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On degree conditions and a dominating longest cycle

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Abstract

Let G be a 2-connected graph. The edge degree $d(e)$ of the edge $e = uv$ is defined as the number of neighbours of e, i.e., $|N(u) \cup N(v)| - 2$. Two edges are called remote if they are disjoint and there is no edge joining them. Here we prove: if $d(e_0) + d(e_1) + d(e_2) > |V(G)| - 2$ for any mutually remote edges e_0, e_1, e_2, then G has a longest cycle C which is dominating, i.e., such that $G - V(C)$ is edgeless. As a corollary we have: if G is a 2-connected triangle-free graph with $|V(G)| < 2\sigma_3(G) - 4$, then G has a longest cycle which is dominating. This generalises a result due to Aung [J. Combin. Theory Ser. B **47** (1989) 171-186].

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1 Introduction

Let G be a simple finite graph and C a cycle in G. If $G - V(C)$ is edgeless, then C is called a *dominating cycle.* In 1971, Nash-Williams [11] showed that if G is 2-connected and $|V(G)| \leq 3\delta(G) - 2$, then any longest cycle of G is dominating. Bondy generalized this fact in 1980 as follows. Let

$$\sigma_3(G) = \min\{d(x) + d(y) + d(z) \mid x, y, z \text{ are independent in } G\}.$$

Theorem 1 (Bondy [5]). *Let G be a 2-connected graph. If $|V(G)| \leq \sigma_3(G) - 2$, then any longest cycle of G is dominating.*

For bipartite graphs, Ash and Jackson [1] showed that: if G is a 2-connected bipartite graph with partite sets B and W and both of $|B|$ and $|W|$ are less than $3\delta(G) - 2$, then there exists a longest cycle which is dominating. This fact implies that a 2-connected balanced bipartite graph with $|V(G)| < 6\delta(G) - 4$ contains a longest cycle which is dominating. They also showed that their conclusion cannot be replaced by the conclusion that every longest cycle is dominating.

Veldman [12] studied the relation between *edge degrees* and the existence of dominating cycles. The edge degree $d(e)$ of the edge $e = uv$ is defined as the number of neighbours of e, i.e., $|N(u) \cup N(v)| - 2$. Actually, the concept of edge degrees is well-suited for studying the existence of dominating cycles. Two edges are called *remote* if they are disjoint and there is no edge joining them. Veldman proved the following result.

Theorem 2 (Veldman [12]). *Let G be a k-connected graph. If $\sum_{l \leq k} d(e_l) > k(|V(G)| - k)/2$ for any $k+1$ edges $e_0, e_1, \ldots, e_k$ which are mutually remote, then G has a dominating cycle.*

This theorem does not imply the existence of a longest cycle which is dominating. However if we carefully read the proof [12], we can easily check that the following result also holds: if G is a 2-connected graph without cycles of length seven, and for any three mutually remote edges e_0, e_1, e_2, the edge degree sum $d(e_0) + d(e_1) + d(e_2)$ is greater than $|V(G)| - 2$, then there exists a longest cycle which is dominating. Since a bipartite graph contains no odd cycles, the theorem by Ash and Jackson is obtained as a corollary. More generally, a similar approach shows that the following also holds. Let

$$\sigma_{1,1}(G) = \min\{d_G(x) + d_G(y) \mid xy \notin E(G), x \in B, y \in W\}.$$

Corollary 3. *Let G be a 2-connected bipartite graph with partite sets B and W. If $|V(G)| < 3\sigma_{1,1}(G) - 4$, then there exists a longest cycle which is dominating.*

From Veldman's theorem, it follows that a 2-connected triangle-free graph has a dominating cycle if $|V(G)| < 2\sigma_3(G) - 4$. For the existence of a longest cycle which is dominating, in 1989 Aung proved the following.

Theorem 4 (Aung [3]). *Let G be a 2-connected triangle-free graph. If $|V(G)| \leq 6\delta(G) - 6$, then there exists a longest cycle which is dominating.*

In the next section, we will prove the following more general result.

Theorem 5. *Let G be a 2-connected graph. If $d(e_0) + d(e_1) + d(e_2) > |V(G)| - 2$ for any mutually remote edges e_0, e_1, e_2, then G has a longest cycle which is dominating.*

Hence we obtain the following corollary which contains Aung's theorem.

