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Spanning Even Subgraphs of 3-edge-connected Graphs

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Abstract

By Petersen's theorem, a bridgeless cubic graph has a 2-factor. H. Fleischner extended this result to bridgeless graphs of minimum degree at least three by showing that every such graph has a spanning even subgraph. Our main result is that, under the stronger hypothesis of 3-edge-connectivity, we can find a spanning even subgraph in which every component has at least five vertices. We show that this is in some sense best possible by constructing an infinite family of 3-edge-connected graphs in which every spanning even subgraph has a 5-cycle as a component.

1 Introduction

A classical result of Petersen [9] is that every bridgeless cubic graph has a 2-factor. This result has been extended in many directions. A related question of Thomassen, see [7], is whether there exists a positive integer k such that every cyclically k-edge-connected cubic graph has a connected 2-factor i.e. a Hamilton cycle. (The Coxeter graph shows that we must take $k \geq 8$ to have an affirmative answer to this question.) We will consider the weaker property of having a 2-factor which contains no short cycles. We show that every 3-edge-connected cubic graph has a 2-factor in which all cycles have length at least five. We also show that our result is best possible by constructing an infinite family of cyclically 4-edge-connected cubic graphs in which every 2-factor has a cycle of length five.

[1]This research was carried out while the second author was visiting Queen Mary, University of London.

We shall in fact consider a more general problem. Fleischner [3] extended the above mentioned result of Petersen to bridgeless graphs of minimum degree at least three, by showing that every such graph has a spanning even subgraph i.e. a spanning subgraph in which each vertex has positive even degree. Jaeger [5] showed that every 4-edge-connected graph has a connected spanning even subgraph. Zhan [11] showed that the same conclusion holds for 3-edge-connected, essentially 7-edge-connected graphs and Chen and Lai [1] conjecture that this result can be extended to 3-edge-connected, essentially 5-edge-connected graphs. We will be concerned with the weaker property of having a spanning even subgraph which has no small components. In this context, we proved the following result in [4].

Theorem 1. *Every bridgeless simple graph G with minimum degree at least three has a spanning even subgraph in which each component has at least four vertices.*

The same conclusion need not hold for graphs which are not simple. Consider a bridgeless graph H with minimum degree at least 3, which contains a 3-edge cut $\{e_1, e_2, e_3\}$. Let G be obtained from H by inserting either a vertex incident to a loop, or two vertices joined by a multiple edge, or a triangle with one edge replaced by a multiple edge, into each edge e_i, $1 \leq i \leq 3$, see Figure 1. Then every spanning

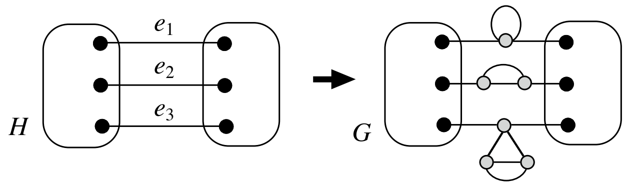

Figure 1:

even subgraph of G contains at least one of the inserted loops, multiple edges, or triangles.

We will show, however, that Theorem 1 can be strengthened when we consider 3-edge-connected graphs.

Theorem 2. *Let G be a 3-edge-connected graph with n vertices. Then G has a spanning even subgraph in which each component has at least* $\min\{n, 5\}$ *vertices.*

The Petersen graph is an example of a 3-edge-connected, essentially 4-edge-connected graph in which every spanning even subgraph has a component with five vertices. We give an infinite family of such graphs in Section 4.

2 Notation and Preliminary Results

All graphs considered are finite and may contain loops and multiple edges. We refer to graphs without loops and multiple edges as *simple graphs*. A graph is said to be *even* if every vertex has positive even degree. All notation and terminology not explained in this paper is given in [2].

The set of neighbours of a vertex x in a graph G is denoted by $N_G(x)$, or simply $N(x)$, and the degree of x by $d_G(x)$, or $d(x)$. The set of edges incident to x is denoted by $E(x)$. For a connected subgraph H of G, we denote by G/H the graph obtained from G by contracting every edge in H and use $[H]$ to denote the vertex of G/H corresponding to H. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. We refer to the number of vertices in a graph as its *order*. We consistently use n to denote the order of a graph G and extend this notation using subscripts and superscripts. Thus we denote the order of a graph G_1' by n_1'. We use $\sigma(G)$ to represent the minimum order of a component of G.

