A 2-factor in which each cycle contains a vertex in a specified stable set

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Abstract

Let G be a graph with order n, and let k be an integer with $1 \leq k \leq n/3$. In this article, we show that if $\sigma_2(G) \geq n+k-1$, then for any stable set $S \subseteq V(G)$ with $|S| = k$, there exists a 2-factor with precisely k cycles $C_1, \ldots, C_k$ such that $|V(C_i) \cap S| = 1$ for all $1 \leq i \leq k$ and at most one of the cycles C_i has length strictly greater than three. The lower bound on σ_2 is best possible.

1 Introduction

All graphs considered are simple and finite. We refer to the number of vertices of G as the *order* of G and denote it by $|G|$. If there is no ambiguity, we let n denote the order of the graph G under consideration. A 2-factor is a spanning subgraph in which every component is a cycle. Let $H_1, H_2, \ldots, H_p$ be pairwise vertex-disjoint subgraphs of G, i.e., $V(H_i) \cap V(H_j) = \emptyset$ for all $i \neq j$. In this article, we always omit the word "pairwise" and simply say that $H_1, \ldots, H_p$ are vertex-disjoint. Notation and terminology not explained in this article can be found in [2].

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Ore [8] proved that a graph G of order $n \geq 3$ with $\sigma_2(G) := \min\{d(x)+d(y) | x \neq y, xy \notin E(G)\} \geq n$ is hamiltonian and, as an extension of it, Brandt et al. [1] showed that a graph G with $\sigma_2(G) \geq n$ has a 2-factor with precisely k cycles for any integer $k \leq n/4$. Furthermore, if the minimum degree is at least $n/2$, then for any set S of $k(\leq (n+3)/6)$ vertices, G contains a 2-factor with precisely k cycles each of which contains a vertex in S (see [4]). However, the natural σ_2-version of this statement does not hold. Let

$$H = K_{2k-1} + (K_k \cup K_{n-(3k-1)}) \text{ and } S = V(K_k) \tag{1.1}$$

(here K_m denotes the complete graph of order m and, for two graphs G_1, G_2 with $V(G_1) \cap V(G_2) = \emptyset$, we let $G_1 \cup G_2$ denote the union of G_1 and G_2, and let $G_1 + G_2$ denote the join of G_1 and G_2, i.e., the graph obtained from $G_1 \cup G_2$ by joining each vertex in $V(G_1)$ to all vertices in $V(G_2)$). Then it is easy to check that $\sigma_2(H) = n + 2(k-1) - 1$ and there is no desired 2-factor. But this is the upper bound of σ_2 for graphs which do not have such a 2-factor. Actually, a much stronger fact holds.

Theorem A ([6]) *Let G be a graph with order n, let k be an integer with $2 \leq k \leq (n+1)/4$, and suppose that $\sigma_2(G) \geq n + 2(k-1)$. Then for any independent edges $e_1, e_2, \ldots, e_k$, there exists a 2-factor with precisely k cycles $C_1, C_2, \ldots, C_k$ such that $e_i \in E(C_i)$ for all $1 \leq i \leq k$.*

The lower bound on σ_2 is best possible. This can be seen from (1.1) by letting $e_1, e_2, \ldots, e_k$ be independent edges joining the K_{2k-1} part and the K_k part.

Ishigami and Wang [7] gave an alternative proof of Theorem A by showing that if G is a graph with order n, k is an integer with $2 \leq k \leq (n+1)/4$, and $\sigma_2(G) \geq n + 2(k-1)$, then for any independent edges $e_1, e_2, \ldots, e_k$, there exists a 2-factor with precisely k cycles $C_1, C_2, \ldots, C_k$ such that $e_i \in E(C_i)$ for all $1 \leq i \leq k$ and at most one of the cycles C_i has length strictly greater than four, unless $\overline{K_{2k}} + (K_p \cup K_{n-(2k+p)}) \subseteq G \subseteq K_{2k} + (K_p \cup K_{n-(2k+p)})$ for some integer p $(2(k-1) < p < n - 4(k-1) - 2)$.