Corollary 6. *Let G be a 2-connected triangle-free graph. If $|V(G)| < 2\sigma_3(G) - 4$, then G has a longest cycle which is dominating.*

A class of examples due to Wang [14] shows that the lower bound in Theorem 5 is best possible, and from the example due to Ash and Jackson [1], we obtain 2-connected graphs satisfying the conditions of Theorem 5, but in which some longest cycles are not dominating.

For the length of longest cycles in bipartite graphs, in 1977 Voss and Zuluaga [13] showed: if G is a 2-connected bipartite graph with partite sets B and W and $\delta(G) \geq 3$, then $c(G) \geq \min\{4\delta(G) - 4, 2|B|, 2|W|\}$, where $c(G)$ is the length of a longest cycle. Recently, Kaneko and Yoshimoto proved the following fact which was conjectured by Wang [14].

Theorem 7 (Kaneko and Yoshimoto [8]). *If G is a 2-connected balanced bipartite graph, then $c(G) \geq 2\sigma_{1,1}(G) - 2$ or G is hamiltonian.*

Wang's examples in [14] imply that we cannot remove "balanced" in Theorem 7, in general. However, the order of these examples is at least $3\sigma_{1,1}(G) - 4$. In fact, we can easily show the following.

Lemma 8. *Let G be a connected bipartite graph with partite sets B and W. If G contains a dominating cycle, then $c(G) \geq \min\{2\sigma_{1,1}(G) - 2, 2|B|, 2|W|\}$.*

Proof. Let C be a longest dominating cycle. If $|V(G-V(C)) \cap B|$ or $|V(G-V(C)) \cap W|$ is empty, then our statement holds. Hence there are $x \in V(G - V(C)) \cap B$ and $y \in V(G - V(C)) \cap W$. Let $\{v_1, v_2, \ldots, v_p\} = N(x) \cup N(y)$ which occur on C in the order of their indices. If there are two vertices v_i, v_j in $N(x)^+ \cap N(y)$, then the cycle $v_i x v_j \cup \overleftarrow{v_j C v_{i+1}} \cup v_{i+1} y v_{j+1} \cup \overrightarrow{v_{j+1} C v_i}$ is longer than C and dominating. Hence $|N(x)^+ \cap N(y)| \leq 1$. By symmetry, $|N(y)^+ \cap N(x)| \leq 1$. Therefore $|V(C)| \geq |N(x) \cup N(x)^+ \cup N(y) \cup N(y)^+| \geq 2|N(x)| + 2|N(y)| - 2 \geq 2\sigma_{1,1}(G) - 2$. $\square$

Hence, by Corollary 3, we obtain the following result in which the upper bound is also best possible by the examples of Wang [14].

Theorem 9. *Let G be a 2-connected bipartite graph with partite sets B and W. If $|V(G)| < 3\sigma_{1,1}(G) - 4$, then $c(G) \geq \min\{2\sigma_{1,1}(G) - 2, 2|B|, 2|W|\}$.*

Finally, before we start our proof in the next section, we give some additional definitions and notations. The set of all the neighbours of a vertex $x \in V(G)$ is denoted by $N_G(x)$ or simply $N(x)$, and its cardinality by $d_G(x)$ or $d(x)$. For a subgraph H of G, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. For simplicity, we denote $|V(H)|$ by $|H|$, and "$u_i \in V(H)$" and "$G - V(H)$" are written by "$u_i \in H$" and "$G - H$" respectively. The set of neighbours $\bigcup_{v \in H} N_G(v) \setminus V(H)$ is written by $N_G(H)$ or $N(H)$, and for a subgraph $F \subset G$, $N_G(H) \cap V(F)$ is denoted by $N_F(H)$. Especially for an edge e, $|N_G(e)|$ is the *edge degree.*

Let $C = u_1 u_2 \ldots u_p u_1$ be a cycle with a fixed orientation. The segment $u_i u_{i+1} \ldots u_j$ is written by $\overrightarrow{u_i C u_j}$, or simply $[u_i, u_j]$ where the subscripts are to be taken modulo $|C|$. The converse segment $u_j u_{j-1} \ldots u_i$ is written by $\overleftarrow{u_j C u_i}$, and we denote $[u_i, u_j] - u_i$ by $(u_i, u_j]$. A path joining x and y in H is denoted by $P_H(x, y)$.