An edge-cut S in a graph G is said to be *essential*, or *cyclic*, if at least two components of $G - S$ contain edges, respectively cycles. The graph G is *essentially k-edge-connected*, or *cyclically k-edge-connected,* if all essential, respectively cyclic, edge-cuts of G have at least k edges.

Given two distinct edges $e_1 = vx_1, e_2 = vx_2$ incident to a vertex v in a graph G, let $G_v^{e_1,e_2}$ be the graph obtained from $G - \{e_1, e_2\}$ by adding a new vertex v' and new edges x_1v' and x_2v'. We say that $G_v^{e_1,e_2}$ has been obtained by *splitting* the vertex v. We will abuse notation somewhat by labeling the edges x_1v' and x_2v' as e_1 and e_2, respectively, so that $E(G_v^{e_1,e_2}) = E(G)$. We will need the following result on splitting in k-edge-connected graphs due to Mader [8, Theorem 10].

Theorem 3 ([8])**.** *Let G be a k-edge-connected graph, $v \in V(G)$ with $d(v) \geq k + 2$.*

Then there exist edges $e_1, e_2 \in E(v)$ such that $G_v^{e_1,e_2}$ is homeomorphic to a k-edge-connected graph.

3 Even Subgraphs

We first prove a slight strengthening of the result of Fleischner mentioned in the Introduction.

Theorem 4. *Suppose G is a bridgeless graph with $\delta(G) \geq 3$ and $f_1, f_2 \in E(G)$. Then G has a spanning even subgraph X with $f_1, f_2 \in E(X)$.*

Proof. We proceed by contradiction. Suppose the theorem is false and choose a counterexample G such that $\Delta = \Delta(G)$ is as small as possible and, subject to this condition, the number of vertices of G of degree Δ is as small as possible. Clearly G is 2-edge-connected.

We first show that G is cubic. Suppose $\Delta \geq 4$ and choose a vertex $v \in V$ with $d(v) = \Delta$. By Theorem 3 we can choose two edges $e_1 = x_1v, e_2 = x_2v \in E$ incident to v such that the graph $G_v^{e_1,e_2}$ is 2-edge-connected, see Figure 2(i). Thus

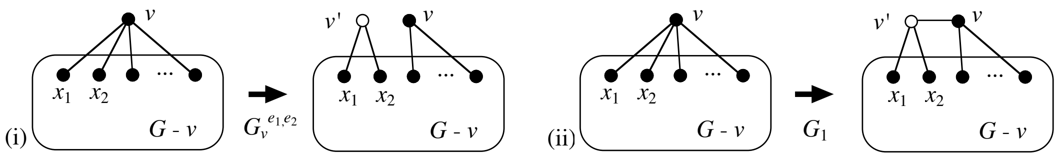

Figure 2:

the graph G_1 obtained from $G_v^{e_1,e_2}$ by adding the new edge vv' is 2-edge-connected, see Figure 2(ii). By induction G_1 has a spanning even subgraph X_1 containing f_1, f_2. If $vv' \notin E(X_1)$, then $x_1v', x_2v' \in E(X_1)$ and we let $X = (X_1 - v') + \{x_1v, x_2v\}$. On the other hand, if $vv' \in E(X_1)$, then relabelling if necessary, we have $x_1v' \in E(X_1)$ and $x_2v' \notin E(X_1)$ and we let $X = X_1 - v' + x_1v$. In both cases X is a spanning even subgraph of G containing f_1, f_2. This contradicts the choice of G.

Thus G is cubic. By a well known strengthening of Petersen's Theorem, see for example Plesník [10], G has a 2-factor containing f_1, f_2. This again contradicts the choice of G. $\square$

Notice that we cannot obtain a similar strengthening of Theorem 2. In the graphs drawn in Figure 3, every spanning even subgraph which contains e_1, e_2 has a 4-cycle as a component. (We know of no other example of a 3-connected graph G of order at least five and edges e_1, e_2 with the property that all spanning even subgraphs of G which contain e_1, e_2 have a component of order at most four.)