We have already mentioned that (1.1) shows that even for a specified vertex set, the lower bound $n + 2(k-1)$ on σ_2 is best possible. However, Dong showed that the situation is different if we assume that the specified set S is stable, i.e., $xy \notin E(G)$ for any $x, y \in S$. He proved the following three theorems.

Theorem B (Dong [3]) *Let G be a graph of order n, and let k be an integer with $1 \leq k \leq n/3$. Suppose that $\sigma_2(G) \geq n + k - 1$, and let S be a stable set of vertices* *with* $|S| = k$. *Then* G *has a 2-factor consisting of precisely* k *cycles* $C_1, C_2, \ldots, C_k$ *such that* $|V(C_i) \cap S| = 1$ *for all* $1 \le i \le k$ *and* $|C_i| \le 4$ *for all* $1 \le i \le k-1$.

Theorem C (Dong [3]) *Let* G *be a graph of order* n, *and let* k *be an integer with* $1 \le k \le n/3$. *Suppose that* $\sigma_2(G) \ge n+k-1$, *and let* S *be a stable set of vertices with* $|S| = k$. *Then there exist* k *vertex-disjoint cycles* $C_1, \ldots, C_k$ *such that* $|V(C_i) \cap S| = 1$ *and* $|C_i| \le 4$ *for all* $1 \le i \le k$.

Theorem D (Dong [4]) *Let* G *be a graph of order* n, *and let* k *be an integer with* $1 \le k \le n/3$. *Suppose that* $\sigma_2(G) \ge n+k-1$, *and let* S *be a stable set of vertices with* $|S| = k$. *Suppose further that there exist vertex-disjoint triangles* $D_1, \ldots, D_k$

$$\text{such that } |V(D_i) \cap S| = 1 \text{ for all } 1 \le i \le k. \tag{1.2}$$

Then G *has a 2-factor consisting of precisely* k *cycles* $C_1, \ldots, C_k$ *such that* $|V(C_i) \cap S| = 1$ *for all* $1 \le i \le k$ *and* $|C_i| = 3$ *for all* $1 \le i \le k-1$.

In Theorems B and D, the lower bound on σ_2 is best possible. To see this, let $H = \overline{K_k} + (K_1 \cup K_{n-k-1})$ and $S = V(\overline{K_k})$. Then $\sigma_2(H) = n+k-2$, but there is no desired 2-factor.

The purpose of this article is to prove a result which is a common refinement of Theorems B and C and, at the same time, implies that the conclusion of Theorem D holds even if we drop the assumption (1.2). Specifically, we prove the following theorem.

Theorem 1 *Let* G *be a graph of order* n, *and let* k *be an integer with* $1 \le k \le n/3$. *Suppose that* $\sigma_2(G) \ge n+k-1$, *and let* S *be a stable set of vertices with* $|S| = k$. *Then one of the following holds:*

(i) *there exist* k *vertex-disjoint triangles* $C_1, \ldots, C_k$ *such that* $|V(C_i) \cap S| = 1$ *for all* $1 \le i \le k$; *or*

(ii) *there exist* $k-1$ *vertex-disjoint triangles* $C_1, \ldots, C_{k-1}$ *such that* $|V(C_i) \cap S| = 1$ *for all* $1 \le i \le k-1$, *and such that if we let* $H = G - \bigcup_{1 \le i \le k-1} V(C_i)$ *and write* $S \cap V(H) = \{v_0\}$, *then* $|H| \ge 4$, $d_H(x) \ge 2$ *for all* $x \in V(H)$, *and* H *contains a vertex* a *with* $a \ne v_0$ *which has the property that* $d_H(x) + d_H(y) \ge |H|$ *for any* $x, y \in V(H) \setminus \{a\}$ *with* $x \ne y$ *and* $xy \notin E(H)$.

