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On longest cycles in a 2-connected balanced bipartite graph with Ore type condition, I

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Abstract

For a balanced bipartite graph G with partite sets B and W, we define an Ore type invariant as follows: $\sigma_{1,1}(G) = \{d(u)+d(v) \mid uv \notin E(G),\ u \in B,\ v \in W\}$. In this article, we shall prove the conjecture of Wang to be correct, i.e., if G is 2-connected, then the length of a longest cycle is at least $2\sigma_{1,1}(G) - 2$ or G is hamiltonian.

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1 Introduction

Let G be a simple graph. Dirac studied the length $c(G)$ of a longest cycle in G and the minimum degree $\delta(G)$ in 1952.

Theorem 1 (Dirac [5]). *If G is a 2-connected graph, then $c(G) \geq 2\delta(G)$ or G is hamiltonian.*

Especially, if $\delta(G) \geq |V(G)|/2$ and $|V(G)| \geq 3$, then G is 2-connected, and hence G is hamiltonian. Ore extended the result in 1960. Let

$$\sigma_2(G) = \min\{d_G(x) + d_G(y) \mid xy \notin E(G)\}$$

where $d_G(x)$ is the degree of x. If non-adjacent vertices do not exist, i.e., the graph is complete, we define $\sigma_2(G) = \infty$. For the invariant, the following was shown:

Theorem 2 (Ore [10]). *If $\sigma_2(G) \geq |V(G)| \geq 3$, then G is hamiltonian.*

On the invariant and a longest cycle, in 1976, Bermond and Linial independently showed:

Theorem 3 (Bermond [1], Linial [8]). *If G is 2-connected and has at least three vertices, then $c(G) \geq \sigma_2(G)$ or G is hamiltonian.*

Recently, for bipartite graphs, the following was proved by Wang:

Theorem 4 (Wang [12]). *If G is a 2-connected bipartite graph with partite sets B and W, then $c(G) \geq \min\{2|B|, 2|W|, 2\sigma_2(G) - 2\}$, unless G belongs to one of two families of exceptional graphs.*

This result improves the degree condition, obtained by Dang and Zhao [3]. In the definition of $\sigma_2(G)$, two non-adjacent vertices are allowed to be chosen from the same partite set of the bipartite graph. However in a bipartite graph any pair of vertices in a partite set are not adjacent. Therefore we define Ore type invariant for bipartite graphs as follows: Let G be a bipartite graph with partite sets B and W, and define:

$$\sigma_{1,1}(G) = \min\{d_G(x) + d_G(y) \mid xy \notin E(G), x \in B, y \in W\}.$$

For the definition, in 1963, the hamiltonicity was shown by Moon and Moser:

Theorem 5 (Moon and Moser [9]). *Let G be a balanced bipartite graph with $2n$ vertices. If $\sigma_{1,1}(G) > n$, then G is hamiltonian.*

About the invariant and the length of a longest cycle, Wang conjectured that $c(G) \geq 2\sigma_{1,1}(G) - 2$ or G is hamiltonian if G is a 2-connected in [12]. In this paper, we shall prove the conjecture to be correct as follows:

Theorem 6. *If G is a 2-connected balanced bipartite graph, then $c(G) \geq 2\sigma_{1,1}(G) - 2$ or G is hamiltonian.*

If $\sigma_{1,1}(G) > n$, then G is 2-connected and $c(G) \geq 2\sigma_{1,1}(G) - 2 > 2n - 2$. This implies that G is hamiltonian because a bipartite graph has no odd cycle and $c(G) \leq 2n$. Hence Theorem 5 is obtained from Theorem 6 as a corollary. In Theorem 6, the length of longest cycles are best possible because there are lots of 2-connected balanced bipartite graphs satisfying $c(G) = 2\sigma_{1,1}(G) - 2$. However in [7], the authors prove that if G is 3-connected, then the exceptional class is uniquely determined. The proof in [7] requires same definitions and lemmas to the proof of Theorem 6. We prepare them in the next section.

