Convergence of the Allen-Cahn equation with Neumann boundary conditions

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Allen-Cahn equation

$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon^2} & t>0,\ x\in\Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0 & t>0, \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & x\in\Omega, \end{cases} \qquad (\mathrm{AC})_\varepsilon$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, ν is an outer unit normal to $\partial\Omega$, $\varepsilon>0$, $W(u^\varepsilon):=\frac{1}{2}(1-(u^\varepsilon)^2)^2$.
For $t>0$, define varifolds V_t^ε as

$$d\mu_t^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon(x,t)|^2 + \frac{W(u^\varepsilon(x,t))}{\varepsilon}\right)dx$$

$$V_t^\varepsilon(\phi) := \frac{1}{\sigma}\int_{\Omega\cap\{|\nabla u^\varepsilon(\cdot,t)|\neq 0\}} \phi\,(x, I-\vec{n}_\varepsilon\otimes\vec{n}_\varepsilon)\ d\mu_t^\varepsilon,$$

where $\phi\in C_c(G_{n-1}(\Omega))$, $\vec{n}_\varepsilon := \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}$, $\sigma := \int_{-1}^{1}\sqrt{2W(\xi)}\,d\xi = \frac{4}{3}$.

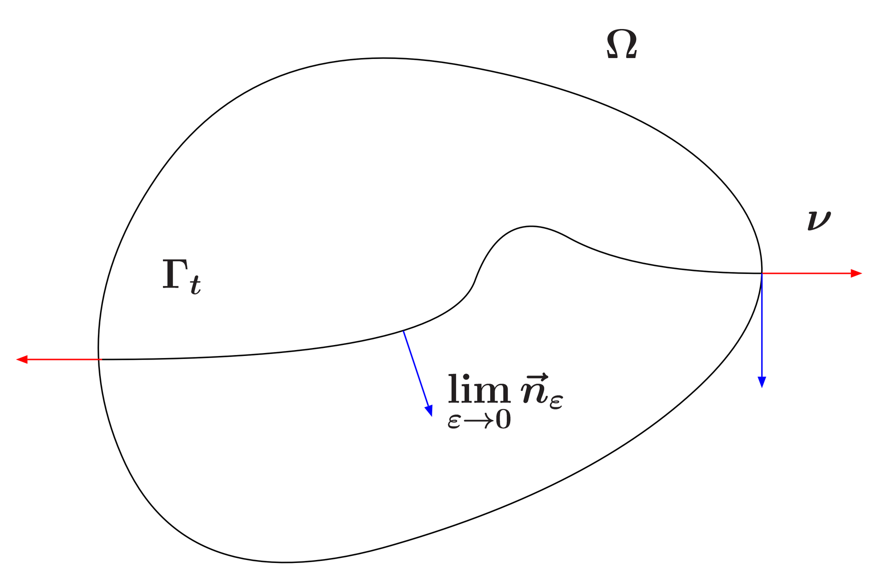
- Ilmanen (1993), Tonegawa (2003), Liu-Sato-Tonegawa (2010), Takasao-Tonegawa (to appear) (without boundary)
  V_t^ε converges to an integral varifold V_t up to subsequence for almost all $t\geq 0$. Moreover, V_t is Brakke's weak solution of Mean Curvature Flow (MCF for short).

Our Aim

To study the boundary behavior of V_t.

Heuristic observation

Formally define $\Gamma_t = \{x\in\Omega : \lim_{\varepsilon\downarrow 0} u^\varepsilon \neq \pm 1\} \simeq \operatorname{spt} V_t$, then Γ_t is solution of MCF and $\vec{n}_\varepsilon$ is an approximate unit normal vector of Γ_t. Since we impose the Neumann boundary condition, Γ_t should intersect $\partial\Omega$ with 90° degree.



1. Does V_t^ε converge up to boundary ?
2. What is the right notion of the boundary of Brakke's weak solutions ?

Related study

- Level set methods (Viscosity solutions)
  - Chen-Giga-Goto (1991), Evans-Spruck (1991)
    Existence of a generalized MCF (without boundary)
  - Giga-Sato (1991), M.-H. Sato (1994)
    Existence of a generalized MCF (with boundary)
  - Evans-Soner-Souganidis (1992)
    Convergence of $(\mathrm{AC})_\varepsilon$ (without boundary)
  - Katsoulakis-Kossioris-Reitich (1995), Barles-Souganidis (1998)
    Convergence of $(\mathrm{AC})_\varepsilon$ (with boundary)
- Matched asymptotic expansion
  - X. Chen (1992)
    Asymptotic behavior of $(\mathrm{AC})_\varepsilon$ as $\varepsilon\to 0$ (without/with boundary)
- Geometric measure theory
  - Brakke (1978)
    Existence and regularity of a weak MCF (without boundary)
  - Hutchinson-Tonegawa (2000),Tonegawa (2004)
    Convergence of the stationary problem of $(\mathrm{AC})_\varepsilon$ (without/with boundary)

It is not well-known how to formulate the boundary conditions for Brakke's weak MCF.

Assumption

- Ω is bounded, strictly convex.
- $\|u_0^\varepsilon\|_\infty \leq 1,\quad \sup_{\varepsilon>0}\int_\Omega(\frac{\varepsilon}{2}|\nabla u_0^\varepsilon|^2 + \frac{W(u_0^\varepsilon)}{\varepsilon})\,dx < \infty$

Theorem (Tonegawa-M.)

