j}}\|u^{-q}\|_{L^{\alpha_j}(D_j)}.$$

Iterating this inequality, we find

$$\sup_{(\tau',T)\times B_{R'}}u^{-q}\le C(n,q,\tau-\tau',R-R')e^{\frac{M\theta(n+2)}{2\varepsilon}}\|u^{-q}\|_{L^1(D_0)}$$

Taking the $-\frac{1}{q}$-th power, we obtain (A.13). □

Almost the same argument, we obtain the following lemma:

LEMMA A.13. *Let u be a nonnegative mild solution in $(0,T)\times B_R$ with $0\le u\le M$. Then, for all $0<q<1\le p$, $0<\tau'<\tau\le T$ and $0<R'<R$, we have*

$$\|u\|_{L^p((0,\tau')\times B_{R'})}\le C\exp\left(\frac{M\theta(n+2)}{2q\varepsilon}\right)\left(\int_0^\tau\int_{B_R}u^q\,dtdx\right)^{\frac{1}{q}}$$

where the constant C depends on $n,q,\tau-\tau',R-R'$ and the constant θ depends on M,q.

Next, we consider the case $\beta=-1$ in the proof of the Lemma A.9 and Lemma A.10.

LEMMA A.14. *Let u be a nonnegative mild solution of (A.1) in $(0,T)\times K_r$. Suppose that $0\le u\le M$ for some $M\ge 0$. Then there exist $C,p_0>0$ such that*

$$\left(\iint_{(0,\frac{1}{8}T)\times K_{\frac{r}{2}}}u^{p_0}\,dtdx\right)^{\frac{1}{p_0}}\le CM\exp\left(\int_0^M\frac{1-e^{-\frac{M}{\varepsilon}s}}{s}\,ds\right)\left(\iint_{(\frac{7}{8}T,T)\times K_{\frac{r}{2}}}u^{-p_0}\,dtdx\right)^{-\frac{1}{p_0}},$$

where the constant C depends on n,T,r only and the constant p_0 depends on n only.

PROOF OF LEMMA A.14. We put $t>0$, $h\in\mathbb{R}$, $\beta=-1$, $t_0=t$ and $t=t+h$ in the inequality (A.9). Replacing B_r with $K_r:=\{x:=(x_i)_i\in\mathbb{R}^n:\max_{1\le i\le n}|x_i|<r\}$, we find

$$\text{(A.14)}\quad -\int_t^{t+h}\int_{K_r}\eta^2e^{-b_0u}u^{-1}\partial_tu\,d\tau dx+\frac{1}{2}\int_t^{t+h}\int_{K_r}\eta^2e^{-b_0u}u^{-2}|\nabla u|^2\,d\tau dx\le 2\int_t^{t+h}\int_{K_r}e^{-b_0u}|\nabla\eta|^2\,d\tau dx.$$

Letting

$$f(u):=-\int_1^u e^{-b_0s}s^{-1}\,ds,$$

then by $\partial_tf(u)=-e^{-b_0u}u^{-1}\partial_tu$ and $\nabla f(u)=-e^{-b_0u}u^{-1}\nabla u$, we see from (A.14) that

$$\int_t^{t+h}\int_{K_r}\eta^2\partial_tf(u)\,d\tau dx+\frac{1}{2}\int_t^{t+h}\int_{K_r}\eta^2e^{b_0u}|\nabla f(u)|^2\,d\tau dx\le 2\int_t^{t+h}\int_{K_r}e^{-b_0u}|\nabla\eta|^2\,d\tau dx.$$

We freeze $\rho > 0$ and $x_0 \in K_r$ so that $K_\rho(x_0) \subset K_r$. We select a cut-off function η such that

$$\begin{aligned}
&\eta = \eta(x) = 1, \quad x \in K_{\frac{\rho}{2}}(x_0)\,, \\
&\operatorname{supp}\,\eta \subset\subset K_\rho(x_0)\,, \\
&0 \le \eta \le 1\,,\ |\nabla \eta| \le \frac{4}{\rho}, \\
&\{x \in \mathbb{R}^n : \eta(x) \ge \lambda\} \text{ is convex for all } \lambda \ge 0.
\end{aligned}$$

Then we obtain

$$\int_{K_\rho(x_0)} \eta^2 f(u)\,dx\Big|_t^{t+h} + \frac{1}{2}\int_t^{t+h}\int_{K_\rho(x_0)} \eta^2 |\nabla f(u)|^2\,d\tau dx \le Ch\rho^{n-2},$$

where the constant C depends on n only.

Applying Lemma B.6 by $g = f(u)$, $\mu = \eta^2$ and $D = K_\rho(x_0)$, we find

$$\int_{K_\rho(x_0)} \eta^2 f(u)\,dx\Big|_t^{t+h} + C_1\frac{\int_{K_\rho(x_0)}\eta^2 dx}{\rho^{n+2}}\int_t^{t+h}\int_{K_\rho(x_0)} (f(u)-V(\tau))^2\eta^2\,d\tau dx \le Ch\rho^{n-2},$$

where C_1 is the constant depending on n and

$$V(\tau) := \frac{\int_{K_\rho(x_0)} f(u(\tau,x))\eta^2\,dx}{\int_{K_\rho(x_0)}\eta^2\,dx}.$$

Dividing by $h\int_{K_\rho(x_0)}\eta^2\,dx$ and letting $h \to 0$, we obtain

$$\frac{dV}{dt} + \frac{C_1}{\rho^{n+2}}\int_{K_{\frac{\rho}{2}}(x_0)} (f(u)-V(t))^2\,dx \le \frac{C\rho^{n-2}}{\int_{K_\rho(x_0)}\eta^2\,dx} \le C_2\rho^{-2}, \quad \text{a.a. } 0 < t < T$$

where the constant C_2 depends on n only. We put $0 < t_0 < T$ such that $0 < t_0 - \frac{\rho^2}{4} < t_0 + \frac{\rho^2}{4} < T$ and set

$$\begin{aligned}
w_1(t,x) &= f(u) - V(t_0) - C_2\rho^{-2}(t-t_0), \\
W_1(t) &= V(t) - V(t_0) - C_2\rho^{-2}(t-t_0).
\end{aligned}$$

Then

$$\text{(A.15)} \qquad \begin{cases} \dfrac{dW_1}{dt} + \dfrac{C_1}{\rho^{n+2}}\displaystyle\int_{K_{\frac{\rho}{2}}(x_0)} (w_1 - W_1)^2\,dx \le 0, \\ \hfill W_1(t_0) = 0. \end{cases}$$

For $s > 0$, we put

$$Q_{\rho,s}(t) := \{x \in K_\rho(x_0) : w_1(t,x) > s\}.$$

Since $W_1(t) \le 0$ for $t_0 \le t \le t_0 + \frac{\rho^2}{4}$ by (A.15), we have

$$w_1 - W_1 \ge s - W_1 > 0, \quad t \ge t_0\,,\ x \in Q_{\frac{\rho}{2},s}(t)$$

hence

$$\frac{dW_1}{dt} + \frac{C_1}{\rho^{n+2}}(s - W_1(t))^2|Q_{\frac{\rho}{2},s}(t)| \le 0.$$

Therefore

$$\frac{|Q_{\frac{\varrho}{2},s}(t)|}{\rho^{n+2}} \le C_1^{-1}(s - W_1(t))^{-2}\frac{d(s - W_1)}{dt} = C_1^{-1}\frac{d}{dt}\{-(s - W_1(t))^{-1}\}.$$

Integrating over $(t_0, t_0 + \frac{\rho^2}{4})$, we find

$$\frac{1}{\rho^{n+2}}\int_{t_0}^{t_0+\frac{\rho^2}{4}} |Q_{\frac{\varrho}{2},s}(t)|\,dt \le C_1^{-1}\left\{\frac{1}{s - W_1(t_0)} - \frac{1}{s - W_1(t_0 + \frac{\rho^2}{4})}\right\} \le \frac{1}{C_1 s}.$$

We set $U_+ := (t_0, t_0 + \frac{\rho^2}{4}) \times K_{\frac{\varrho}{2}}(x_0)$, then

$$\begin{aligned}
&\frac{1}{|U_+|}\iint_{U_+} \sqrt{(f(u) - V(t_0))_+}\,dtdx \\
&= \frac{1}{|U_+|}\iint_{U_+} \sqrt{(w_1(t,x) + C_2\rho^{-2}(t - t_0))_+}\,dtdx \\
&\le \frac{1}{|U_+|}\left(\iint_{U_+} \sqrt{w_1(t,x)_+}\,dtdx + \iint_{U_+} \sqrt{C_2\rho^{-2}(t - t_0)}\,dtdx\right) \\
&\le \frac{1}{|U_+|}\left(\frac{1}{2}\int_{t_0}^{t_0+\frac{\rho^2}{4}}\left(\int_0^\infty s^{-\frac{1}{2}}|Q_{\frac{\varrho}{2},s}(t)|\,ds\right)dt + \sqrt{\frac{C_2}{4}}|U_+|\right).
\end{aligned} \tag{A.16}$$

Here we write

$$\begin{aligned}
&\int_{t_0}^{t_0+\frac{\rho^2}{4}}\left(\int_0^\infty s^{-\frac{1}{2}}|Q_{\frac{\varrho}{2},s}(t)|\,ds\right)dt \\
&= \int_{t_0}^{t_0+\frac{\rho^2}{4}}\left(\int_0^1 s^{-\frac{1}{2}}|Q_{\frac{\varrho}{2},s}(t)|\,ds + \int_1^\infty s^{-\frac{1}{2}}|Q_{\frac{\varrho}{2},s}(t)|\,ds\right)dt \\
&=: I_1 + I_2.
\end{aligned}$$

Using the following estimates

$$I_1 \le \int_{t_0}^{t_0+\frac{\rho^2}{4}}\left(\int_0^1 s^{-\frac{1}{2}}|K_{\frac{\varrho}{2}}|\,ds\right)dt = 2|U_+|\,,$$

$$I_2 \le \int_1^\infty s^{-\frac{1}{2}}\left(\int_{t_0}^{t_0+\frac{\rho^2}{4}}|Q_{\frac{\varrho}{2},s}(t)|\,dt\right)ds \le \int_1^\infty s^{-\frac{1}{2}}\frac{\rho^{n+2}}{C_1 s}\,ds = \frac{8}{C_1}|U_+|,$$

we obtain

$$\frac{1}{|U_+|}\iint_{U_+} \sqrt{(f(u) - V(t_0))_+}\,dtdx \le C,$$

where C is the constant depending on n only.