2 Preparations

In this section, we prepare some definitions, notations and lemmas. The set of all vertices adjacent to x in a graph G is denoted by $N_G(x)$, and we denote $\bigcup_{x \in V(H)} N_G(x)$ by $N_G(H)$ and $N_G(x) \cap V(H)$ by $N_H(x)$ for a subgraph H. The cardinalities $|N_G(H)|$ and $|N_H(x)|$ are denoted by $d_G(H)$ and $d_H(x)$, respectively. For simplicity, we denote $N_G(H) \setminus \{u\}$ by $N_G(H) \setminus u$

Let $P = (u_1, u_2 \cdots, u_p)$ be a path. For two vertices u_i and $u_j \in V(P)$, the subpath joining u_i and u_j in P is denoted by $P[u_i, u_j]$, and we denote the paths $P[u_i, u_j] - u_i$, $P[u_i, u_j] - u_j$ and $P[u_i, u_j] - \{u_i, u_j\}$ by $P(u_i, u_j]$, $P[u_i, u_j)$ and $P(u_i, u_j)$, respectively. The vertices u_{i-1} and u_{i+1} are denoted by u_i^- and u_i^+. For a vertex x which is adjacent to P, let $a = \min\{l \mid u_l \in N_P(x)\}$ and $b = \max\{l \mid u_l \in N_P(x)\}$. The vertices u_a and u_b are denoted by $\alpha_P(x)$ and $\beta_P(x)$. For any vertex $u_i \in V(P)$, let $c = \min\{l \mid u_l \in N_P(u_1)$ and $l > i\}$ and $d = \max\{l \mid u_l \in N_P(u_1)$ and $l < i\}$. We denote the vertices u_c and u_d by $\psi_{P+}^{u_1}(u_i)$ and $\psi_{P-}^{u_1}(u_i)$, respectively. For vertex disjoint subgraphs H_1 and H_2 in G, a *path joining H_1 and*

H_2 is a path such that the ends are contained in H_1 and H_2, respectively, and except the ends, the path and $H_1 \cup H_2$ have no common vertex.

If all of the neighbours of u_1 and u_p are contained in P, then we call P a *maximal path*. Notice that a longest path is maximal. If there are two vertices $u_i \in N(u_1)$ and $u_j \in N(u_p)$ such that $i > j$, then P is called a *crossing path*. If u_1 and u_p are contained in different partite sets and $N_P(u_1)^- \cap N_P(u_p) = \emptyset$, then P is called an *essential* path. The following fact is obvious because G is bipartite.

Fact 1. *Let G be a bipartite graph. If $P = (u_1, u_2, \ldots, u_p)$ is an essential path, then $N_P(u_1), N_P(u_1)^-, N_P(u_p)$ and $N_P(u_p)^+$ are pairwise disjoint.*

At first, we prepare the following lemma:

Lemma 7. *Let G be a bipartite graph and $P = (u_1, u_2 \cdots, u_p)$ a crossing maximal path of G. If P is essential, then $c(G) \geq 2(d(u_1) + d(u_p)) - 2$.*

Proof. Let $u_i \in N(u_1)$ and $u_j \in N(u_p)$ such that $i-j = \min\{k-l \mid u_k \in N(u_1),\ u_l \in N(u_p),\ k > l\}$. Then the cycle $C = P[u_1, u_j] \cup u_j u_p \cup P[u_p, u_i] \cup u_i u_1$ contains all of the vertices in $N(u_1) \cup (N(u_1) \setminus u_i)^- \cup N(u_p) \cup (N(u_p) \setminus u_j)^+$. Therefore

$$|C| \geq d(u_1) + (d(u_1) - 1) + d(u_p) + (d(u_p) - 1) \geq 2(d(u_1) + d(u_p)) - 2$$

since P is maximal and essential. □

Similarly, for a maximal path of which ends are contained in the same partite set, we have the following lemma.