In Thereom 1, the lower bound on σ_2 is best possible. Assume that $n+k$ is even, and let $G' = K_{k-2} + K_{(n-k+2)/2,(n-k+2)/2}$ (here $K_{l,m}$ denotes the complete bipartite graph with partite sets having cardinalities l and m). Then $\sigma_2(G') = n+k-2$, and G' does not contain $k-1$ vertex-disjoint triangles. Thus neither (i) nor (ii) holds.

In view of Theorem D, we obtain the following two corollaries as consequences of Thereom 1 (see Section 3). Note that Corollaries 2 and 3 are refinements of Theorems B and C, respectively, and Corollary 2 also shows that in Theorem D, the assumption (1.2) is not necessary.

Corollary 2 *Let G be a graph of order n, and let k be an integer with $1 \le k \le n/3$. Suppose that $\sigma_2(G) \ge n+k-1$, and let S be a stable set of vertices with $|S| = k$. Then G has a 2-factor consisting of precisely k cycles $C_1, \dots, C_k$ such that $|V(C_i) \cap S| = 1$ for all $1 \le i \le k$ and $|C_i| = 3$ for all $1 \le i \le k-1$.*

Corollary 3 *Let G be a graph of order n, and let k be an integer with $1 \le k \le n/3$. Suppose that $\sigma_2(G) \ge n+k-1$, and let S be a stable set of vertices with $|S| = k$. Then there exist k vertex-disjoint cycles $C_1, \dots, C_k$ such that $|V(C_i) \cap S| = 1$ for all $1 \le i \le k$, $|C_i| = 3$ for all $1 \le i \le k-1$, and $|C_k| = 3$ or 4.*

We establish Theorem 1 in Section 2 by proving the following two propositions (note that the graph H in Proposition 4 (ii) satisfies the conditions stated in (ii) of Theorem 1).

Proposition 4 *Let G be a graph of order n, and let k be an integer with $1 \le k \le n/3$. Suppose that $\sigma_2(G) \ge n+k-1$, and let S be a stable set of vertices with $|S| = k$. Suppose further that each $v \in S$ is contained in a triangle. Then one of the following holds:*

(i) *there exist k vertex-disjoint triangles $C_1, \dots, C_k$ such that $|V(C_i) \cap S| = 1$ for all $1 \le i \le k$; or*

(ii) *$n+k$ is odd, $d(v) = (n+k-1)/2$ for all $v \in S$, and there exist $k-1$ vertex-disjoint triangles $C_1, \dots, C_{k-1}$ such that $|V(C_i) \cap S| = 1$ for all $1 \le i \le k-1$, and such that if we let $H = G - \bigcup_{1 \le i \le k-1} V(C_i)$, then $|H| \ge 4$ and H contains a spanning subgraph isomorphic to $K_{(n-3(k-1))/2,(n-3(k-1))/2}$.*

Proposition 5 *Let G be a graph of order n, and let k be an integer with $1 \le k \le n/3$. Suppose that $\sigma_2(G) \ge n+k-1$, and let S be a stable set of vertices with $|S| = k$. Suppose further that there exists $v_0 \in S$ such that v_0 is not contained in a triangle. Then there exist $k-1$ vertex-disjoint triangles $C_1, \dots, C_{k-1}$ such that*

$|V(C_i) \cap S| = 1$ *for all* $1 \le i \le k-1$, *and such that if we let* $H = G - \bigcup_{1 \le i \le k-1} V(C_i)$, *then* $|H| \ge 4$, $d_H(x) \ge 2$ *for all* $x \in V(H)$, *and* H *contains a vertex* a *with* $a \ne v_0$ *which has the property that* $d_H(x) + d_H(y) \ge |H|$ *for any* $x, y \in V(H) \setminus \{a\}$ *with* $x \ne y$ *and* $xy \notin E(H)$.

In the rest of this section, we prepare notations which we use in subsequent sections. The set of all neighbours of a vertex x in a graph G is denoted by $N_G(x)$, or simply by $N(x)$, and its cardinality is denoted by $d_G(x)$ or $d(x)$. For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. For simplicity, we denote $|V(H)|$ by $|H|$, and $G - V(H)$ by $G - H$. Also we write "$u \in H$" to mean that $u \in V(H)$.