We set $U_- = (t_0 - \frac{\rho^2}{4}, \tau) \times K_{\frac{\varrho}{2}}(x_0)$ and by the same argument, we have

$$\frac{1}{|U_-|}\iint_{U_-} \sqrt{(V(t_0) - f(u))_+}\,dtdx \le C.$$

Consequently, for $0 < t_0 < T$, $x_0 \in K_r$ and $\rho > 0$ with $(t_0 - \frac{\rho^2}{4}, t_0 + \frac{\rho^2}{4}) \times K_{\frac{\rho}{2}}(x_0) \subset (0,T) \times K_r$ we have

$$\frac{1}{|U_+|} \iint_{U_+} \sqrt{(f(u) - V(t_0))_+}\, dtdx \le C,$$
$$\frac{1}{|U_-|} \iint_{U_-} \sqrt{(V(t_0) - f(u))_+}\, dtdx \le C.$$

By the parabolic John-Nirenberg estimate, Lemma B.8, we have

$$\left(\iint_{(0,\frac{1}{8}T) \times K_{\frac{r}{2}}} e^{-p_0 f(u)}\, dtdx \right) \left(\iint_{(\frac{7}{8}T,T) \times K_{\frac{r}{2}}} e^{-p_0 f(u)}\, dtdx \right) \le C. \tag{A.17}$$

Now, we give the following lemma.

Lemma A.15. *Let*

$$A = \exp\left(- \int_1^M \frac{1 - e^{-b_0 s}}{s}\, ds \right), \quad B = \exp\left(\int_0^1 \frac{1 - e^{-b_0 s}}{s}\, ds \right).$$

Then we have

$$-\log B\xi \le f(\xi) \le -\log A\xi$$

for all $0 < \xi \le M$.

Proof of Lemma A.15. We show that

$$F_1(\xi) := -\log A\xi - f(\xi) \ge 0$$

for all $0 < \xi \le M$. By differentiating F_1, we have

$$F_1'(\xi) := -\frac{1}{\xi} + \frac{e^{-b_0 \xi}}{\xi} \le 0.$$

Therefore $F_1(\xi) \ge F_1(M)$ for $0 < \xi \le M$. Since

$$F_1(M) = -\log A - \int_1^M \frac{1 - e^{-b_0 s}}{s}\, ds,$$

we have $F_1(M) = 0$ if and only if $A = \exp\left(- \int_1^M \frac{1-e^{-b_0 s}}{s}\, ds \right)$ and hence $F_1(\xi) \ge 0$ for all $0 < \xi \le M$.

As the similar argument, we obtain $-\log B\xi \le f(\xi)$ for all $0 < \xi \le M$. □

By Lemma A.15 and the estimate (A.17), we have

$$\left(\iint_{(0,\frac{1}{8}T) \times K_{\frac{r}{2}}} e^{p_0 \log Au}\, dtdx \right) \left(\iint_{(\frac{7}{8}T,T) \times K_{\frac{r}{2}}} e^{-p_0 \log Bu}\, dtdx \right) \le C,$$

or

$$\left(\iint_{(0,\frac{1}{8}T) \times K_{\frac{r}{2}}} u^{p_0}\, dtdx \right)^{\frac{1}{p_0}} \le C \frac{B}{A} \left(\iint_{(\frac{7}{8}T,T) \times K_{\frac{r}{2}}} u^{-p_0}\, dtdx \right)^{-\frac{1}{p_0}}.$$

□

Using Lemma A.12, A.13 and A.14, we obtain Proposition A.6.

3. Existence of a mild solution

We show Proposition A.2, namely the existence of the mild solution of the following initial value problem:

$$
\text{(A.18)} \qquad \begin{cases} \partial_t u - \Delta u + \dfrac{u}{\varepsilon}(|\nabla u|^2 - 1) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^n, \\ \qquad\qquad u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases}
$$

To prove Proposition A.2, we give key estimates.

Lemma A.16. *Let $1 \le q \le p \le \infty$. Then for all $\phi \in L^q(\mathbb{R}^n)$ we have*

$$
\|e^{tA_\varepsilon}\phi\|_p \le C_1 e^{\frac{t}{\varepsilon}} t^{-\gamma}\|\phi\|_q,
$$
$$
\|\nabla e^{tA_\varepsilon}\phi\|_p \le C_2 e^{\frac{t}{\varepsilon}} t^{-(\gamma+\frac{1}{2})}\|\phi\|_q,
$$

where

$$
\gamma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right).
$$

and C_1, C_2 are constants depending on p, q, n only.

Using the L^p-L^q estimate for $e^{t\Delta}$, we obtain Lemma A.16. In Lemma A.16, we may take

$$
C_1 = (4\pi)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}, \quad C_2 = C_0 4^{-\gamma}\left(|\mathbb{S}^{n-1}|\Gamma\left(\frac{n(n-2\gamma+1)}{2n(n-2\gamma)}\right)\right)^{1-\frac{2\gamma}{n}},
$$

where the constant C_0 depends on n only, $|\mathbb{S}^{n-1}|$ is the area of the $(n-1)$-dimensional unit sphere and Γ is the gamma function, namely

$$
\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt.
$$

In this section, the constants C_1, C_2 are as in Lemma A.16. To construct the contraction mapping, we set the following function spaces.

Definition A.17. Let $1 \le p, r \le \infty$, $T, M > 0$. We define

$$
\begin{aligned}
X_M(T) = X_{M,p,r}(T) := \{u \in C([0,T]\,; L^p(\mathbb{R}^n)) : \nabla u \in C([0,T]\,; L^r(\mathbb{R}^n))\,, \\
\|u\|_{X_M} := \|u\|_{C([0,T]\,;L^p(\mathbb{R}^n))} + \|\nabla u\|_{C([0,T]\,;L^r(\mathbb{R}^n))} \le M\}.
\end{aligned}
$$

We define the distance of $X_M(T)$ by

$$
d(u,v) := \|u - v\|_{C([0,T]\,;L^p(\mathbb{R}^n))} + \|\nabla(u-v)\|_{C([0,T]\,;L^r(\mathbb{R}^n))}.
$$

We denote the homogeneous Sobolev space by $\dot{W}^{1,q}(\mathbb{R}^n)$. Since $X_M(T)$ is closed in $C([0,T]\,; L^p(\mathbb{R}^n)) \cap C([0,T]\,; \dot{W}^{1,q}(\mathbb{R}^n))$ and $C([0,T]\,; L^p(\mathbb{R}^n)) \cap C([0,T]\,; \dot{W}^{1,q}(\mathbb{R}^n))$ is complete, $X_M(T)$ is a complete metric space.

3.1. Estimate of perturbation.

Definition A.18. Using e^{tA_ε}, we define

$$
\text{(A.19)} \qquad \Phi(u) := e^{tA_\varepsilon}u_0 - \frac{1}{\varepsilon}\int_0^t e^{(t-\tau)A_\varepsilon}u(\tau)|\nabla u(\tau)|^2\,d\tau
$$

for $u \in X_M(T)$.

We show the existence of a fixed point for Φ. First, we take $T > 0$ such that we define Φ on $X_M(T)$.

LEMMA A.19. *Let $1 \leq p, q \leq \infty$ be satisfying*

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{n}, \quad \frac{1}{p} + \frac{2}{q} \leq 1,$$

and let M, γ be

$$M := 2(\|u_0\|_p + \|\nabla u_0\|_q), \quad \gamma = \frac{n}{2}\left(\frac{1}{p} + \frac{1}{q}\right) + \frac{1}{2}.$$

Let $0 < T_0 < 1$ be small enough such that

$$CT_0^{1-\gamma}M^2 \ll 1, \quad e^{\frac{T_0}{\varepsilon}} < \frac{3}{2},$$

where C is the constant depending on n, p, q, ε only. Then $\Phi(u) \in X_M(T)$ for all $T < T_0$ and $u \in X_M(T)$.