Lemma 8. *Let G be a bipartite graph and $P = (u_1, u_2 \cdots, u_p)$ a crossing maximal path of G. If u_1 and u_p are contained in the same partite set, then $c(G) \geq \min\{|P|-1, 2(d(u_1) + d(u_p)) - 4\}$.*

Proof. Let $u_i \in N(u_1)$ and $u_j \in N(u_p)$ such that $i-j = \min\{k-l \mid u_k \in N(u_1),\ u_l \in N(u_p),\ k > l\}$. If $i - j = 2$, then the cycle $C = P[u_1, u_j] \cup u_j u_p \cup P[u_p, u_i] \cup u_i u_1$ contains all vertices in P except u_{j+1}. Therefore $|C| \geq |P| - 1$. If $i - j \geq 3$, then C contains all of the vertices in

$$N(u_1) \cup (N(u_1) \setminus u_i)^- \cup (N(u_p) \setminus u_j)^+ \cup (N(u_p) \setminus \{u_j, u_{p-1}\})^{++}.$$

Because the vertex subsets are pairwise disjoint, $|C| \geq 2(d(u_1) + d(u_p)) - 4$. □

Next we consider the case of a maximal path which is non-crossing. The following fact is obtained from Lemma of Perfect [11].

Fact 2. *Let G be a 2-connected graph and P a path joining u_1 and u_p in G. For any two vertices u_i and $u_{j(>i)}$ in $P(u_1, u_p)$, there are two vertex disjoint paths Q_1 and Q_2 joining $P[u_1, u_i]$ and $P[u_j, u_p]$ such that u_i and u_j are ends of Q_1 or Q_2.*

In the following, Q_1 and Q_2 are called $(P; u_i, u_j)$-*links*.

Lemma 9. *Let G be a bipartite graph and $P = (u_1, u_2 \cdots, u_p)$ a non-crossing maximal path of G. Then $c(G) \geq 2(d(u_1) + d(u_p)) - 2$. Especially, if u_1 and u_p are contained in the same partite set and $N(u_1) \cap N(u_p) = \emptyset$, then $c(G) \geq 2(d(u_1) + d(u_p))$.*

Proof. As P is non-crossing, $u_1 u_p \notin E(G)$. Let $u_i = \beta_P(u_1)$ and $u_j = \alpha_P(u_p)$. Suppose $i \neq j$. From Fact 2, there are $(P; u_i, u_j)$-links Q_1 and Q_2. Let $\{u_a, u_i, u_j, u_b\}$ be the set of the end vertices of Q_1 and Q_2. We may assume that Q_1 contains u_i. Using the links, we obtain a cycle as follows;

$$C = P[u_1, u_a] \cup P[\psi_{P^+}^{u_1}(u_a), u_i] \cup P[u_j, \psi_{P^-}^{u_p}(u_b)] \cup P[u_b, u_p] \cup$$
$$Q_1 \cup Q_2 \cup \{u_1 \psi_{P^+}^{u_1}(u_a), \psi_{P^-}^{u_p}(u_b) u_p\}$$

See Figure 1. Because $V(C) \supset N(u_1) \cup (N(u_1) \setminus u_c)^- \cup N(u_p) \cup (N(u_p) \setminus u_d)^+$ and

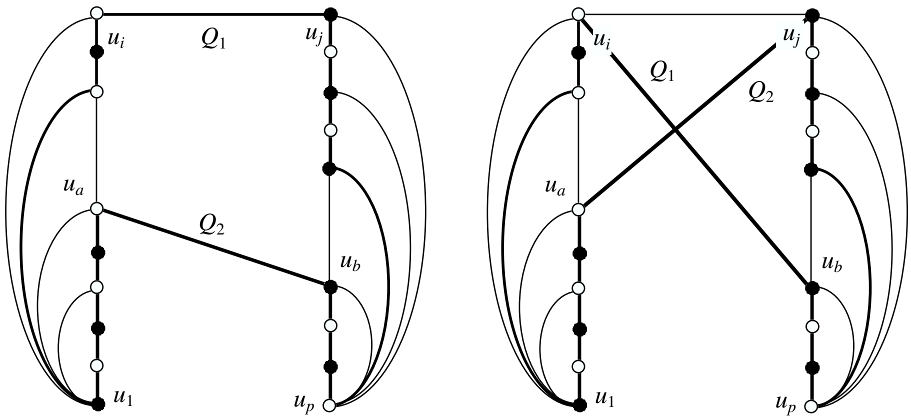

Figure 1:

the vertex subsets are pairwise disjoint, we have:

$$\begin{aligned} |C| &\geq |N(u_1)| + |(N(u_1) \setminus u_c)^-| + |N(u_p)| + |(N(u_p) \setminus u_d)^+| \\ &\geq 2(d(u_1) + d(u_p)) - 2. \end{aligned}$$