There exist a subsequence $\mu_t^{\varepsilon_i}$ and a family of Radon measures $\{\mu_t\}_{t>0}$ such that for all $t>0$, $\mu_t^{\varepsilon_i} \rightharpoonup \mu_t$ as $\varepsilon_i\to 0$ on $\overline{\Omega}$. Moreover, μ_t is rectifiable on $\overline{\Omega}$ for almost all $t\geq 0$.

For almost all $t\geq 0$, let V_t be an associated rectifiable varifold with μ_t such that $\|V_t\| = \mu_t$ on $\overline{\Omega}$.

Theorem (Tonegawa-M.)

- *(Boundedness of first variation) First variation of V_t, which denote by δV_t, is bounded up to boundary for almost all $t\geq 0$. In fact, for $T>0$*
  $$\int_0^T \|\delta V_t\|(\overline{\Omega})\,dt < \infty.$$
- *(Generalized 90° degree condition) Let*
  $$\delta V_t\lfloor_{\partial\Omega}^{\top}(g) := \delta V_t\lfloor_{\partial\Omega}(g-(g\cdot\nu)\nu)$$
  for $g\in C(\partial\Omega:\mathbb{R}^n)$. Then for almost all $t\geq 0$, $\|\delta V_t\lfloor_\Omega + \delta V_t\lfloor_{\partial\Omega}^{\top}\| \ll \|V_t\|$, and there exists $h=h(t)\in L^2(\|V_t\|)$ such that
  $$\delta V_t\lfloor_\Omega + \delta V_t\lfloor_{\partial\Omega}^{\top} = -h(t)\|V_t\|.$$
- *(Brakke's inequality) For $\phi\in C^1(\overline{\Omega}\times[0,\infty)\,;\mathbb{R}^+)$ with $\nabla\phi(\cdot,t)\cdot\nu = 0$ on $\partial\Omega$ and for any $0\leq t_1<t_2<\infty$, we have*
  $$\int_{\overline{\Omega}}\phi(\cdot,t)\,d\|V_t\|\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2}\int_{\overline{\Omega}}\left(-\phi|h|^2 + \nabla\phi\cdot h + \partial_t\phi\right)d\|V_t\|dt.$$

Generalized 90° degree condition

Assume V_t is an associated varifold with some smooth hypersurface M_t. Then by Gauss' divergence theorem,

$$\delta V_t(g) = \int_{M_t}\operatorname{div}_{M_t} g\,d\mathscr{H}^{n-1} = -\int_{M_t} g\cdot h\,d\mathscr{H}^{n-1} + \int_{\partial M_t} g\cdot\gamma\,d\sigma$$

for $g\in C^1(\overline{\Omega}:\mathbb{R}^n)$, where γ is a binormal vector of M_t. Hence if $M_t\perp\partial\Omega$, then $\int_{\partial M_t} g\cdot\gamma\,d\sigma = 0$ for any vector field g, which satisfies $g(x)\in\operatorname{Tan}(\partial\Omega,x)$ for all $x\in\partial\Omega$. Therefore $\|\delta V_t\lfloor_{\partial\Omega}^{\top}\| \ll \|V_t\|$.

How to prove Brakke's inequality ?

Let $\phi\in C^\infty(\overline{\Omega})$ be a non-negative test function with $\nabla\phi\cdot\nu\equiv 0$ and let

$$d\xi_t^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon(x,t)|^2 - \frac{W(u^\varepsilon(x,t))}{\varepsilon}\right)dx.$$

Then we get

$$\begin{aligned} \frac{d}{dt}\int_\Omega \phi\,d\mu_t^\varepsilon = &-\int_\Omega \varepsilon\phi\left(-\Delta u^\varepsilon + \frac{W'(u^\varepsilon)}{\varepsilon}\right)^2 dx \\ &-\int_\Omega (D^2\phi : I-\vec{n}_\varepsilon\otimes\vec{n}_\varepsilon)\,d\mu_t^\varepsilon \\ &+\int_\Omega (D^2\phi:\vec{n}_\varepsilon\otimes\vec{n}_\varepsilon)\,d\xi_t^\varepsilon \\ &+\int_{\partial\Omega}(\nabla\phi\cdot\nu)\left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon}\right)d\sigma. \\ =:&\ I_1^\varepsilon(t) + I_2^\varepsilon(t) + I_3^\varepsilon(t) + I_4^\varepsilon(t). \end{aligned}$$

We may obtain for almost all $t\geq 0$

- $\limsup_{\varepsilon\to 0} I_1^\varepsilon(t) \leq -\int_\Omega \phi|h|^2\,d\|V_t\|;$
- $I_2^\varepsilon(t) = -\delta V_t^\varepsilon(\nabla\phi) \to -\delta V_t(\nabla\phi) = \int_\Omega \nabla\phi\cdot h\,d\|V_t\|;$
- $d\xi_t^\varepsilon dt \rightharpoonup 0$, hence $\int_{t_1}^{t_2} I_3^\varepsilon(t)\,dt \to 0;$
- $I_4^\varepsilon(t)\equiv 0$ since $\nabla\phi\cdot\nu = 0$ on $\partial\Omega$.

Reference

M. Mizuno and Y. Tonegawa, SIAM J. Math. Anal. **47** (2015), 1906–1932.