REMARK A.20. We can take T_0 explicitly so that

$$e^{\frac{T_0}{\varepsilon}} \leq \frac{3}{2}, \quad \frac{1}{\varepsilon}\left(\frac{C_1 r T_0^{1-\frac{n}{q}}}{r-n} + \frac{C_2 T_0^{1-\gamma}}{1-\gamma}\right) \leq \frac{1}{4M^2}. \tag{A.20}$$

PROOF OF LEMMA A.19. First, we consider the estimate of $\|\Phi(u)\|_{C([0,T]\,;L^p(\mathbb{R}^n))}$. We put $r \geq 1$ as $\frac{1}{r} = \frac{1}{p} + \frac{2}{q}$. By Lemma A.16, we have

$$\begin{aligned} \|\Phi(u(t))\|_p &\leq \|e^{tA}u_0\|_p + \frac{1}{\varepsilon}\int_0^t \|e^{(t-\tau)A}u(\tau)|\nabla u(\tau)|^2\|_p\,d\tau \\ &\leq \|e^{tA}u_0\|_p + \frac{C_1}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\frac{n}{q}}\|u(\tau)|\nabla u(\tau)|^2\|_r\,d\tau. \end{aligned} \tag{A.21}$$

Using the Hölder inequality, we have $\|u(\tau)|\nabla u(\tau)|^2\|_r \leq \|u(\tau)\|_p\|\nabla u(\tau)\|_q^2$ hence

$$\|\Phi(u(\tau))\|_p \leq \|e^{tA}u_0\|_p + \frac{C_1}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\frac{n}{q}}\|u(\tau)\|_p\|\nabla u(\tau)\|_q^2\,d\tau.$$

We remark $q > n$ since $\frac{1}{n} > \frac{1}{q} + \frac{1}{p}$. Therefore taking a supremum for t in (A.21), we find

$$\begin{aligned} \sup_{0\leq t\leq T}\|\Phi(u(t))\|_p &\leq e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}\sup_{0\leq t\leq T}\int_0^t (t-\tau)^{-\frac{n}{q}}\|u(\tau)\|_p\|\nabla u(\tau)\|_q^2\,d\tau \\ &\leq e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}\sup_{0\leq t\leq T}\|u(t)\|_p\sup_{0\leq t\leq T}\|\nabla u(t)\|_q^2\sup_{0\leq t\leq T}\int_0^t (t-\tau)^{-\frac{n}{q}}\,d\tau \\ &\leq e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}M^3\sup_{0\leq t\leq T}\int_0^t (t-\tau)^{-\frac{n}{q}}\,d\tau. \end{aligned}$$

Since

$$\int_0^t (t-\tau)^{-\frac{n}{q}}\,d\tau = \frac{q}{q-n}\left[-(t-\tau)^{-\frac{n}{q}+1}\right]_0^t = \frac{q}{q-n}t^{1-\frac{n}{q}},$$

we obtain

$$\sup_{0\leq t\leq T}\|\Phi(u(t))\|_p \leq e^{\frac{T}{\varepsilon}}\|u_0\|_p + \frac{C_1}{\varepsilon}e^{\frac{T}{\varepsilon}}M^3\frac{qT^{1-\frac{n}{q}}}{q-n}.$$

Next, we consider $\|\nabla\Phi(u)\|_{C([0,T]\,;L^q(\mathbb{R}^n))}$. Differentiating (A.19), we can write

$$\nabla\Phi(u(t)) = \nabla e^{tA}u_0 - \frac{1}{\varepsilon}\int_0^t e^{(t-\tau)A}\nabla(u(\tau)|\nabla u(\tau)|^2)\,d\tau.$$

Considering the L^p-L^q estimate of the derivative in Lemma A.16, we find

$$\begin{aligned}\|\nabla\Phi(u(t))\|_q &\le \|e^{tA}\nabla u_0\|_q + \frac{1}{\varepsilon}\int_0^t \left\|\nabla e^{(t-\tau)A}\big(u(\tau)|\nabla u(\tau)|^2\big)\right\|_q d\tau \\ &\le \|e^{tA}\nabla u_0\|_q + \frac{C_2}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\gamma}\|u(\tau)|\nabla u(\tau)|^2\|_r\,d\tau,\end{aligned}$$

where $\frac{1}{r}=\frac{1}{p}+\frac{2}{q}$. Using the Hölder inequality for the integrand, we have $\|u|\nabla u|^2\|_r \le \|u\|_p\|\nabla u\|_q^2$. Since

$$\gamma = \frac{n}{2}\left(\frac{1}{p}+\frac{1}{q}\right)+\frac{1}{2} < 1,$$

we have

$$\|\nabla\Phi(u(t))\|_q \le \|e^{tA}\nabla u_0\|_q + \frac{C_2}{\varepsilon}e^{\frac{T}{\varepsilon}}\int_0^t (t-\tau)^{-\gamma}\|u(\tau)\|_p\|\nabla u(\tau)\|_q^2\,d\tau.$$

As the previous estimate, taking the supremum for t, we obtain

$$\sup_{0\le t\le T}\|\nabla\Phi(u(t))\|_q \le e^{\frac{T}{\varepsilon}}\|\nabla u_0\|_q + \frac{C_2}{\varepsilon}e^{\frac{T}{\varepsilon}}M^3\frac{T^{1-\gamma}}{1-\gamma}.$$

From the above estimate, we have

$$\|\Phi(u)\|_{X_M} \le \frac{M}{2}e^{\frac{T}{\varepsilon}} + \frac{M^3e^{\frac{T}{\varepsilon}}}{\varepsilon}\left(\frac{C_1qT^{1-\frac{n}{q}}}{q-n}+\frac{C_2T^{1-\gamma}}{1-\gamma}\right).$$

Taking T_0 as (A.20), we obtain

$$\|\Phi(u)\|_{X_M} \le \frac{3M}{4}+\frac{M}{4} \le M$$

for $T<T_0$, therefore if $u\in X_M(T)$, then $\Phi(u)\in X_M(T)$. □

3.2. Contraction of Φ.

Lemma A.21. *Let p,q be as Lemma A.19. Then for small $T>0$, Φ is a contraction mapping on $X_M(T)$.*

Proof of Lemma A.21. By Lemma A.16 we find

$$\begin{aligned}\|\Phi(u(t))-\Phi(v(t))\|_p &\le \frac{1}{\varepsilon}\int_0^t \|e^{(t-\tau)A}(u(\tau)|\nabla u(\tau)|^2 - v(\tau)|\nabla v(\tau)|^2)\|_p\,d\tau \\ &\le \frac{C_1}{\varepsilon}\int_0^t e^{\frac{t-\tau}{\varepsilon}}(t-\tau)^{-\frac{n}{q}}\|u(\tau)|\nabla u(\tau)|^2 - v(\tau)|\nabla v(\tau)|^2\|_r\,d\tau\end{aligned}$$

for $u, v \in X_M(T)$, where $\frac{1}{r} = \frac{1}{p} + \frac{2}{q}$. By the Hölder inequality, we have

$$\begin{aligned}
\|u(t)|\nabla u(t)|^2 - v(t)|\nabla v(t)|^2\|_r &\leq \|(u(t)-v(t))|\nabla u(t)|^2\|_r + \|(|\nabla u(t)|^2 - |\nabla v(t)|^2)v(t)\|_r \\
&\leq \|(u(t)-v(t))\|_p \|\nabla u(t)\|_q^2 \\
&\quad + \|\nabla u(t) + \nabla v(t)\|_q \|\nabla(u(t)-v(t))\|_q \|v(t)\|_p \\
&\leq M^2 \|u(t)-v(t)\|_p + 2M^2 \|\nabla(u(t)-v(t))\|_q,
\end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|\Phi(u(t)) - \Phi(v(t))\|_p \\
&\leq \frac{2M^2 C_1 q e^{\frac{T}{\varepsilon}} T^{1-\frac{n}{q}}}{\varepsilon(q-n)} \left(\sup_{0\leq t\leq T} \|u(t)-v(t)\|_p + \sup_{0\leq t\leq T} \|\nabla(u(t)-v(t))\|_q \right).
\end{aligned}$$

As the similar estimate, putting $\gamma = \frac{n}{2}\left(\frac{1}{p} + \frac{1}{q}\right) + \frac{1}{2}$ we find

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|\nabla(\Phi(u(t)) - \Phi(v(t)))\|_q \\
&\leq \frac{2M^2 C_2 e^{\frac{T}{\varepsilon}} T^{1-\gamma}}{\varepsilon(1-\gamma)} \left(\sup_{0\leq t\leq T} \|u(t)-v(t)\|_p + \sup_{0\leq t\leq T} \|\nabla(u(t)-v(t))\|_q \right).
\end{aligned}$$

From the above estimate, we obtain

$$\|\Phi(u) - \Phi(v)\|_{X_M} \leq \frac{2M^2 e^{\frac{T}{\varepsilon}}}{\varepsilon} \left(\frac{C_1 q T^{1-\frac{n}{q}}}{q-n} + \frac{C_2 T^{1-\gamma}}{1-\gamma} \right) \|u - v\|_{X_M}.$$

Therefore, taking $T > 0$ small enough so that

$$\frac{2M^2 e^{\frac{T}{\varepsilon}}}{\varepsilon} \left(\frac{C_1 q T^{1-\frac{n}{q}}}{q-n} + \frac{C_2 T^{1-\gamma}}{1-\gamma} \right) \leq \frac{3}{4}, \tag{A.22}$$

we have

$$\|\Phi(u) - \Phi(v)\|_{X_M} \leq \frac{3}{4} \|u - v\|_{X_M}.$$

□

REMARK A.22. We take $T_0 > 0$ satisfying (A.20). Then the inequality (A.22) is satisfied for all $T < T_0$.