All notation and terminology not explained here is given in [7].

2 The Proof of Theorem 5

Let $C = u_1 u_2 \dots u_p u_1$ be a longest cycle such that $|E(G - C)|$ is smallest among all longest cycles in G. For each vertex $u_i \in N_C(H)$, we define an edge $e_i = u_{i+1}v_{i+1}$ as follows: if u_{i+1} has neighbours in $G - C$, then $v_{i+1} = w$ for an arbitrary vertex $w \in N_{G-C}(u_{i+1})$; otherwise $v_{i+1} = u_{i+2}$. When $v_{i+1} \notin C$, the path $e_i \cup [u_{i+1}, u_l]$ is denoted by $[v_{i+1}, u_l]$.

Fact 1. *Let $u_i \in N_C(H)$ and $x \in N_H(u_i)$. Then:*

1. *There is no edge joining e_i and H.*
2. *$u_{i+1} \notin N(e_l)$ for all $u_l \in N_C(H)$.*
3. *e_i is remote to e_l for all $u_l \in N_C(H - x)$.*
4. *If $e_i \notin E(C)$, then e_i is remote to e_l for all $u_l \in N_C(H)$.*

Proof. (1) As C is longest, clearly $u_{i+1} \notin N(H)$, and $v_{i+1} \notin N(H)$ if $v_{i+1} \notin C$. If $v_{i+1} = u_{i+2}$ and $u_{i+2} \in N(H)$, then $N_{G-C}(u_{i+1}) = \emptyset$, and thus $C' = u_i x \cup P_H(x, w) \cup w u_{i+2} \cup [u_{i+2}, u_i]$ is longest and $|E(G - C')| < |E(G - C)|$ where $w \in N_H(u_{i+2})$. This contradicts the assumption of C. (2) Let $y \in N_H(u_l)$ $(l \neq i)$. Clearly $u_{i+1}u_{l+1} \notin E(G)$ because C is longest. If $v_{l+1} = u_{l+2}$ and $u_{i+1}u_{l+2} \in E(G)$, then the cycle $u_{i+1}u_{l+2} \cup [u_{l+2}, u_i] \cup u_i x \cup P_H(x, y) \cup y u_l \cup \overleftarrow{u_l C u_{i+1}}$ is longer than C or $|C'| = |C|$ and $|E(G - C')| < |E(G - C)|$. See Figure 1i. The case that

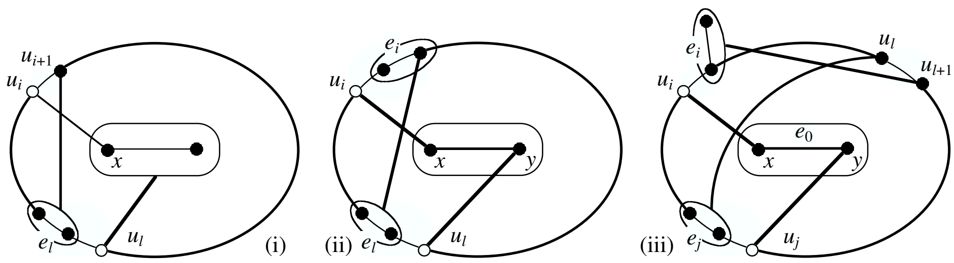

Figure 1:

$v_{l+1} \notin C$ can be shown similarly. (3) By (2), we may assume $v_{i+1}v_{l+1} \in E(C)$. Then $C' = v_{i+1}v_{l+1} \cup [v_{l+1}, u_i] \cup u_i x \cup P_H(x, y) \cup y u_l \cup \overleftarrow{u_l C v_{i+1}}$ is longer than C or $|C'| = |C|$ and $|E(G - C')| < |E(G - C)|$ because $|P_H(x, y)| \geq 2$. See Figure 1ii. (4) The proof is similar to the proofs of (2) and (3). $\square$

Define a bijection σ on $V(C) \cup \{v_{l+1} \mid u_l \in N_C(H)\}$ as follows: $\sigma(u_l) = u_{l+1}$ for $u_l \notin \{u_{m+1}, v_{m+1} \mid u_m \in N_C(H)\}$, and for each $u_l \in N_C(H)$:

$$\begin{cases} \sigma(u_{l+1}) = v_{l+1} & \\ \sigma(v_{l+1}) = u_{l+2} & \text{if } v_{l+1} \notin C \\ \sigma(v_{l+1}) = u_{l+3} & \text{if } v_{l+1} = u_{l+2} \in C. \end{cases}$$

Let u_i, u_j be two vertices in $N_C(H)$ and $I = (u_i, u_j]$ and $J = (u_j, u_i]$, and let $\widetilde{I} = I \cup \sigma(I) - u_{j+1}$ and $\widetilde{J} = I \cup \sigma(J) - u_{i+1}$.