2 Proof of Propositions

We first prove Proposition 4. Let n, k, G, S be an in Proposition 4. We proceed by induction on k. If $k = 1$, (i) clearly holds. Thus let $k \ge 2$, and assume that the proposition holds for $k - 1$. We may assume (i) does not hold. Let S' be a subset of S with cardinality $k - 1$. Note that if $k \ge 3$, then by the assumption that $\sigma_2(G) \ge n+k-1$, it is not possible that $d(v) = (n+(k-1)-1)/2$ for all $v \in S'$, and hence it follows from the induction assumption that there exist $k - 1$ vertex-disjoint triangles $C_1, \dots, C_{k-1}$ such that $|V(C_i) \cap S'| = 1$ for all $1 \le i \le k - 1$; if $k = 2$, then $|S'| = 1$, and hence there exists a triangle C_1 such that $|V(C_1) \cap S'| = |S'| = 1$. Write $S = \{v_1, \dots, v_k\}$ so that $d(v_1) \le d(v_2) \le \cdots \le d(v_k)$. Note that if there exists $v \in S$ with $v \ne v_1$ such that $d(v) = (n + k - 1)/2$, then we also have $d(v_1) = (n + k - 1)/2$ by the assumption that $\sigma_2(G) \ge n - k + 1$. Thus the proposition follows if we prove the following lemma.

Lemma 2.1 *Let* $n, k, G, S, v_1, \dots, v_k$ *be as above, and suppose that* (i) does not hold. Fix i_0 with $2 \le i_0 \le k$, and set $S' = S \setminus \{v_0\}$. Further let $C_1, \dots, C_{k-1}$ be vertex-disjoint triangles such that $|V(C_i) \cap S'| = 1$ for all $1 \le i \le k - 1$, and set $H = \bigcup_{1 \le i \le k-1} C_i$. Then $n+k$ is odd, $d(v_{i_0}) = (n+k-1)/2$, $|H| \ge 4$, and H contains a spanning subgraph isomorphic to $K_{(n-3(k-1))/2,(n-3(k-1))/2}$.

Proof of Lemma 2.1. Recall that S is stable. Thus $d_{C_i}(v_{i_0}) \le 2$ for every $1 \le i \le k - 1$. Since $d_G(v_1) \le d_G(v_{i_0})$, we also have

$$d_G(v_{i_0}) \ge (n + k - 1)/2. \tag{2.1}$$

Hence

$$d_H(v_{i_0}) \geq d_G(v_{i_0}) - 2(k-1) \geq (n - 3(k-1))/2 = |H|/2. \tag{2.2}$$

In particular, $d_H(v_{i_0}) \geq 2$. Note that from the assumption that (i) does not hold, it follows that $N_H(v_{i_0})$ is stable. Hence

$$N_H(x) \cap N_H(v_{i_0}) = \emptyset \text{ for all } x \in N_H(v_{i_0}), \tag{2.3}$$

which implies

$$d_H(x) + d_H(v_{i_0}) \leq |H| \text{ for all } x \in N_H(v_{i_0}). \tag{2.4}$$

Take $x_1, x_2 \in N_H(x)$ with $x_1 \neq x_2$. If there exists i with $1 \leq i \leq k-1$ such that $d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(v_{i_0}) \geq 7$, then in the subgraph induced by $V(C_i) \cup \{v_{i_0}, x_1, x_2\}$, we can easily find two disjoint triangles C_i' and D such that $V(C_i') \cap S' = V(C_i) \cap S'$ and $v_{i_0} \in D$, which contradicts the assumption that (i) does not hold. Thus $d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(v_{i_0}) \leq 6$ for every $1 \leq i \leq k-1$. Consequently it follows from (2.1) that

$$\begin{aligned} d_H(x_1) + d_H(x_2) + d_H(v_{i_0}) &\geq \frac{3}{2}(n+k-1) - 6(k-1) \\ &= \frac{3}{2}(n - 3(k-1)) = \frac{3}{2}|H|. \end{aligned}$$