PROOF OF PROPOSITION A.2. By Lemma A.19 and Lemma A.21, we find that Φ is a contraction mapping on $X_M(T)$. Since Cauchy's fixed point theorem, Φ has a fixed point, namely there uniquely exists $u \in X_M(T)$ such that $\Phi(u) = u$. This u satisfies (A.3) and is unique in $\{u \in C([0,T]\,; L^p(\mathbb{R}^n)) : \nabla u \in C([0,T]\,; L^q(\mathbb{R}^n))\}$. □

REMARK A.23. We consider the following initial-boundary problem:

$$\begin{cases} \partial_t u - \Delta u + \dfrac{u}{\varepsilon}(|\nabla u|^2 - 1) = 0, \quad (t,x) \in (0,T) \times \Omega, \\ \qquad\qquad u(0,x) = u_0(x), \quad x \in \Omega, \\ \qquad\qquad u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{A.23}$$

If Lemma A.16 holds, then we may use our argument and show the existence of a solution of (A.23).

APPENDIX B

Some fundamental calculus

Their results are well-known, however we give the proof for self-containedness.

1. Algebraic inequalities

LEMMA B.1 (DiBenedetto [**18**, Lemma 4.4 in p.13]). *Let $p>2$ and $d\in\mathbb{N}$. Then there exists $C_0>0$ depending only on p such that*

$$(|a|^{p-2}a-|b|^{p-2}b)\cdot(a-b)\geq C_0|a-b|^p \tag{B.1}$$

for all $a,b\in\mathbb{R}^d$.

PROOF OF LEMMA B.1. Since $p>2$, we have

$$\begin{aligned}(|a|^{p-2}a-|b|^{p-2}b)\cdot(a-b)&=\left(\int_0^1\frac{d}{ds}\{|sa+(1-s)b|^{p-2}(sa+(1-s)b)\}\,ds\cdot(a-b)\right)\\&\geq\int_0^1|sa+(1-s)b|^{p-2}|a-b|^2\,ds.\end{aligned}$$

Either if $|a|\geq|b-a|$, we have

$$|sa+(1-s)b|^{p-2}=|a-(1-s)(a-b)|\geq\Big||a|-(1-s)|a-b|\Big|\geq s|a-b|$$

and hence we obtain (B.1). Otherwise, namely if $|a|<|b-a|$, we obtain

$$|sa+(1-s)b|\leq|a|+(1-s)|b-a|\leq(2-s)|b-a|$$

and hence

$$\begin{aligned}|a-b|^2\int_0^1|sa+(1-s)b|^{p-2}\,ds&\geq|a-b|^2\int_0^1\frac{(|sa+(1-s)b|^2)^{\frac{p}{2}}}{(2-s)^2|b-a|^2}\,ds\\&\geq\frac14\left(\int_0^1|sa+(1-s)b|^2\,ds\right)^{\frac{p}{2}}\\&=\frac14\frac{1}{3^{\frac{p}{2}}}(|a|^2+|b|^2+(a\cdot b))^{\frac{p}{2}}.\end{aligned}$$

Remarking that $|a|^2+|b|^2+(a\cdot b)=\frac14|a-b|^2+\frac34|a+b|^2$, we obtain (B.1). $\square$

2. Sobolev type inequalities

PROPOSITION B.2 (Ladyženskaja-Solonnikov-Ural'ceva [**29**, p.74]). *Let $I\subset\mathbb{R}$ be an open interval and let $\Omega\subset\mathbb{R}^n$ be a domain. Then for $f\in L^\infty(I\,;L^2(\Omega))\cap L^2(I\,;H^1_0(\Omega))$*

and $p,q \geq 2$ satisfying

$$\begin{aligned} &\frac{2}{q}+\frac{n}{p}=\frac{n}{2} & & \textit{if} \quad n \neq 2, \\ &\frac{2}{q}+\frac{n}{p}=\frac{n}{2} \quad \textit{without} \quad q=2,\ p=\infty & & \textit{if} \quad n=2, \end{aligned}$$

we obtain

$$\|f\|_{L^q(I;L^p(\Omega))} \leq C(n,p,q)(\|f\|_{L^\infty(I;L^2(\Omega))}+\|\nabla f\|_{L^2(I\times\Omega)}). \tag{B.2}$$

Proof of Proposition B.2. By the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\|f(t)\|_{L^p(\Omega)} \leq C(n,p)\|\nabla f(t)\|_{L^2(\Omega)}^{\frac{n}{2}-\frac{n}{p}}\|f(t)\|_{L^2(\Omega)}^{1-(\frac{n}{2}-\frac{n}{p})} \quad \text{a.a. } t \in I.$$

Taking $L^q(I)$ norm on both side, we obtain (B.2). $\square$

Remark B.3. If $r=q$ in the Ladyženskaja inequality, we obtain

$$\|f\|_{L^{2(1+\frac{2}{n})}(I\times\Omega)} \leq C(\|f\|_{L^\infty(I;L^2(\Omega))}+\|f\|_{L^2(I\times\Omega)}). \tag{B.3}$$

Proposition B.4 (Ladyženskaja-Solonnikov-Ural'ceva [**29**, p.91]). *Let $f \in W^{1,1}(B_\rho)$ be a non-negative function and let $l > k$. Then there exists a constant $C > 0$ depending on n only such that*

$$(l-k)|\{f>l\}| \leq \frac{C\rho^{n+1}}{|B_\rho|-|\{f>k\}|}\int_{\{k<f\leq l\}}|\nabla f|\,dx,$$

where the n-dimensional Lebesgue measure of $A \subset \mathbb{R}^n$ is denoted by $|A|$.

For the proof of Proposition B.4, we need the following weighted Poincaré inequality:

Lemma B.5 (Ladyženskaja-Solonnikov-Ural'ceva [**29**, Lemma 5.1 in p.89]). *Let g be a non-negative function in $W^{1,1}(B_\rho)$ and let $N_0 := \{g=0\}$. Let $\eta(x)=\eta(|x|)$ be a decreasing function of $|x|$ satisfying $0 \leq \eta \leq 1$ and $\eta\big|_{N_0} \equiv 1$. Then for measurable set $N \subset B_\rho$, we have*

$$\int_N g(x)\eta(x)\,dx \leq \frac{C_n\rho^n}{|N_0|}|N|^{\frac{1}{n}}\int_{B_\rho}|\nabla g(x)|\eta(x)\,dx.$$

Proof of Lemma B.5. We firstly consider the case $n \geq 2$. For $x \in N$, $x' \in N_0$, we have

$$g(x)=g(x)-g(x')=-\int_0^{|x'-x|}\frac{d}{dr}g(x+r\omega)\,dr \leq \int_0^{|x'-x|}|\nabla g(x+r\omega)|\,dr$$

where $\omega=\frac{x'-x}{|x'-x|}$. We show

$$\eta(x) \leq \eta(x+r\omega) \quad \text{for} \quad 0<r\leq|x'-x|. \tag{B.4}$$

Either if $|x| \leq |x'|$, then $x+r\omega \in B_{|x'|}$ by the convexity of $B_{|x'|}$. By the monotonicity of η, we have $\eta(x+r\omega) \geq \eta(x')=1$. Otherwise, if $|x|>|x'|$, then $x+r\omega \in B_{|x|}$. Since $\eta(x+r\omega) \geq \eta(x)$, we obtain (B.4).

By (B.4), we have

$$g(x)\eta(x) \leq \int_0^{|x'-x|}|\nabla g(x+r\omega)|\eta(x+r\omega)\,dr.$$

Integrating over $x \in N$ and $x' \in N_0$, we have

$$|N_0| \int_N g(x)\eta(x)\,dx \le \int_N dx \int_{N_0} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\eta(x+r\omega)\,dr.$$

Let $g(x) = \eta(x)$ be zero on $x \in \mathbb{R}^n \setminus B_\rho$. Introducing the polar coordinate, we obtain

$$\begin{aligned}
&\int_{N_0} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\eta(x+r\omega)\,dr \\
&\le \int_{B_{2\rho(x)}} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\eta(x+r\omega)\,dr \\
&\le \int_{B_{2\rho(x)}} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\eta(x+r\omega)\,dr \\
&= \int_0^{2\rho} s^{n-1}\,ds \int_{\mathbb{S}^{n-1}} d\sigma \int_0^s \frac{|\nabla g(x+r\omega)|\eta(x+r\omega)}{r^{n-1}} r^{n-1}\,dr \quad \begin{pmatrix} x' = s\sigma + x, \\ \sigma \in \mathbb{S}^{n-1}, s > 0 \end{pmatrix} \\
&= \int_0^{2\rho} s^{n-1}\,ds \int_{B_s(x)} \frac{|\nabla g(y)|\eta(y)}{|x-y|^{n-1}}\,dy \quad (y = x + r\sigma) \\
&\le \frac{(2\rho)^n}{n} \int_{B_\rho} \frac{|\nabla g(y)|\eta(y)}{|x-y|^{n-1}}\,dy,
\end{aligned}$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional unit sphere. Therefore,

$$\begin{aligned}
|N_0| \int_N g(x)\eta(x)\,dx &\le \frac{(2\rho)^n}{n} \int_N dx \int_{B_\rho} \frac{|\nabla g(y)|\eta(y)}{|x-y|^{n-1}}\,dy \\
&= \frac{(2\rho)^n}{n} \int_{B_\rho} |\nabla g(y)|\eta(y)\,dy \int_N \frac{1}{|x-y|^{n-1}}\,dx.
\end{aligned}$$

We show the following estimate:

$$\int_N \frac{1}{|x-y|^{n-1}}\,dx \le (1 + |\mathbb{S}^{n-1}|)|N|^{\frac{1}{n}} \tag{B.5}$$

where $|\mathbb{S}^{n-1}|$ is the area of the $(n-1)$-dimensional unit sphere. To show (B.5), let $\delta > 0$ be chosen later. We split the integral

$$\begin{aligned}
\int_N \frac{1}{|x-y|^{n-1}}\,dx &\le \int_{N\cap\{|x-y|\le\delta\}} \frac{1}{|x-y|^{n-1}}\,dx + \int_{N\cap\{|x-y|\ge\delta\}} \frac{1}{|x-y|^{n-1}}\,dx \\
&=: I_1 + I_2.
\end{aligned}$$