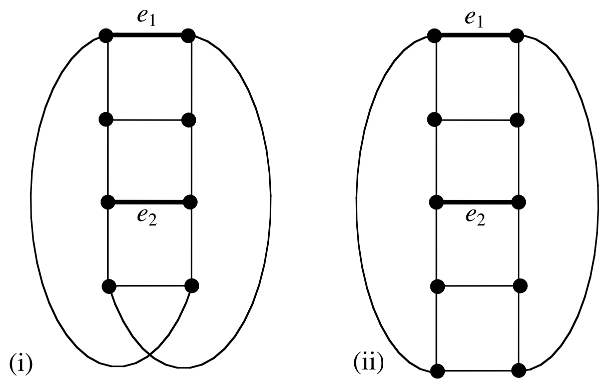

Figure 3:

We will show, however, that we can find an even subgraph X with $\sigma(X) \geq 5$ which contains two specified edges e_1, e_2 in a 3-connected graph G as long as e_1, e_2 are incident to a common vertex of degree three. Indeed, we need this stronger statement for our inductive proof.

Theorem 5. *Let G be a 3-edge-connected graph with n vertices, u_2 be a vertex of G with $d(u_2) = 3$, and $e_1 = u_1u_2, e_2 = u_2u_3$ be edges of G. (We allow the possibility that $u_1 = u_3$.) Then G has a spanning even subgraph X with $\{e_1, e_2\} \subset E(X)$ and $\sigma(X) \geq \min\{n, 5\}$.*

Proof. Suppose the theorem is false and choose a counterexample G such that:
(a) $\Delta = \Delta(G)$ is as small as possible;
(b) subject to (a), the number of vertices of degree Δ in G is as small as possible;
(c) subject to (a) and (b), $|E(G)|$ is as small as possible.

Claim 1. $\Delta \leq 4$.

Proof. Suppose $\Delta \geq 5$ and let x be a vertex with $d(x) = \Delta$. By Theorem 3, there exist two edges $f_1 = xy_1, f_2 = xy_2 \in E(x)$ such that the graph $G' = G - \{f_1, f_2\} + y_1y_2$ is 3-edge-connected. Note that, since $d(u_2) = 3$, $u_2 \neq x$. Furthermore, since G' is 3-edge-connected, we cannot have $y_1 = y_2 = u_2$ and $u_1 = u_3 = x$ so $\{e_1, e_2\} \neq \{f_1, f_2\}$. Relabelling if necessary, we may suppose that $e_1 \notin \{f_1, f_2\}$. Let $e_2' = y_1y_2$ if $e_2 \in \{f_1, f_2\}$, and otherwise let $e_2' = e_2$. By induction, G' has a spanning even subgraph X' such that $\{e_1, e_2'\} \subset E(X')$ and $\sigma(X') \geq \min\{n', 5\}$. Then X' readily gives rise to the required even subgraph of G. □

Claim 2. *G is essentially 4-edge-connected.*

Proof. Suppose that $\{f_1, f_2, f_3\}$ is an essential 3-edge-cut in G. Let G_1', G_2' be the two components of $G - \{f_1, f_2, f_3\}$ and let $G_1 = G/G_2'$ and $G_2 = G/G_1'$. We denote by f_i^j the edge in G_j corresponding to f_i for $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

By symmetry, we may assume that $u_2 \in V(G_1')$. Let e_1^1, e_2^1 be the edges of G_1 corresponding to e_1, e_2, respectively. By induction, G_1 has a spanning even subgraph X_1 such that $\{e_1^1, e_2^1\} \subset E(X_1)$ and $\sigma(X_1) \geq \min\{n_1, 5\}$. By symmetry, we may suppose that:

$$E(X_1) \cap \{f_1^1, f_2^1, f_3^1\} = \{f_1^1, f_2^1\}.$$

By induction, G_2 has a spanning even subgraph X_2 such that $\{f_1^2, f_2^2\} \subset E(X_2)$ and $\sigma(X_2) \geq \min\{n_2, 5\}$. Then $((X_1 - [G_2']) \cup (X_2 - [G_1']) + \{f_1, f_2\}$ is the required spanning even subgraph of G. □