If $|C| = 2(d(u_1) + d(u_p)) - 2$, then both of Q_1 and Q_2 are edges joining $N(u_1)$ and $N(u_p)$. Therefore if u_1 and u_p are contained in the same partite set, then $|C| \geq 2(d(u_1) + d(u_p))$. Thus in the case of $i = j$, we look for $(P; \psi^{u_1}_{P^-}(u_i), u_j)$-links and repeat same argument. Then we can obtain $|C| \geq 2(d(u_1) - 1 + d(u_p))$. □

Lemma 7 and 9 imply that if G has a maximal essential path, then $c(G) \geq 2\sigma_{1,1}(G) - 2$ because the ends of an essential path are not adjacent. The following lemma is important in our proof.

Lemma 10. *Let G be a bipartite graph with partite sets B and W. If $|B| < |W|$, then there exists a maximal path joining vertices in W.*

Proof. Let M be a maximum matching of G and $w \in W \backslash V(M)$. As M is maximum, w is not adjacent to $B \backslash V(M)$. If w is not adjacent to $V(M)$, then the path consisting of one single vertex w is the desired path. Suppose that w is adjacent to $V(M)$. Let P be a maximal path containing edges in M alternately such that w is the end. Then the other end w' is contained in $W \cap V(M)$. Because P is maximal, w' is not adjacent to $V(M) \setminus V(P)$. If w' is adjacent to $B \setminus V(M)$, then we can obtain a matching which is larger than M. Thus $N(w') \subset V(P)$. If w is not adjacent to $V(M) \setminus V(P)$, then P is a desired path. Assume that w is adjacent to $V(M) \setminus V(P)$, and let Q be a maximal path containing edges in $E(M) \setminus E(P)$ alternately such that w is the end. As the path P, the end $w''(\neq w)$ of Q is not adjacent to $(V(M) \setminus V(P \cup Q)) \cup (B \setminus V(M))$, i.e., $N(w'') \subset V(P \cup Q)$. Since $N(w') \subset V(P)$, the path $P \cup Q$ is a desired path. □

3 The Proof of Theorem 6

Suppose that G is not hamiltonian, and let $P = (u_1, u_2, \ldots, u_p)$ be a longest path in G with $d(u_1) \leq d(u_p)$. In the following, we denote $\sigma_{1,1}(G)$ by simply $\sigma_{1,1}$. At first, we show the following claim.

Claim 3. *If $c(G) \geq |P| - 1$, then G is hamiltonian or $c(G) \geq 2\sigma_{1,1}$.*

Proof. If there is a cycle C of length at least $|P|$, then $G - V(C)$ is empty because P is a longest path and G is connected. Hence G is hamiltonian. Suppose that there exists a cycle C of length $|P| - 1$. If there is a component in $G - V(C)$ containing at least two vertices, then there exists a path joining it and C since G is connected. This contradicts the assumption that P is a longest path. Therefore $G-V(C)$ is a set of isolated vertices. As G is balanced, there exist vertices $x \in B\cap(V(G)\setminus V(C))$ and $y \in W \cap (V(G)\setminus V(C))$. Since x and y are not adjacent, we have $d(x)+d(y) \geq \sigma_{1,1}$. Because P is longest, it holds that $N_C(x) \cap N_C(y)^+ = \emptyset$ and $N_C(x)^+ \cap N_C(y) = \emptyset$. This implies that $|C| \geq 2(d(x) + d(y)) \geq 2\sigma_{1,1}$ since the cycle contains all vertices in $N(x) \cup N(x)^+ \cup N(y) \cup N(y)^+$. □

If $u_1 \in B$ and $u_p \in W$, then from Claim 3, P is an essential path; otherwise there exists a cycle of length $|P|$. Because a longest path is maximal, from Lemma 7 and 9, we have $c(G) \geq 2(d(u_1) + d(u_p)) - 2 \geq 2\sigma_{1,1} - 2$.