By the simple calculation, we obtain

$$I_1 \le \int_0^\delta \frac{r^{n-1}}{r^{n-1}}\,dr \int_{\mathbb{S}^{n-1}} d\sigma = \delta|\mathbb{S}^{n-1}|, \quad I_2 \le \int_N \frac{1}{\delta^{n-1}}\,dx \le \delta^{1-n}|N|.$$

Taking $\delta = |N|^{\frac{1}{n}}$, we have $I_1 + I_2 \le (1 + |\mathbb{S}^{n-1}|)|N|^{\frac{1}{n}}$. Using (B.5), we have

$$|N_0| \int_N g(x)\eta(x)\,dx \le \frac{2^n(1 + |\mathbb{S}^{n-1}|)}{n} \rho^n |N|^{\frac{1}{n}} \int_{B_\rho} |\nabla g(y)|\eta(y)\,dy.$$

We consider the case $n = 1$. For $x \in N$ and $x' \in N_0$, we have

$$g(x) = g(x) - g(x') = \int_{x'}^{x} \frac{d}{dy} g(y)\, dy \leq \left| \int_{x'}^{x} \left| \frac{d}{dy} g(y) \right| dy \right|.$$

Since

$$g(x)\eta(x) \leq \left| \int_{x'}^{x} \left| \frac{d}{dy} g(y)\eta(y) \right| dy \right| \leq \int_{1}^{-1} \left| \frac{d}{dy} g(y)\eta(y) \right| dy,$$

we obtain

$$\int_N g(x)\eta(x)\, dx \leq |N| \int_{-1}^{1} |\nabla g(x)| \eta(x)\, dx.$$

□

Proof of Proposition B.4. Let

$$\begin{aligned} &g(x) := \max\{l - k\,,\, (f-k)_+\} \in W^{1,1}(B_\rho), \quad &N_0 := \{f < k\}, \\ &\eta(x) \equiv 1, &N := \{f > l\}. \end{aligned}$$

Then, by Lemma B.5, we have

$$\int_N g(x)\, dx \leq \frac{C_n \rho^n |N|^{\frac{1}{n}}}{|N_0|} \int_{B_\rho} |\nabla g(x)|\, dx,$$

hence

$$(l-k)|\{f > l\}| \leq \frac{C_n \rho^n |\{f > l\}|^{\frac{1}{n}}}{|\{f < k\}|} \int_{\{k < f \leq l\}} |\nabla f(x)|\, dx.$$

□

Next, we give another type of the weighted Poincaré inequality.

Lemma B.6 (Lieberman [**30**, p.113 Lemma 6.12]). *Let μ be a nonnegative continuous function in a bounded convex domain D with compact support. Furthermore $\{x \in D : \mu(x) \geq \lambda\}$ is convex for all $\lambda \geq 0$. Then*

$$\int_D (g(x) - k)^2 \mu(x)\, dx \leq C \frac{(\operatorname{diam} D)^{n+2}}{A} \|\mu\|_{L^\infty(D)} \int_D |\nabla g(x)|^2 \mu(x)\, dx$$

for all $g \in H^1(D)$, where

$$A = \int_D \mu(x)\, dx, \quad k = \frac{\int_D g(x)\mu(x)\, dx}{A}.$$

Proof of Lemma B.6. Considering $\frac{\mu}{A}$, we may assume $A = 1$. By the Hölder inequality, we have

$$\begin{aligned} \int_D (g(x) - k)^2 \mu(x)\, dx &= \int_D \left(\int_D (g(x) - g(y))\mu(y)\, dy \right)^2 dx \\ &\leq \int_D dx \int_D |g(x) - g(y)|^2 \mu(x)\mu(y)\, dy. \end{aligned}$$

We fix $x, y \in \operatorname{supp} \mu$ with $x \neq y$ and let $\omega = \frac{y-x}{|y-x|}$. Then

$$|g(x) - g(y)|^2 = \left| \int_0^{|y-x|} \frac{d}{dr} g(x + r\omega)\, dr \right|^2 \leq |y - x| \int_0^{|y-x|} |\nabla g(x + r\omega)|^2\, dr.$$

Since $\{z \in D : \mu(z) \geq \min\{\mu(x), \mu(y)\}\}$ is convex, we find

$$\mu(x + r\omega) \geq \min\{\mu(x), \mu(y)\}$$

for $0 < r < |y - x|$. Using $\mu(x)\mu(y) \leq \|\mu\|_{L^\infty(D)} \min\{\mu(x), \mu(y)\}$, we have

$$|g(x) - g(y)|^2 \mu(x)\mu(y) \leq \|\mu\|_{L^\infty(D)} |y - x| \int_0^{|y-x|} |\nabla g(x + r\omega)|^2 \mu(x + r\omega)\, dr.$$

We let $d = \operatorname{diam} D$ and $g(z) = \mu(z) = 0$ if $z \in \mathbb{R}^n \setminus D$. Then for all $x \in \operatorname{supp} \mu$,

$$\begin{aligned}
&\int_D |g(x) - g(y)|^2 \mu(x)\mu(y)\, dy \\
&\leq \|\mu\|_{L^\infty(D)} \int_{B_d(x)} |y - x|\, dy \int_0^{|y-x|} |\nabla g(x + r\omega)|^2 \mu(x + r\omega)\, dr \\
&= \|\mu\|_{L^\infty(D)} \int_0^d s^n\, ds \int_{\mathbb{S}^{n-1}} d\sigma \int_0^s \frac{|\nabla g(x + r\omega)|^2 \mu(x + r\omega)}{r^{n-1}} r^{n-1}\, dr \qquad \begin{pmatrix} y = s\sigma + x \\ 0 \leq s \leq d,\ \sigma \in \mathbb{S}^{n-1} \end{pmatrix} \\
&= \|\mu\|_{L^\infty(D)} \int_0^d s^n\, ds \int_{B_s(x)} \frac{|\nabla g(z)|^2 \mu(z)}{|z - x|^{n-1}}\, dz \qquad \begin{pmatrix} \omega = \frac{y-x}{|y-x|} = \sigma \\ z = x + r\omega \end{pmatrix} \\
&= \frac{d^{n+1}}{n+1} \|\mu\|_{L^\infty(D)} \int_D \frac{|\nabla g(z)|^2 \mu(z)}{|z - x|^{n-1}}\, dz.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_D (g(x) - k)^2 \mu(x)\, dx &\leq \int_D dx \int_D |g(x) - g(y)|^2 \mu(x)\mu(y)\, dy \\
&\leq \frac{d^{n+1}}{n+1} \|\mu\|_{L^\infty(D)} \int_D dx \int_D \frac{|\nabla g(z)|^2 \mu(z)}{|z - x|^{n-1}}\, dz \\
&= \frac{d^{n+1}}{n+1} \|\mu\|_{L^\infty(D)} \int_D |\nabla g(z)|^2 \mu(z)\, dz \int_D |z - x|^{n-1}\, dx.
\end{aligned}$$

As the same argument of the proof of Lemma B.5, we have

$$\int_D |z - x|^{n-1}\, dx \leq C(n) |D|^{\frac{1}{n}} \leq C(n) \operatorname{diam} D$$

and we obtain Lemma B.6. □

3. Parabolic John-Nirenberg estimates

Before giving the parabolic John-Nirenberg estimate, we introduce some notations: Let $N \in \mathbb{N}$ be a space dimension. For $r > 0$, we put $K_r = [-r, r]^N$, $U_r = (-r^2, r^2) \times K_r$ and

$$\begin{aligned}
U_r^+ &= (0, r^2) \times K_r, & U_r^- &= (-r^2, 0) \times K_r, \\
V_r^+ &= \left(\frac{1}{2} r^2, r^2\right) \times K_r, & V_r^- &= \left(-r^2, -\frac{1}{2} r^2\right) \times K_r, \\
W_r^+ &= \left(\frac{3}{4} r^2, r^2\right) \times K_r, & W_r^- &= \left(-r^2, -\frac{3}{4} r^2\right) \times K_r.
\end{aligned}$$

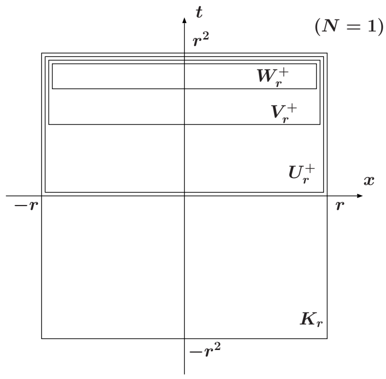

FIGURE B.1. Definition of K_r, U_r^+, V_r^+ and W_r^+ for $N=1$

DEFINITION B.7. For $C \subset \mathbb{R}^{N+1}$, we call C a parabolic rectangle if there exist $(t_0,x_0) \in \mathbb{R}^{N+1}$ and $r>0$ such that $C=(t_0,x_0)+U_r$. For parabolic rectangle C, we write

$$\begin{aligned} C^+ &= (t_0,x_0)+U_r^+, \quad C^- = (t_0,x_0)+U_r^-, \\ D^+ &= (t_0,x_0)+V_r^+, \quad D^- = (t_0,x_0)+V_r^-, \\ E^+ &= (t_0,x_0)+W_r^+, \quad E^- = (t_0,x_0)+W_r^-. \end{aligned}$$

We write $U=U_1$ for short and $U^\pm, V^\pm, W^\pm$ are defined in a similar manner. In this section, we show the following lemma:

LEMMA B.8 (Moser [**37**], Fabes-Garofalo [**22**]). *Let f be a function on U. Suppose that there exists constant $A>0$ such that for all parabolic rectangle $C \subset U$ we have*

$$\text{(B.6)} \qquad \begin{aligned} \frac{1}{|C^+|}\iint_{C^+}\sqrt{(f(t,x)-V_C)_+}\,dtdx \le A, \\ \frac{1}{|C^-|}\iint_{C^-}\sqrt{(V_C-f(t,x))_+}\,dtdx \le A, \end{aligned}$$

for some constant $a_C \in \mathbb{R}$ depending on C only. Then there exist $p_0, C_0>0$ such that

$$\left(\iint_{W^+} e^{p_0 f(t,x)}\,dtdx\right)\left(\iint_{W^-} e^{-p_0 f(t,x)}\,dtdx\right) \le C|W^+||W^-|,$$

where the constant C_0 depends on N and the constant p_0 depends on N, A.