Claim 3. *No edge of G is incident to two vertices of degree four.*

Proof. Suppose there is an edge $f = xy$ incident to two vertices of degree four. Then $G_1 = G - f$ is 3-edge-connected by Claim 2. Since $d(u_2) = 3$, $f \notin \{e_1, e_2\}$. By induction, G_1 has a spanning even subgraph X such that $\{e_1, e_2\} \subset E(X)$ and $\sigma(X) \geq \min\{n_1, 5\}$. Then X is the required subgraph of G. □

Claim 4. *G is simple and hence $u_1 \neq u_3$.*

Proof. This follows easily from Claims 1, 2 and 3. □

Claim 5. *Let x be a vertex of G of degree four and $f_1, f_2 \in E(x)$. Then the graph G' obtained from $G_x^{f_1,f_2}$ by adding the edge xx' is 3-edge-connected.*

Proof. This follows easily from Claim 2. □

Claim 6. *G is cubic.*

Proof. Suppose that G has a vertex x of degree four. Let $N(x) = \{z_1, z_2, z_3, z_4\}$ and $f_i = xz_i$. Let G' be the graph obtained from $G_x^{f_2,f_3}$ by adding the edge xx'. See Figure 4(i),(ii). Since $d(u_2) = 3$, $u_2 \neq x$. By induction, G' has a spanning

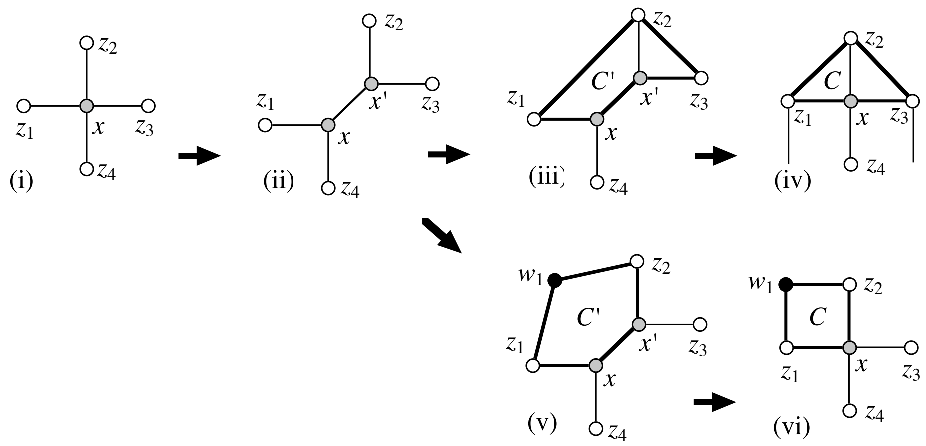

Figure 4:

even subgraph X' such that $\{e_1, e_2\} \subset X'$ and $\sigma(X') \geq 5$. Then $xx' \in E(X')$; otherwise X' gives rise to the required subgraph of G. Let C' be the component of X' passing through xx'. Since $X = X'/xx'$ is a spanning even subgraph of G containing $\{e_1, e_2\}$, $C = C'/xx'$ has exactly four vertices; otherwise X would be the required subgraph of G. Since G is simple, C is a 4-cycle.

Suppose C contained three vertices in $N(x)$, say z_1, z_2, z_3. Then, since each neighbour of x has degree three by Claim 3, the edges joining C and $G - C$ form a 3-edge-cut of G. See Figure 4(iii)-(iv). Claim 2 now implies that $G - C$ has exactly one vertex, and hence G is a wheel on five vertices. Since the theorem holds for the wheel on five vertices this gives a contradiction.