Suppose that the ends of P are contained in the same partite set, say B. From Claim 3, we may assume that $c(G) < |P| - 1$. Then from Lemma 8 and 9, $c(G) \geq 2(d(u_1) + d(u_p)) - 4$. If $c(G) \leq 2\sigma_{1,1} - 4$, we have $d(u_1) \leq \sigma_{1,1}/2$. Because G is a balanced graph, $G - V(P)$ is not balanced. Thus there exists a component D in $G-V(P)$ such that $|V(D)\cap W| > |V(D)\cap B|$. From Lemma 10, there is a maximal path Q joining vertices in W.

Suppose that Q is one vertex z. Then $N(z) \subset V(P)$ and $N(u_1)^- \cap N(z) = \emptyset$ as P is longest. Hence if $\beta_P(z) \in P[\beta_P(u_1), u_p]$, then the path $P[u_1, \beta_P(z)] \cup \beta_P(z)z$ is maximal and essential, i.e., $c(G) \geq 2\sigma_{1,1} - 2$. On the other hand, if $\beta_P(z) \in P[u_1, \beta_P(u_1))$, then $P[u_1, \beta_P(u_1)] \cup \beta_P(u_1)u_1$ is a desired cycle because the cycle contains all vertices in $N(u_1) \cup N(u_1)^- \cup N(z) \cup N(z)^+$ and $u_1 z \notin E(G)$.

Assume that $Q = (z_1, z_2, \ldots, z_q)$ contains at least three vertices. As $z_1 u_1 \notin E(G)$ and $d(u_1) \leq \sigma_{1,1}/2$, it holds that $d(z_1) \geq \sigma_{1,1}/2$. Because G is connected, there is a path joining Q and P. Suppose that there exists a path joining $Q[z_1, \alpha_Q(z_q)] \cup Q[\beta_Q(z_1), z_q]$ and P. In such paths, we choose a path R such that the end u_m is nearest to u_p on P. By symmetry, we may assume that another end z_a of R is in $Q[\beta_Q(z_1), z_q]$. Then $N(z_1) \subset P[u_1, u_m] \cup Q[z_1, z_a]$ and $N_P(u_1)^- \cap N_P(z_1) = \emptyset$ since P is a longest path. If $\beta_P(u_1) \in P[u_1, u_m]$, then the path $P[u_1, u_m] \cup R \cup Q[z_a, z_1]$ is maximal and essential. Hence, we have $c(G) \geq 2\sigma_{1,1} - 2$ from Lemma 7 and 9.

If $\beta_P(u_1) \in P(u_m, u_p]$, then $|P(u_m, \psi_{P+}^{u_1}(u_m))| \geq |Q[z_1, \beta_Q(z_1)]| \geq 2d_Q(z_1)$ because $N_Q(z_1) \cup N_Q(z_1)^- \subset Q[z_1, \beta_Q(z_1)]$ and P is longest. See Figure 2i. Since $P[u_1, \beta_P(u_1)]$ contains all vertices in$N(u_1) \cup N(u_1)^- \cup N_P(z_1) \cup N_P(z_1)^+$, the cycle

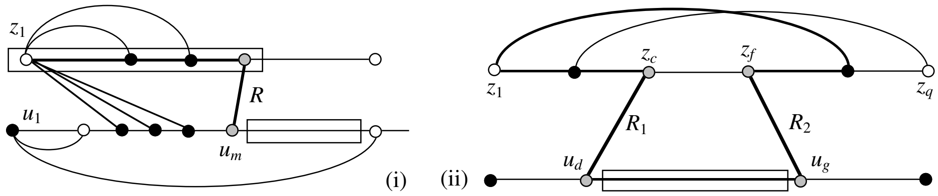

Figure 2:

$P[u_1, \beta_P(u_1)] \cup \beta_P(u_1)u_1$ is longer than or equal to

$$2d(u_1) + 2(d(z_1) - d_Q(z_1)) + 2d_Q(z_1) - 2 \geq 2\sigma_{1,1} - 2.$$

Thus there is no path joining $Q[z_1, \alpha_Q(z_q)] \cup Q[\beta_Q(z_1), z_q]$ and P. In particular, $N_P(z_1) = N_P(z_q) = \emptyset$.