Fact 2. *It holds that $\sigma(\widetilde{I}) \cap \sigma(\widetilde{J}) = \emptyset$ and for any $e_0 \in E(H)$:*

$$\begin{aligned} \sigma^{-1}(N_{\tilde{I}}(e_i)) \cup N_{\tilde{I}}(e_j) \cup \sigma(N_{\tilde{I}}(e_0)) &\subset \sigma(\widetilde{I}) \\ \sigma^{-1}(N_{\tilde{J}}(e_j)) \cup N_{\tilde{J}}(e_i) \cup \sigma(N_{\tilde{J}}(e_0)) &\subset \sigma(\widetilde{J}) \end{aligned}$$

Proof. Because σ is a bijection and $\widetilde{I} \cap \widetilde{J} = \emptyset$, we get the first expression. By Fact 1(2), $u_{i+1} \notin N(e_i) \cup N(e_j)$, and thus the second expression holds. Similarly, the third expression follows from $u_{j+1} \notin N(e_i) \cup N(e_j)$ □

Claim 3. *Let $e_0 \in E(H)$ and $x \in N_H(u_i)$, $y \in N_H(u_j)$. If $x \neq y$, then one of $\sigma(N_{\tilde{I}}(e_0)) \cap \sigma^{-1}(N_{\tilde{I}}(e_i))$ and $\sigma(N_{\tilde{J}}(e_0)) \cap \sigma^{-1}(N_{\tilde{J}}(e_j))$ is not empty.*

Proof. By Fact 1(1)(3), e_0, e_i, e_j are mutually remote. By Fact 1(2), we have $N_{\tilde{I}}(e_j) \cap \sigma(N_{\tilde{I}}(e_0)) = \emptyset$. Assume that there is a vertex $u_l \in N_{\tilde{I}}(e_j) \cap \sigma^{-1}(N_{\tilde{I}}(e_i))$, and let $w_i \in V(e_i) \cap N(u_{l+1})$ and $w_j \in V(e_j) \cap N(u_l)$. Then the cycle $C' = u_l w_j \cup [w_j, u_i] \cup u_i x \cup P_H(x, y) \cup y u_j \cup \overleftarrow{u_j C u_{l+1}} \cup u_{l+1} w_i \cup [w_i, u_l]$ is longer than C or $|C'| = |C|$ and $|E(G - C')| < |E(G - C)|$ because $|P_H(x, y)| \geq 2$. See Figure 1iii. This is a contradiction. Hence:

$$N_{\tilde{I}}(e_j) \cap (\sigma(N_{\tilde{I}}(e_0)) \cup \sigma^{-1}(N_{\tilde{I}}(e_i))) = \emptyset.$$

By symmetry, we have:

$$N_{\tilde{J}}(e_i) \cap (\sigma(N_{\tilde{J}}(e_0)) \cup \sigma^{-1}(N_{\tilde{J}}(e_j))) = \emptyset.$$

Suppose both of $\sigma(N_{\tilde{I}}(e_0)) \cap \sigma^{-1}(N_{\tilde{I}}(e_i))$ and $\sigma(N_{\tilde{J}}(e_0)) \cap \sigma^{-1}(N_{\tilde{J}}(e_j))$ are empty, and let $D = \widetilde{I} \cup \widetilde{J}$. Because σ is a bijection, by Fact 2, we have:

$$|D| = |\sigma(\widetilde{I}) \cup \sigma(\widetilde{J})| = |\sigma(\widetilde{I})| + |\sigma(\widetilde{J})|$$