Thus C contains exactly two vertices in $N(x)$, one from $\{z_1, z_4\}$ and one from $\{z_2, z_3\}$. Relabelling if necessary we may suppose that $C = xz_1w_1z_2x$. See Figure 4(v)-(vi). Since $\{e_1, e_2\} \subset E(X)$ we have,

$$\{e_1, e_2\} \subset E(C) \text{ or } \{e_1, e_2\} \cap E(C) = \emptyset. \qquad (1)$$

Let G'' be the graph obtained from $G_x^{f_3,f_4}$ by adding the edge xx'', where x'' is the vertex of degree two which is 'split' from x in $G_x^{f_3,f_4}$. See Figure 5(i). We may

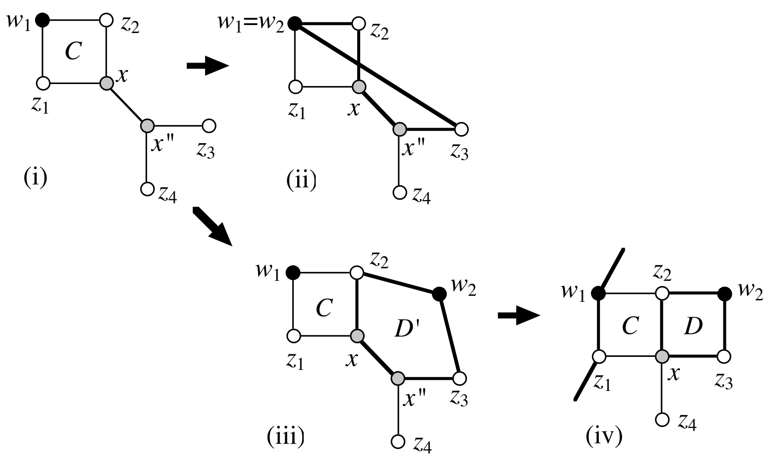

Figure 5:

apply the above argument to G'', and relabel z_1 and z_2, and z_3 and z_4 if necessary, to deduce that G has a spanning even subgraph Y with $\{e_1, e_2\} \subset E(Y)$, and such that $D = xz_2w_2z_3x$ is a component of Y. If $w_1 = w_2$, then since $z_1x, z_1w_1 \notin E(Y)$ and $d_G(z_1) = 3$ we would have $z_1 \notin V(Y)$, see Figure 5(ii). This would contradict the fact that Y is a spanning even subgraph of G. Thus $w_1 \neq w_2$. See Figure 5(iii). Since $z_1x, z_2w_1 \notin E(Y)$ we have $\{z_1x, z_2w_1\} \cap \{e_1, e_2\} = \emptyset$. Now (1) implies that $E(C) \cap \{e_1, e_2\} = \emptyset$. Since $d_G(z_1) = 3$ and $z_1x \notin E(Y)$, the component of Y containing z_1 passes through the edge z_1w_1. See Figure 5(iv). Hence $Y - \{z_1w_1, z_2x\} + \{w_1z_2, z_1x\}$ is the required even subgraph of G. □

Claim 7. *G is triangle-free.*

Proof. This follows immediately from Claims 2 and 6. □

Claim 8. *G contains no 4-cycles.*

Proof. Suppose $C = x_1x_2x_3x_4x_1$ is a 4-cycle in G. For $1 \leq i \leq 4$, let y_i be the neighbour of x_i in $G - C$. Let $G^* = G - \{x_3, x_4\} + \{x_1y_3, x_2y_4\}$. See Figure 7(i),(ii). We abuse notation somewhat by labeling the edges x_1y_3 and x_2y_4 in G^* with the same labels as x_3y_3 and x_2y_2, respectively, in G. Thus $E(G^*) \subseteq E(G)$.

Suppose G^* has a 2-edge-cut $\{e, f\}$. If $x_1x_2 \notin \{e, f\}$ then $\{e, f\}$ would be a 2-edge-cut of G and would contradict the hypothesis that G is 3-edge-connected. Relabeling if necessary, we may suppose that $e = x_1x_2$ and $f = z_1z_2$. See Figure 6. By Claim 2, neither $\{x_1y_1, x_3y_3, f\}$ nor $\{x_2y_2, x_4y_4, f\}$ are essential 3-edge-cuts of

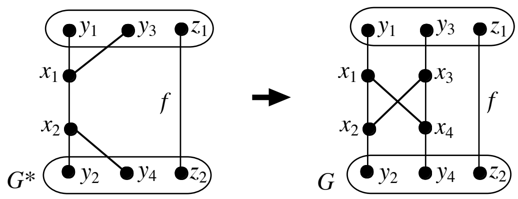

Figure 6:

G. This implies that $y_1 = y_3 = z_1$, $y_2 = y_4 = z_2$, and hence that G is isomorphic to the complete bipartite graph $K_{3,3}$. Since the theorem holds for $K_{3,3}$, this gives a contradiction.