To show Lemma B.8, we firstly give the estimate of distribution functions of f.

LEMMA B.9. *As for the same assumption of Lemma B.8, we obtain*

$$\text{(B.7)} \qquad \begin{aligned} \left|\left\{(t,x)\in V^+ : (f(t,x)-a_U)_+ > \alpha\right\}\right| \le Be^{-b\sqrt{\frac{\alpha}{A}}}|V^+| \\ \left|\left\{(t,x)\in V^- : (f(t,x)-a_U)_- > \alpha\right\}\right| \le Be^{-b\sqrt{\frac{\alpha}{A}}}|V^+| \end{aligned}$$

for all $\alpha>0$, where the constants B, b depend only on N.

PROOF OF LEMMA B.9. Considering $-f(-t,x)$, we only show the first inequality of (B.7). Without loss of generality, we assume $A=1$ and $a_U=0$ by considering $\frac{f-a_U}{A^2}$. For $\alpha \le 1$, we have $e^{-b\alpha^{\frac{1}{2}}} \ge e^{-b}$ and hence for $\alpha>1$, we show

$$\left|\left\{(t,x)\in V^+ : f_+(t,x) > \alpha\right\}\right| \le Be^{-b\sqrt{\frac{\alpha}{A}}}|V^+|.$$

We give a decomposition procedure. We fix $\beta > 0$. We subdivide $D_0^+ = V^+$ into 4^{N+2} congruent sub-rectangles with disjoint interiors. Let $\{D_{1,i}^+(\beta)\}_i$ denote the family of those sub-rectangles and $C_{1,i}(\beta)$ denote the corresponding parabolic rectangle of $D_{1,i}^+(\beta)$. Next, for $D_{1,i}^+(\beta)$ satisfying $\beta \geq a_{C_{1,i}(\beta)}$, we similarly subdivide $D_{1,i}(\beta)$ and the process is repeated indefinitely. Then we obtain $\{D_{n,i}^+(\beta)\}_{n,i}$ and we put

$$D(\beta) := \bigcup_{n=1}^{\infty} \bigcup_i D_{n,i}^+(\beta).$$

We show

$$\sqrt{f_+(t,x)} \leq 1 + \sqrt{\beta} \quad \text{a.a. } (t,x) \in D_0 \setminus D(\beta). \tag{B.8}$$

In fact, if $(t,x) \in D_0 \setminus D(\beta)$, then we obtain the sequence of parabolic rectangles $\{C_n\}_{n=1}^{\infty}$ such that

$$(t,x) \in D_n^+, \;\; a_{C_n} \leq \beta \;\text{ and }\; |C_n| \to 0 \;\text{ as }\; n \to \infty.$$

Therefore, by (B.6) and $A = 1$, we have

$$\frac{1}{|C_n^+|} \int_{C_n^+} \sqrt{f_+(t,x)}\, dtdx \leq \frac{1}{|C_n^+|} \int_{C_n^+} \sqrt{(f(t,x) - a_{C_n})_+}\, dtdx + \sqrt{\beta} \leq 1 + \sqrt{\beta}.$$

By the Lebesgue differentiation theorem, we obtain

$$\frac{1}{|C_n^+|} \int_{C_n^+} \sqrt{f_+(t,x)}\, dtdx \to \sqrt{f_+(t,x)} \;\text{ as }\; n \to \infty$$

for almost all $(t,x) \in D_0 \setminus D(\beta)$. From (B.8), for $\beta > 0$, we have

$$\{(t,x) \in D_0^+ : f(t,x) \geq (1 + \sqrt{\beta})^2\} \subset D(\beta).$$

Claim B.10. *There exists a constant C_0 depending only on N such that if $\sqrt{\beta} \geq \sqrt{\alpha} + 1$, then*

$$|D(\beta)| \leq \frac{C_0}{\sqrt{\beta} - \sqrt{\alpha} - 1} |D(\alpha)|. \tag{B.9}$$

We show Lemma B.9 by temporary admitting Claim B.10. Put $L := 2C_0 + 1$. Then for $\gamma > 0$, we have

$$\left| D\big((\gamma + L)^2\big) \right| \leq \frac{C_0}{(\gamma + L) - \gamma - 1} \big| D(\gamma^2) \big| \leq \frac{1}{2} \big| D(\gamma^2) \big|.$$

For $\beta > 0$, we take $n_0 \in \mathbb{N} \cup \{0\}$ satisfying $n_0 L \leq \beta \leq (n_0 + 1)L$. Then we find

$$\big| D(\beta^2) \big| \leq \big| D((n_0 L)^2) \big| \leq \left(\frac{1}{2}\right)^{n_0 - 1} \big| D(L^2) \big| \leq 4 \left(\frac{1}{2}\right)^{\frac{\alpha}{L}} \big| D_0^+ \big| = 4e^{-\frac{\beta}{L}\log 2} \big| D_0^+ \big|.$$

Therefore, for $\alpha > 1$, we have

$$\left| \big\{(t,x) \in V^+ : f_+(t,x) > \alpha \big\} \right| \leq \left| D\big((\sqrt{\alpha} - 1)^2\big) \right| \leq 4e^{-\frac{\sqrt{\alpha} - 1}{L}\log 2} |V^+|.$$

and we obtain Lemma B.9.

We show the inequality (B.9). $\{D^+_{n,i}(\beta)\}$ is disjoint interiors but $\{C^-_{n,i}(\beta)\}$ may not be. Hence we make a disjoint family of $\{C^-_{n,i}(\beta)\}$ as follows: First, we take $\{{}^*C^-_{1,j}(\beta)\} \subset \{C^-_{1,i}(\beta)\}$ with disjoint interiors and

$$\operatorname{Int} C^-_{1,i}(\beta) \bigcap \left(\bigcup_j \operatorname{Int} {}^*C^-_{1,j}(\beta) \right) \neq \emptyset$$

for all i. Second, we take $\{{}^*C^-_{2,j}(\beta)\} \subset \{C^-_{2,i}(\beta)\}$ such that $\{{}^*C^-_{m,j}(\beta)\}_{1\le m\le 2\,,\,j}$ is disjoint interiors and

$$\operatorname{Int} C^-_{2,i}(\beta) \bigcap \left(\bigcup_{1\le m\le 2} \bigcup_j \operatorname{Int} {}^*C^-_{m,j}(\beta) \right) \neq \emptyset$$

for all i. Similarly, for $n \in \mathbb{N}$, we take $\{{}^*C^-_{n,j}(\beta)\} \subset \{C^-_{n,i}(\beta)\}$ such that $\{{}^*C^-_{m,j}(\beta)\}_{1\le m\le n\,,\,j}$ is disjoint interiors and

$$\operatorname{Int} C^-_{n,i}(\beta) \bigcap \left(\bigcup_{1\le m\le n} \bigcup_j \operatorname{Int} {}^*C^-_{m,j}(\beta) \right) \neq \emptyset$$

for all i.

For $0 < \alpha < \beta$, we take $\{D^+_{n,i}(\alpha)\}$ and $\{D^+_{n,i}(\beta)\}$ by the decomposition procedure. Let $\{{}^*C^-_{m,j}(\beta)\}$ be the disjoint family of $\{C^-_{n,j}(\beta)\}$. For $m \in \mathbb{N}$, we put

$$I_m := \left\{ (n,i) : \operatorname{Int} C^-_{n,i}(\beta) \bigcap \left(\bigcup_{1\le m'\le m} \bigcup_j \operatorname{Int} {}^*C^-_{m',j}(\beta) \right) \neq \emptyset \right.$$
$$\left. \text{and} \ \operatorname{Int} C^-_{n,i}(\beta) \bigcap \left(\bigcup_{1\le m'\le m-1} \bigcup_j \operatorname{Int} {}^*C^-_{m',j}(\beta) \right) = \emptyset \right\}.$$

Since $(n,i) \in I_m$ for some $m \le n$, we have

$$|D(\beta)| = \sum_{m=1}^{\infty} \sum_{(n,i)\in I_m} |D^+_{n,i}(\beta)|.$$

For a rectangle $C \subset \mathbb{R}^{N+1}$ and $a > 0$, let aC be a rectangle which length are the length of C a times. Since

$$D^+_{n,i}(\beta) \subset 4C^-_{n,i}(\beta) \subset \bigcup_j 16\,{}^*C^-_{m,j}(\beta) \quad \text{for} \ \ (n,i) \in I_m,$$

we have

$$\sum_{m=1}^{\infty} \sum_{(n,i)\in I_m} |D^+_{n,i}(\beta)| \le 16^{N+1} \sum_{m=1}^{\infty} \sum_j |{}^*C^-_{m,j}(\beta)|.$$