$$\begin{aligned} &\geq |\sigma^{-1}(N_{\widetilde{I}}(e_i))| + |N_{\widetilde{I}}(e_j)| + |\sigma(N_{\widetilde{I}}(e_0))| + |\sigma^{-1}(N_{\widetilde{J}}(e_j))| + |N_{\widetilde{J}}(e_i)| + |\sigma(N_{\widetilde{J}}(e_0))| \\ &\geq |N_{\widetilde{I}}(e_i)| + |N_{\widetilde{I}}(e_j)| + |N_{\widetilde{I}}(e_0)| + |N_{\widetilde{J}}(e_j)| + |N_{\widetilde{J}}(e_i)| + |N_{\widetilde{J}}(e_0)| \\ &= |N_D(e_i)| + |N_D(e_j)| + |N_D(e_0)|. \end{aligned}$$

If there is a common vertex in $N_{G-D}(e_0), N_{G-D}(e_i)$ and $N_{G-D}(e_j)$, then we can easily find a cycle C' such that $|C'| > |C|$ or $|C'| = |C|$ and $|E(G - C')| < |E(G - C')|$. Moreover, by Fact 1(1), $V(e_0) \cap (N(e_i) \cup N(e_j) \cup N(e_0)) = \emptyset$. Thus:

$$\begin{aligned} |G| - 2 &\geq |D| + |N_{G-D}(e_0)| + |N_{G-D}(e_i)| + |N_{G-D}(e_j)| \\ &\geq |N(e_0)| + |N(e_i)| + |N(e_j)| > |G| - 2. \end{aligned} \quad (1)$$

This is a contradiction. □

Claim 4. *$e_i \in E(C)$ for all $u_i \in N_C(H)$.*

Proof. Assume that there exists a vertex $u_j \in N_C(H)$ such that $e_j \notin E(C)$. Because G is 2-connected and $|H| \geq 2$, there is a vertex $y \in N_H(u_j)$ such that $N_{H-y}(C - u_j)$ is not empty. Let $x \in N_{H-y}(C - u_j)$ and $u_i \in N_{C-u_j}(x)$ such that $\widetilde{I} \cap N_C(x) \subset N_C(H - x)$. Then for any edge e_0 in H, by Fact 1(3), $\sigma(N_{\widetilde{I}}(e_0)) \cap \sigma^{-1}(N_{\widetilde{I}}(e_i)) = \emptyset$. Moreover, by Fact 1(4), $\sigma(N_{\widetilde{J}}(e_0)) \cap \sigma^{-1}(N_{\widetilde{J}}(e_j)) = \emptyset$. This contradicts Claim 3. □

Therefore $\widetilde{I} = I$ and $\widetilde{J} = J$. Suppose first that $|H| \geq 3$. Because G is 2-connected, there exists a vertex $x \in N_H(C)$ and $u_i \in N_C(x)$ such that $E(H - x) \neq \emptyset$ and $N_{C-u_i}(H - x) \neq \emptyset$. Let $e_0 \in E(H - x)$ which is adjacent to $C - u_i$, and $u_j \in N_{C-u_i}(e_0)$ such that $[u_{j+1}, u_{i-1}) \cap N_C(e_0) = \emptyset$. Then, by Fact 1(3), we have $\sigma(N_J(e_0)) \cap \sigma^{-1}(N_J(e_j)) = \emptyset$ and $\sigma(N_I(e_0)) \cap \sigma^{-1}(N_I(e_i)) = \emptyset$. This contradicts Claim 3.

Assume now that $|H| = 2$, and let $e_0 = x_1x_2 = H$ and $N_C(e_0) = \{u_{\tau(l)} \mid l \leq |N(e_0)|\}$ which occur on C in the order of their indices.

Remark. If we suppose the minimum edge degree is greater than $(|G| - 2)/3$, then our proof can be completed immediately as follows: By Claim 4, all $e_{\tau(l)} \in E(C)$, and so $|(u_{\tau(l)}, u_{\tau(l+1)}]| \geq 3$ by Fact 1(1). Hence:

$$|G| - 2 \geq |C| \geq \sum_{l \leq |N(e_0)|} |(u_{\tau(l)}, u_{\tau(l+1)}]| \geq 3d(e_0) > |G| - 2.$$

This is a contradiction. However we do not use this fact in the following.

Let $f_l = u_{\tau(l+1)-2}u_{\tau(l+1)-1}$ for all $u_{\tau(l)}$. We prove the following claims simultaneously.