Because G is 2-connected, there are two vertex disjoint paths R_1 and R_2 joining $Q(\alpha_Q(z_q), \beta_Q(z_1))$ and P. Let $\{z_c, u_d\}$ and $\{z_f, u_g\}$ be the ends of R_1 and R_2, respectively. We may assume $c < f$. If there exists a path Q' joining z_c and z_f of length at least $\sigma_{1,1} - 2$ which is vertex disjoint to P, then the length of the cycle $Q' \cup R_1 \cup R_2 \cup P[u_d, u_g]$ is at least $2\sigma_{1,1} - 2$ because $|P(u_d, u_g)| \geq |Q'|$, otherwise P is not longest.

If $Q(z_c, z_f) \cap N(z_1) = \emptyset$, then the path $Q[z_c, z_1] \cup z_1\beta_Q(z_1) \cup Q[\beta_Q(z_1), z_f]$ joins z_c and z_f and contains at least $2d(z_1) - 1 \geq \sigma_{1,1} - 1$ vertices because $V(Q') \supset N(z_1) \cup (N(z_1) \setminus z_f)^-$. See Figure 2ii. Hence $c(G) \geq 2\sigma_{1,1} - 2$. By symmetry, we have that $Q(z_c, z_f) \cap N(z_1) \neq \emptyset$ and $Q(z_c, z_f) \cap N(z_q) \neq \emptyset$.

Let $h = \min\{l \mid z_l \in N(u_1) \cup N(z_q) \text{ and } c < l\}$ and $k = \max\{l \mid z_l \in N(z_1) \cup N(z_q) \text{ and } l < f\}$. Suppose that $z_k \in N(z_q)$. If $u_g \in P[u_1, \beta_P(u_1))$, then both of $|P[u_1, u_g)|$ and $|P(u_g, \beta_P(u_1))|$ are greater than or equal to the length of the path $Q' = Q[z_1, z_k] \cup z_k z_q \cup Q[z_q, z_f]$ because P is longest. See Figure 3i. Since $N(z_1) \cup$

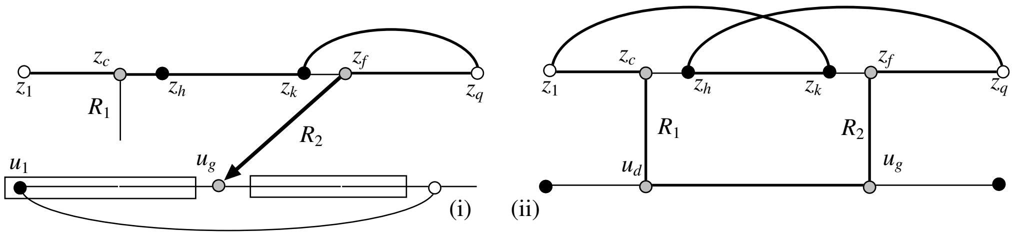

Figure 3:

$(N(z_1) \setminus z_f)^- \subset V(Q')$ and $z_q \in V(Q')$, we have $|Q'| \geq \sigma_{1,1}$. Hence $P[u_1, \beta_P(u_1)] \cup \beta_P(u_1)u_1$ is a desired cycle. By symmetry, we have $u_g \notin P(\alpha_P(u_p), u_p]$. This implies that the path $Q' \cup R_2 \cup P[u_g, u_1]$ is maximal and essential, i.e., $c(G) \geq 2\sigma_{1,1} - 2$ from Lemma 7 and 9. Hence by symmetry, it holds that $z_k \in N(z_1)$ and $z_h \in N(z_q)$. See Figure 3ii. In this case, $Q[z_c, z_1] \cup z_1 z_k \cup Q[z_k, z_h] \cup z_h z_q \cup Q[z_q, z_f]$ is a path joining z_c and z_f of length at least $2d(z_1) \geq \sigma_{1,1}$ because the path contains all vertices in $N(z_1) \cup (N(z_1) \setminus z_f)^-$ and z_q. Thus $c(G) \geq 2\sigma_{1,1} - 2$.

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