Since $\{D^+_{l,k}(\alpha)\}$ is disjoint interiors, we find

$$|D(\beta)| \le 16^{N+1} \sum_{l=1}^{\infty} \sum_{(m,j)\in J_l} |{}^*C^-_{m,j}(\beta)|,$$

where $J_l := \big\{(m,j) : D^+_{m,j}(\beta) \subset \bigcup_k D^+_{l,k}(\alpha)\big\}$. For $(m,j) \in J_l$, we have

$$\begin{aligned}\sqrt{\beta} &< \sqrt{a_{{}^*C^-_{m,j}(\beta)}}\\ &\le \frac{1}{|{}^*C^-_{m,j}(\beta)|}\int_{{}^*C^-_{m,j}(\beta)} \sqrt{(f(t,x)-a_{{}^*C^-_{m,j}(\beta)})_-}\,dtdx + \frac{1}{|{}^*C^-_{m,j}(\beta)|}\int_{{}^*C^-_{m,j}(\beta)} \sqrt{f_+(t,x)}\,dtdx\\ &\le 1 + \frac{1}{|{}^*C^-_{m,j}(\beta)|}\int_{{}^*C^-_{m,j}(\beta)} \sqrt{f_+(t,x)}\,dtdx\end{aligned}$$

and hence

$$(\sqrt{\beta}-1)\sum_{(m,j)\in J_l} |{}^*C^-_{m,j}(\beta)| \le \sum_{(m,j)\in J_l} \int_{{}^*C^-_{m,j}(\beta)} \sqrt{f_+(t,x)}\,dtdx.$$

Furthermore, since $D^+_{m,j}(\beta) \subset D^+_{l,k}(\alpha) \subset D^+_{l-1,k'}(\alpha)$ for some $(l,k), k'$ and $a_{C_{l-1,k'}(\alpha)} \le \alpha$, we have

$$\begin{aligned}&\sum_{(m,j)\in J_l} \int_{{}^*C^-_{m,j}(\beta)} \sqrt{f_+(t,x)}\,dtdx\\ &\qquad\le \sum_{(m,j)\in J_l} \int_{{}^*C^-_{m,j}(\beta)} \sqrt{(f(t,x)-a_{C_{l-1,k'}(\alpha)})_+}\,dtdx + \sqrt{\alpha}|{}^*C^-_{m,j}(\beta)|.\end{aligned}$$

For (l,k), we put

$$J_{l,k} := \{(m,i) : D^+_{m,i}(\beta) \subset D^+_{l,k}(\alpha)\}.$$

Then ${}^*C^-_{m,j}(\beta) \subset C^+_{l-1,k'}(\alpha)$ for all $(m,j) \in J_{l,k}$ and for some k' depending only on k. By the disjointness of $\{{}^*C^-_{m,j}(\beta)\}$, we obtain

$$\begin{aligned}&\sum_{(m,j)\in J_l} \int_{{}^*C^-_{m,j}(\beta)} \sqrt{(f(t,x)-a_{C_{l-1,k'}(\alpha)})_+}\,dtdx\\ &=\sum_k \sum_{(m,j)\in J_{l,k}} \int_{{}^*C^-_{m,j}(\beta)} \sqrt{(f(t,x)-a_{C_{l-1,k'}(\alpha)})_+}\,dtdx\\ &=\sum_k \int_{\cup_{(m,j)\in J_{l,k}} {}^*C^-_{m,j}(\beta)} \sqrt{(f(t,x)-a_{C_{l-1,k'}(\alpha)})_+}\,dtdx\\ &\le \sum_k \int_{C^+_{l-1,k'}(\alpha)} \sqrt{(f(t,x)-a_{C_{l-1,k'}(\alpha)})_+}\,dtdx\\ &\le \sum_k |C^+_{l-1,k'}(\alpha)| = 2\sum_k |D^+_{l-1,k'}(\alpha)| = 2\cdot 4^{N+2}\sum_k |D^+_{l,k}(\alpha)|.\end{aligned}$$

Finally, we obtain

$$|D(\beta)| \le C(N)\sum_{l=1}^{\infty}\sum_k \frac{1}{\sqrt{\beta}-\sqrt{\alpha}-1}|D^+_{l,k}(\alpha)| \le C(N)\frac{1}{\sqrt{\beta}-\sqrt{\alpha}-1}|D(\alpha)|$$

and proof of (B.9) is complete. □

LEMMA B.11. *As the same assumption of Lemma B.8, there exists $A' > 0$ depending only on N, A such that*

$$\begin{aligned} &\frac{1}{|D^+|}\iint_{D^+} (f(t,x) - V_C)_+ \, dtdx \le A', \\ &\frac{1}{|D^-|}\iint_{D^-} (V_C - f(t,x))_+ \, dtdx \le A', \end{aligned} \tag{B.10}$$

for all parabolic cylinder $C \subset U$.

PROOF OF LEMMA B.11. We only show the first inequality of (B.10). By Lemma B.9, we have

$$\left|\left\{(t,x) \in D^+ : (f(t,x) - a_C)_+ > \alpha\right\}\right| \le Be^{-b\sqrt{\frac{\alpha}{A}}}|D^+|.$$

Therefore

$$\frac{1}{|D^+|}\iint_{D^+} (f(t,x) - V_C)_+ \, dtdx \le B\int_0^\infty e^{-b\sqrt{\frac{\alpha}{A}}}\, d\alpha < \infty.$$

$\square$

As the same argument of the proof of Lemma B.9, we obtain the following lemma:

LEMMA B.12. *Assume that there exists $A' > 0$ such that* (B.10) *holds for all parabolic rectangle $C \subset U$. Then there exist constants $B', b' > 0$ depending only on N such that*

$$\begin{aligned} &\left|\left\{(t,x) \in W^+ : (f(t,x) - a_U)_+ > \alpha\right\}\right| \le B'e^{-b'(\frac{\alpha}{A})}|W^+|, \\ &\left|\left\{(t,x) \in W^- : (f(t,x) - a_U)_- > \alpha\right\}\right| \le B'e^{-b'(\frac{\alpha}{A})}|W^-|. \end{aligned}$$

PROOF OF LEMMA B.8. Let $p_0 < \frac{b'}{A'}$ where A' is as Lemma B.11 and b' is as Lemma B.12. Then

$$\begin{aligned} \int_{W^+} e^{p_0 f(t,x)}\, dtdx &\le e^{p_0 a_U}\int_{W^+} e^{p_0 (f(t,x)-a_U)_+}\, dtdx \\ &\le e^{p_0 a_U}\int_{W^+} p_0\left(\int_0^{(f(t,x)-a_U)_+} e^{p_0\alpha}\, d\alpha + 1\right) dtdx \\ &\le e^{p_0 a_U} p_0\left(\int_0^\infty e^{p_0\alpha}\left|\left\{(t,x) \in W^+ : (f(t,x) - a_U)_+ > \alpha\right\}\right| d\alpha + |W^+|\right) \\ &\le e^{p_0 a_U} p_0\left(\int_0^\infty e^{p_0\alpha}(B'e^{-\frac{b'}{A'}\alpha})\, d\alpha + 1\right)|W^+| \\ &\le e^{p_0 a_U} p_0\left(B'\int_0^\infty e^{(p_0-\frac{b'}{A'})\alpha}\, d\alpha + 1\right)|W^+|. \end{aligned}$$

Similarly,

$$\int_{W^-} e^{-p_0 f(t,x)}\, dtdx \le e^{-p_0 a_U} p_0\left(B'\int_0^\infty e^{(p_0-\frac{b'}{A'})\alpha}\, d\alpha + 1\right)|W^-|.$$

Therefore,

$$\int_{W^+} e^{p_0 f(t,x)}\, dtdx \int_{W^-} e^{-p_0 f(t,x)}\, dtdx \le p_0\left(B'\int_0^\infty e^{(p_0-\frac{b'}{A'})\alpha}\, d\alpha + 1\right)^2 |W^+||W^-|$$

and proof of Lemma B.8 is complete. $\square$

4. Recursive inequalities

Lemma B.13 (Ladyženskaja-Solonnikov-Ural'ceva [**29**, p.96]). *Let $C, \varepsilon, \delta > 0$, $b \geq 1$ and let $\{Y_n\}_{n=0}^{\infty}$, $\{Z_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfy*

$$\begin{aligned} Y_{n+1} &\leq Cb^n (Y_n^{1+\delta} + Y_n^{\delta} Z_n^{1+\varepsilon}), \\ Z_{n+1} &\leq Cb^n (Y_n + Z_n^{1+\varepsilon}). \end{aligned} \tag{B.11}$$

Set

$$d := \min\left\{\delta, \frac{\varepsilon}{1+\varepsilon}\right\} \, , \, \lambda = \min\left\{(2C)^{-\frac{1}{\delta}} b^{-\frac{1}{\delta d}} \, , \, (2C)^{-\frac{1+\varepsilon}{\varepsilon}} b^{-\frac{1}{\varepsilon d}}\right\}.$$

Then, if $Y_0 \leq \lambda$ and $Z_0 \leq \lambda^{\frac{1}{1+\varepsilon}}$, we obtain

$$Y_n \leq \lambda b^{-\frac{n}{d}}, \quad Z_n \leq (\lambda b^{-\frac{n}{d}})^{\frac{1}{1+\varepsilon}}. \tag{B.12}$$

In particular, $Y_n, Z_n \to 0$ as $n \to \infty$.