On the other hand, since it follows from (2.2) and (2.4) that $d_H(x_1) \leq |H|/2$, we get $d_H(x_1) + d_H(x_2) + d_H(v_{i_0}) \leq |H|/2 + |H|$ by (2.4). Since x_1 and x_2 are arbitrary, this means that equality holds in (2.2) and (2.4). Therefore $|H|$ is even, $d_H(v_{i_0}) = |H|/2$, and $d_H(x) = |H|/2$ for all $x \in N_H(v_{i_0})$. In view of (2.3), this implies that H contains a spanning subgraph isomorphic to $K_{|H|/2,|H|/2} \cong K_{(n-3(k-1))/2),(n-3(k-1))/2}$. Since $|H| = n - 3(k-1) \geq 3$ and $|H|$ is even, it follows that $|H| \geq 4$ and $n+k$ is odd. Finally the equality in (2.2) together with (2.1) implies $d_G(v_{i_0}) = (n+k-1)/2$.

Thus Lemma 2.1 is proved, and this completes the proof of Proposition 4.

We proceed to the proof of Proposition 5. Let n, k, G, S, v_0 be as in Proposition 5. If $k = 1$, then the proposition clearly holds because the assumption $\sigma_2(G) \geq n$ implies that $d(x) \geq 2$ for all $x \in G$. Thus assume $k \geq 2$. From the assumption that v_0 is not contained in a triangle, it follows that $N(v_0)$ is stable. Hence

$$d(x) + d(y) \geq n + k - 1 \text{ for all } x, y \in N(v_0) \text{ with } x \neq y. \tag{2.5}$$

In particular, there exists $a \in N_G(v_0)$ such that

$$d(x) \geq (n+k-1)/2 \text{ for all } x \in N(v_0) \setminus \{a\}. \tag{2.6}$$

Thus

$$d(v_0) \leq (n-(k-1))/2. \tag{2.7}$$

This implies that for each $v \in S \setminus \{v_0\}$, $d(v) \geq (n+3(k-1))/2$ and v is contained in a triangle. Hence applying Proposition 4 with k and S replaced by $k-1$ and $S \setminus \{v_0\}$, we see that there exist $k-1$ vertex-disjoint triangles $C_1, \ldots, C_{k-1}$ such that $|V(C_i) \cap S| = 1$ for all $1 \leq i \leq k-1$. We choose $C_1, \ldots, C_{k-1}$ so that the number p of edges joining v_0 and $\bigcup_{1 \leq i \leq k-1} V(C_i)$ is as large as possible. Set $H = G - \bigcup_{1 \leq i \leq k-1} C_i$. We have $|H| = n - 3(k-1) \geq 3$. By (2.7),

$$d_G(w) \geq \frac{n+3(k-1)}{2} \text{ for all } w \in V(G) \setminus N_G(v_0),$$

and hence

$$d_H(w) \geq d_G(w) - 3(k-1) \geq (n-3(k-1))/2 = |H|/2 \quad \text{for all } w \in V(H) \setminus N_H(v_0). \tag{2.8}$$

From the fact that $N_G(v_0)$ is stable, it follows that $|N_{C_i}(v_0)| \leq 1$ for every $1 \leq i \leq k-1$. Hence

$$d_H(v_0) + d_H(w) \geq \sigma_2(G) - 4(k-1) \geq n - 3(k-1) = |H| \quad \text{for all } w \in V(H) \setminus N_H(v_0). \tag{2.9}$$