Thus G^* is 3-edge-connected. Consider the following three cases.

Case 1 $E(C) \cap \{e_1, e_2\} = \emptyset$.

By induction, G^* has a 2-factor F^* such that $\{e_1, e_2\} \subset F^*$ and $\sigma(F^*) \geq \min\{n^*, 5\}$.

Suppose F^* passes through the edge x_1x_2. If F^* contains x_1y_1, x_2y_2, then the set of edges $(E(F^*) - \{x_1x_2\}) \cup \{x_1x_4, x_4x_3, x_3x_2\}$ induces the required 2-factor of G. See Figure 7(iii),(iv). A similar contradiction can be obtained if F^* contains x_1y_1, x_2y_4, or x_1y_3, x_2y_2, or x_1y_3, x_2y_4.

Thus $x_1x_2 \notin E(F^*)$. Let F be the subgraph of G induced by $E(F^*) \cup \{x_1x_2, x_3x_4\}$. Then F is a 2-factor of G containing $\{e_1, e_2\}$. See Figure 7(v)-(vi). Let D_1, D_2 be the cycles of F passing through x_1x_2 and x_3x_4, respectively. (We allow the possibility $D_1 = D_2$.) If neither D_1 nor D_2 is a 4-cycle, then F is the required 2-factor

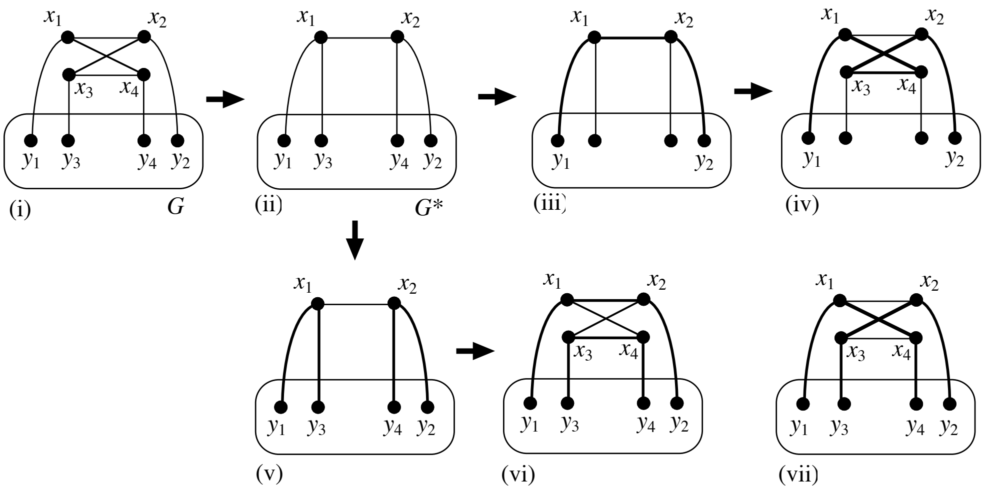

Figure 7:

of G. Hence either D_1 or D_2 is a 4-cycle, and so $D_1 \neq D_2$. We may now deduce that the set of edges $E(F^*) \cup \{x_1x_4, x_3x_2\}$ induces the required 2-factor of G, see Figure 7(vii).

Case 2 $\{e_1, e_2\} \subset E(C)$.