Proof of Lemma B.13. Inequalities (B.12) are valid for $n = 0$. We prove (B.12) by induction. If (B.12) hold for n, then by (B.11), we have

$$Y_{n+1} \leq 2C\lambda^{1+\delta} b^{n(1-\frac{1+\delta}{d})}, \quad Z_{n+1} \leq 2C\lambda b^{n(1-\frac{1}{d})}.$$

Since $\lambda \leq (2C)^{-\frac{1}{\delta}} b^{-\frac{1}{\delta d}}$ and $d \leq \delta$, we have

$$2C\lambda^{1+\delta} b^{n(1-\frac{1+\delta}{d})} \leq \lambda b^{-\frac{1}{d}} b^{-\frac{n}{d}+n(1-\frac{\delta}{d})} \leq \lambda b^{-\frac{n+1}{d}}.$$

Similarly, since $\lambda \leq (2C)^{-\frac{1+\varepsilon}{\varepsilon}} b^{-\frac{1}{\varepsilon d}}$, we obtain

$$2C\lambda b^{n(1-\frac{1}{d})} = 2C\lambda^{\frac{\varepsilon}{1+\varepsilon}} \lambda^{\frac{1}{1+\varepsilon}} b^{-\frac{n+1}{(1+\varepsilon)d}} b^{n(1-\frac{1}{d})+\frac{n+1}{(1+\varepsilon)d}} \leq (\lambda b^{-\frac{n+1}{d}})^{\frac{1}{1+\varepsilon}} b^{n(1-\frac{\varepsilon}{(1+\varepsilon)d})}.$$

Since $d \leq \frac{\varepsilon}{1+\varepsilon}$, we find $1 - \frac{\varepsilon}{(1+\varepsilon)d} \leq 0$ and hence we have (B.12) for $n+1$. $\square$

Lemma B.14 (Giaquinta [**23**, Lemma 2.1 in p.86]). *Let $\phi = \phi(s)$ be a non-negative function on $[0, \infty)$. Assume that for some constants $R_0, A_0, A_1, B, \alpha, \beta > 0$ with $\beta < \alpha$, the function ϕ satisfies*

$$\begin{aligned} \phi(\rho) &\leq A_0 \phi(R), \\ \phi(\rho) &\leq A_1 \left(\frac{\rho}{R}\right)^{\alpha} \phi(R) + BR^{\beta} \end{aligned}$$

for all $0 < \rho \leq R \leq R_0$. Then, there exists a constant $C > 0$ depending only on A_0, A_1, α, β such that

$$\phi(\rho) \leq C\left\{\left(\frac{\rho}{R}\right)^{\beta} \phi(R) + B\rho^{\beta}\right\}.$$

Proof of Lemma B.14. Let $r < 1$ be chosen later. For $R \leq R_0$, taking $\rho = rR$, we have

$$\phi(rR) \leq A_1 r^{\alpha} \phi(R) + BR^{\beta}.$$

We fix $\beta < \gamma < \alpha$ and we put $r = r_0$ satisfying $A_1 r_0^{\alpha} \leq r_0^{\gamma}$. Then

$$\phi(r_0 R) \leq A_1 r_0^{\alpha} \phi(R) + BR^{\beta} \leq r_0^{\gamma} \phi(R) + BR^{\beta}.$$

Therefore, for $k \in \mathbb{N} \cup \{0\}$, we have

$$\phi(r_0^{k+1} R) \le r_0^{(k+1)\gamma} \phi(R) + B r_0^{k\beta} R^\beta \sum_{j=0}^{k} r_0^{j(\gamma-\beta)}.$$

Since

$$\sum_{j=0}^{k} r_0^{j(\gamma-\beta)} \le \sum_{j=0}^{\infty} r_0^{j(\gamma-\beta)} = \frac{r_0^\beta}{r_0^\beta - r_0^\gamma},$$

we obtain

$$\phi(r_0^k R) \le r_0^{k\gamma} \phi(R) + r_0^{k\beta} \frac{BR^\beta}{r_0^\beta - r_0^\gamma}$$

for $k \in \mathbb{N} \cup \{0\}$. For $\rho > 0$, there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $r_0^{k_0+1} R \le \rho \le r_0^{k_0} R$. Therefore, we obtain

$$\begin{aligned}\phi(\rho) \le A_0 \phi(r_0^{k_0} R) &\le A_0 \left(r_0^{k\gamma} \phi(R) + r_0^{k\beta} \frac{BR^\beta}{r_0^\beta - r_0^\gamma} \right) \\ &\le A_0 \left(r_0^{-\gamma} \left(\frac{\rho}{R}\right)^\gamma \phi(R) + r_0^{-\beta} \left(\frac{\rho}{R}\right)^\beta \frac{BR^\beta}{r_0^\beta - r_0^\gamma} \right).\end{aligned}$$

□

5. Weak L^p spaces and Lorentz spaces

Let $\Omega \subset \mathbb{R}^n$ be a domain (not necessary bounded).

DEFINITION B.15. For $1 \le p < \infty$, we define the Lorentz space $L^{p,\infty}(\Omega)$ by

$$L^{p,\infty}(\Omega) := \{ f \in L^1_{\mathrm{loc}}(\Omega) \,:\, \lambda^p \mu_{|f|}(\lambda) \text{ is bounded for all } \lambda > 0 \}$$

where $\mu_{|f|}(\lambda) := |\{ x \in \Omega \,:\, |f(x)| > \lambda \}|$.

PROPOSITION B.16 (cf. Benilan-Brezis-Crandall [**6**, p.548]). *For $1 < p < \infty$, we have*

$$\frac{p-1}{p^{1+\frac{1}{p}}} \|f\|_{L^p_{\mathrm{w}}(\Omega)} \le \sup_{\lambda > 0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}} \le \|f\|_{L^p_{\mathrm{w}}(\Omega)}.$$

PROOF OF PROPOSITION B.16. We firstly show $\sup_{\lambda>0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}} \le \|f\|_{L^p_{\mathrm{w}}(\Omega)}$. For $\rho, \lambda > 0$, we take $K = \{ x \in \Omega : |f(x)| > \lambda \} \cap B_\rho$. Then we have

$$\begin{aligned}\|f\|_{L^p_{\mathrm{w}}(\Omega)} &\ge \left| \left\{ x \in \Omega \cap B_\rho : |f(x)| > \lambda \right\} \right|^{\frac{1}{p}-1} \int_{\{|f|>\lambda\} \cap B_\rho} |f(x)| \, dx \\ &\ge \lambda \left| \left\{ x \in \Omega \cap B_\rho : |f(x)| > \lambda \right\} \right|^{\frac{1}{p}},\end{aligned}$$

where $\{|f| > \lambda\} = \{ x \in \Omega : |f(x)| > \lambda \}$. Letting $\rho \to \infty$, we find

$$\lambda \mu_{|f|}(\lambda)^{\frac{1}{p}} \le \|f\|_{L^p_{\mathrm{w}}(\Omega)}.$$

We show $\frac{p-1}{p^{1+\frac{1}{p}}} \|f\|_{L^p_{\mathrm{w}}(\Omega)} \le \sup_{\lambda>0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}}$. We fix $\lambda_0 > 0$. For measurable set $K \subset \Omega$, we have

$$\int_K |f(x)| \, dx \le \lambda_0 |K| + \int_{\{|f| > \lambda_0\}} |f(x)| \, dx.$$

By the above inequality, we have

$$\begin{aligned}
\int_{\{|f|>\lambda_0\}} |f(x)|\,dx &= \int_0^\infty \Big|\Big\{x \in \{|f|>\lambda_0\} \,:\, |f(x)|>\lambda\Big\}\Big|\,d\lambda \\
&= \int_0^{\lambda_0} \big|\{|f|>\lambda_0\}\big|\,d\lambda + \int_{\lambda_0}^\infty \big|\{|f|>\lambda\}\big|\,d\lambda \\
&= \lambda_0 \mu_{|f|}(\lambda_0) + \int_{\lambda_0}^\infty \mu_{|f|}(\lambda)\,d\lambda \\
&\le \lambda_0^{1-p} \sup_{\lambda>0} \lambda^p \mu_{|f|}(\lambda) + \sup_{\lambda>0} \lambda^p \mu_{|f|}(\lambda) \int_{\lambda_0}^\infty \lambda^{-p}\,d\lambda \\
&= \sup_{\lambda>0} \lambda^p \mu_{|f|}(\lambda) \frac{p}{p-1} \lambda_0^{1-p}.
\end{aligned}$$

Taking $\lambda_0^p |K| = p \sup_{\lambda>0} \lambda^p \mu_{|f|}(\lambda)$, we find

$$\int_K |f(x)|\,dx \le \frac{p}{p-1} (p \sup_{\lambda>0} \lambda^p \mu_{|f|}(\lambda))^{\frac{1}{p}} |K|^{1-\frac{1}{p}}$$

or

$$\|f\|_{L^p_{\mathrm{w}}(\Omega)} \le \frac{p^{1+\frac{1}{p}}}{p-1} \sup_{\lambda>0} \lambda \mu_{|f|}(\lambda)^{\frac{1}{p}}.$$

$\square$

From Proposition B.16, we immediately obtain the following corollary:

Corollary B.17. *Let $p, q > 1$ and let $f \in L^{pq}_{\mathrm{w}}(\Omega)$. Then $|f|^q \in L^p_{\mathrm{w}}(\Omega)$.*

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