Since $|H| \geq 3$, (2.9) in particular implies $d_H(v_0) \geq 2$. Take $x \in N_H(v_0)$. Suppose that there exists i with $1 \leq i \leq k-1$ such that $d_{C_i}(x) = 3$. Write $C_i = vu_1u_2$ with $v \in S$. Since $N_G(v_0)$ is stable, we have $d_{C_i}(v_0) = 0$. But then replacing C_i by the triangle vu_1x, we get a contradiction to that maximality of p. Thus $d_{C_i}(x) \leq 2$ for each $x \in N_H(v_0)$ and each $1 \leq i \leq k-1$. Therefore it follows from (2.5) and (2.6) that

$$d_H(x) + d_H(y) \geq d_G(x) + d_G(y) - 4(k-1) \geq n - 3(k-1) = |H| \quad \text{for all } x, y \in N_H(v_0) \text{ with } x \neq y, \tag{2.10}$$

and

$$d_H(x) \geq d_G(x) - 2(k-1) \geq (n-3(k-1))/2 = |H|/2 \quad \text{for all } x \in N_H(v_0) \setminus \{a\} \tag{2.11}$$

(it is possible that $a \notin H$). Recall that $d_H(v_0) \geq 2$. Thus (2.10) in particular implies that $d_H(x) \geq 2$ for all $x \in N_H(v_0)$. Consequently we see from (2.8) that $d_H(x) \geq 2$ for all $x \in V(H)$. Since v_0 is not contained in a triangle, this implies $|H| \geq 4$. Finally, combining (2.8), (2.9) and (2.11), we see that $d_H(x) + d_H(y) \geq |H|$ for any $x, y \in V(H) \setminus \{a\}$ with $x \neq y$ and $xy \notin E(G)$.

This completes the proof of Proposition 5.

3 A Lemma

For completeness, we here include the proof of the following lemma, which shows that Theorem 1 implies Corollaries 2 and 3.

Lemma 3.1 *Let H be a graph such that $|H| \geq 4$, and $d_H(x) \geq 2$ for all $x \in H$. Let $a \in H$, and suppose that $d_H(x) + d_H(y) \geq |H|$ for any $x, y \in V(H) \setminus \{a\}$ with $x \neq y$ and $xy \notin E(H)$. Then the following hold.*

(1) *H is hamiltonian.*

(2) *For each $v \in V(H) \setminus \{a\}$, there exists a cycle C such that $v \in C$ and $|C| = 4$.*

Proof. We first prove (1). Take a path P such that $a \in P$ and a is not an endvertex of P. We choose P so that $|P|$ is as large as possible. Write $P = x_1 x_2 \dots x_l$. Then $N_H(x_1), N_H(x_l) \subseteq V(P)$. This implies that if $x_1 x_l \notin E(H)$, then there exists i with $2 \leq i \leq l$ such that $x_{i-1} x_l, x_1 x_i \in E(H)$. Thus H contains a cycle D with $V(D) = V(P)$. Since H is connected by the assumption of the lemma, it follows from the maximality of $|P|$ that $V(P) = V(H)$, and hence D is a hamiltonian cycle of H, as desired. We now prove (2). If $|H| = 4$, the desired conclusion follows from (1). Thus we may assume $|H| \geq 5$. Let $v \in V(H) \setminus \{a\}$. First assume $d_H(v) \leq |H| - 3$, and take $x \in V(H) \setminus (\{v\} \cup N_H(v) \cup \{a\})$. Then $d_H(v) + d_H(x) \geq |H|$, which implies $|N_H(v) \cap N_H(x)| \geq 2$. Hence v, x and two vertices in $N_H(v) \cap N_H(x)$ form a cycle with the desired properties. Next assume $d_H(v) = |H| - 2$, and write $V(H) \setminus (\{v\} \cup N_H(v)) = \{x\}$. Then $|N_H(v) \cap N_H(x)| = |N_H(x)| \geq 2$, and hence we can again find a desired cycle. Finally assume $d_H(v) = |H| - 1$. Then $|N_H(v) - \{a\}| = |H| - 2 \geq 3$. Hence if $N_H(v) - \{a\}$ induces a complete graph, then the desired conclusion clearly holds. Thus we may assume there exist $x, y \in N_H(v) - \{a\}$ with $x \neq y$ such that $xy \notin E(H)$. Then $|N_H(x) \cap N_H(y)| \geq 2$. Consequently v, x, y and a vertex in $(N_H(x) \cap N_H(y)) \setminus \{v\}$ form a desired cycle.

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