By symmetry, we may assume that $e_1 = x_1x_4$ and $e_2 = x_4x_3$. Let $e_1^* = x_1x_2$ and $e_2^* = x_2y_2$ in G^*. By induction, G^* has a 2-factor F^* such that $\{e_1^*, e_2^*\} \subset E(F^*)$ and $\sigma(F^*) \geq \min\{n^*, 5\}$. If $x_1y_1 \in E(F^*)$, then $(E(F^*) - \{x_1x_2\}) \cup \{x_1x_4, x_4x_3, x_3x_2\}$ induces the required 2-factor of G. See Figure 7(iii)-(iv). Thus $x_1y_3 \in E(F^*)$, and $(E(F^*) - \{x_1x_2\}) \cup \{x_2x_1, x_1x_4, x_4x_3\}$ induces the required 2-factor of G. See Figure 8.

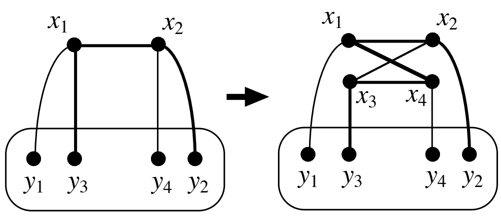

Figure 8:

Case 3 $|E(C) \cap \{e_1, e_2\}| = 1$.

By symmetry, we may assume that $e_1 = y_1x_1$ and $e_2 = x_1x_2$. Let $e_2^* = x_1y_3$. By induction, G^* has a 2-factor F^* such that $\{e_1, e_2^*\} \subset E(F^*)$ and $\sigma(F^*) \geq \min\{n^*, 5\}$. See Figure 7(v). Then $E(F^*) \cup \{x_1x_2, x_3x_4\}$ induces a 2-factor F of G with $\{e_1, e_2\} \subset E(F)$. See Figure 7(vi). Since G is a counterexample to the theorem, F must contain a 4-cycle C'. Since $\sigma(F^*) \geq \min\{n^*, 5\}$, C' passes through x_1x_2 or x_3x_4. If the first alternative holds then C' is a 4-cycle of G with $\{e_1, e_2\} \subset E(C')$. If the second alternative holds then C' is a 4-cycle of G with $\{e_1, e_2\} \cap E(C') = \emptyset$. We can now obtain a contradiction by returning to Case 1 or 2 with C replaced by C'. □

We can now complete the proof of the theorem. By the above-mentioned strengthening of Petersen's theorem, G has a 2-factor F with $\{e_1, e_2\} \subset E(F)$. Since G has girth at least 5, $\sigma(F) \geq 5$. □

Proof of Theorem 2

We use induction on the number of edges of G. If $G - e$ is 3-edge-connected for some $e \in E(G)$ then we are through by induction. Thus $G - e$ is not 3-edge-connected for all $e \in E(G)$. By a result of Mader [8, Lemma 13], G has a vertex u_2 of degree three. We can now choose a pair of edges incident to u_2 and apply Theorem 5. □

4 Closing Remarks

The construction illustrated in Figure 9 shows that there exists an infinite family of 3-edge-connected, essentially 4-edge-connected graphs G in which every spanning even subgraph has a component with at most five vertices. To see this let X be a spanning even subgraph of G. Since u, v have degree three in G we have $d_X(u) = 2 = d_X(v)$. Hence, by symmetry, we may suppose that X contains at most one edge from $\{e_1, e_2\}$. If X contains exactly one edge from $\{e_1, e_2\}$, then X must also contain exactly one of f_1, f_2 and exactly one of g_1, g_2. The fact that every 2-factor of the Petersen graph contains a 5-cycle now implies that $X \cap H_2$ contains a 5-cycle. Thus we may assume that $E(X) \cap \{e_1, e_2\} = \emptyset$. Then either $E(X) \cap \{f_1, f_2\} = \emptyset$ or $\{f_1, f_2\} \subset E(X)$. In both cases we have that $X \cap H_1$ contains a 5-cycle. Thus $\sigma(X) \leq 5$.

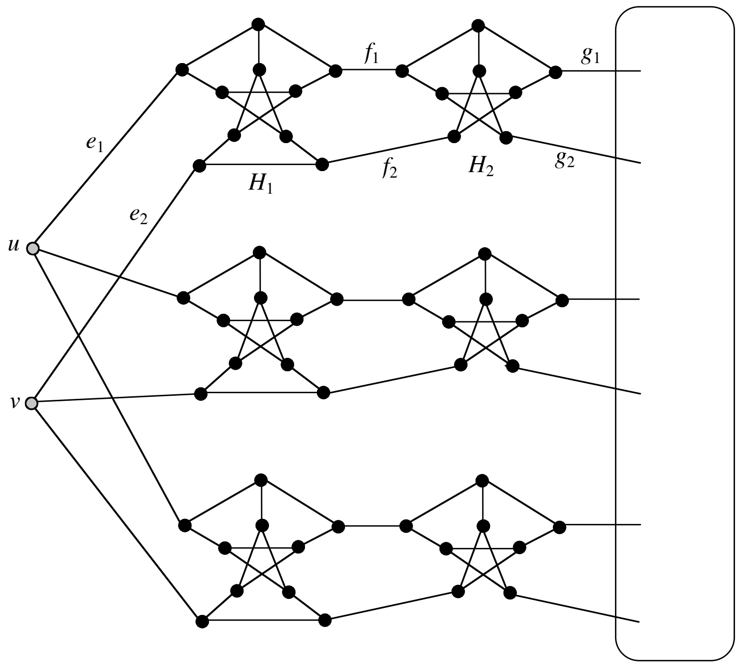

Figure 9:

As mentioned in the Introduction, Chen and Lai [1] conjecture that every 3-edge-connected, essentially 5-edge-connected graph has a spanning connected even subgraph. We propose the following problem which is significantly weaker than their conjecture.

Problem 6. *Does there exist an unbounded function* $f : \mathbb{N} \to \mathbb{N}$ *such that every* 3-*edge-connected, essentially 6-edge-connected graph* G *has a spanning even subgraph* X *with* $\sigma(X) \geq f(n)$*?*

One could also ask whether a cubic graph with high cyclic edge-connectivity must contain a 2-factor in which all cycles are long.

Problem 7. *Is there a value of* k *for which there exists an unbounded function* $g : \mathbb{N} \to \mathbb{N}$ *such that every cyclically* k*-edge-connected cubic graph* G *has a* 2*-factor* X *with* $\sigma(X) \geq g(n)$*?*

Kochol [6, Theorem10.5] has constructed an infinite family of cyclically 6-edge-connected cubic graphs in which every 2-factor has at least $\lfloor n/118 \rfloor$ components.

One of these components must therefore be a cycle of length at most 118. Hence we must take $k \geq 7$ to have an affirmative answer to Problem 7.

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References

[1] Z. H. Chen and H.–J. Lai, Reduction techniques for super-Eulerian graphs and related topics-a survey, Combinatorics and Graph Theory'95, Vol. 1 (Hefei), 53-69, World Sci. Publishing, River Edge, NJ, 1995.

[2] R. Diestel, Graph Theory, Second edition, Graduate Texts in Mathematics 173, Springer (2000)

[3] H. Fleischner, Spanning Eulerian subgraphs, the splitting lemma, and Petersen's theorem, Disc. Math. 101 (1992) 33-37.

[4] B. Jackson and K. Yoshimoto, Even subgraphs of bridgeless graphs and 2-factors of line graphs, Discrete Math. 307 (2007), 2775-2785.

[5] F. Jaeger, Flows and generalized coloring theorems in graphs, J. Combin. Theory Ser. B 26 (1979) 205-216.

[6] M. Kochol, Superposition and constructions of graphs without nowhere-zero k-flows, European J. Combin. (2002) 23, 281-306.

[7] E. Máčajová and M. Škoviera, Constructing hypohamiltonian snarks with cyclic connectivity 5 and 6, Electronic J. Combin. 14 (2007), R18.

[8] W. Mader, A reduction method for edge-connectivity in graphs, Ann. Discrete Math. 3 (1978), 145–164.

[9] J. Petersen, Die Theorie der regulären Graphs, Acta Math. 15 (1891) 193-220.

[10] J. Plesník, Connectivity of regular graphs and the existence of 1-factors, Mat. Časopis Sloven. Akad. Vied 22 (1972), 310–318.

[11] S. Zhan, On hamiltonian line graphs and connectivity, Discrete Math. 89 (1991) 89-95.