Claim 5. *If $u_{\tau(i)}$ and $u_{\tau(j)}$ are adjacent to distinct vertices in H, then:*

$$N_K(e_{\tau(i)}) \cap \{\sigma^{-1}(N_K(f_{\tau(j)})) \cup \sigma(N_K(f_{\tau(j)}))\} = \emptyset$$

where $K = [u_{\tau(j)}, u_{\tau(i)}]$.

Claim 6. $|(u_{\tau(j)}, u_{\tau(j+1)}]| \geq 5$ *for all* $u_{\tau(j)} \in N(e_0)$.

Proof. Suppose there is a vertex $u_{\tau(j)}$ such that $|(u_{\tau(j)}, u_{\tau(j+1)}]| \leq 4$. By symmetry, we may assume that $u_{\tau(j)} \in N(x_2)$. Let $u_{\tau(i)} \in N(x_1)$ such that $(u_{\tau(i)}, u_{\tau(j)}] \cap N(e_0) \subset N(x_2)$, and let $I = (u_{\tau(i)}, u_{\tau(j)}]$ and $J = (u_{\tau(j)}, u_{\tau(i)}]$. Then, $\sigma(N_I(e_0)) \cap \sigma^{-1}(N_I(e_{\tau(i)})) = \emptyset$ and $\sigma(N_J(x_1)) \cap \sigma^{-1}(N_J(e_{\tau(j)})) = \emptyset$ by Fact 1(3). Therefore $\sigma(N_J(x_2)) \cap \sigma^{-1}(N_J(e_{\tau(j)})) \neq \emptyset$ by Claim 3. Let:

$$\begin{aligned}
L &= \sigma(N_J(e_0) \setminus N_J(x_1)) \cap \sigma^{-1}(N_J(e_{\tau(j)})) \\
S &= \sigma(L) \setminus \sigma^{-1}(N_J(e_{\tau(j)})) \\
T &= \sigma(J) \setminus \{\sigma^{-1}(N_J(e_{\tau(j)})) \cup N_J(e_{\tau(i)}) \cup \sigma(N_J(e_0)) \cup S\}.
\end{aligned}$$

By Fact 1(3)(1), $S \cap \{N(e_{\tau(i)}) \cup \sigma(N(e_0)) \cup \sigma^{-1}(N(e_{\tau(j)}))\} = \emptyset$. As $u_{\tau(i)+1} \notin L$, we have $S \subset \sigma(J)$, and thus the following inequalities hold:

$$\begin{aligned}
|C| &= |\sigma(I) \cup \sigma(J)| = |\sigma(I)| + |\sigma(J)| \\
&\geq |\sigma^{-1}(N_I(e_{\tau(i)}))| + |N_I(e_{\tau(j)})| + |\sigma(N_I(e_0))| \\
&\quad + |\sigma^{-1}(N_J(e_{\tau(j)}))| + |N_J(e_{\tau(i)})| + |\sigma(N_J(e_0))| - |L| + |S| + |T| \\
&\geq |N_I(e_{\tau(i)})| + |N_I(e_{\tau(j)})| + |N_I(e_0)| + |N_J(e_{\tau(j)})| + |N_J(e_{\tau(i)})| + |N_J(e_0)| \\
&\quad - |L| + |S| + |T| \\
&= |N_C(e_{\tau(i)})| + |N_C(e_{\tau(j)})| + |N_C(e_0)| - |L| + |S| + |T|.
\end{aligned}$$

Because $|C| + (|N_{G-C}(e_0)| + |N_{G-C}(e_{\tau(i)})| + |N_{G-C}(e_{\tau(j)})|) \leq |G| - 2$, we have $|L| - |S| \geq |T| + 1$ as in (1). Hence $|\sigma(L) \cap \sigma^{-1}(N(e_{\tau(j)}))| = |\sigma(L)| - |S| \geq |T| + 1$. Therefore there exists a vertex $u_{\tau(l)+2} \in \sigma(L)$ such that $u_{\tau(l)+3}$ is adjacent to $e_{\tau(j)}$. Let M be the set of all such vertices, i.e., $M = \sigma(L) \cap \sigma^{-1}(N(e_{\tau(j)}))$, and then $|M| \geq |T|+1$. By Fact 1(2), $u_{\tau(l)+2} \in M$ is adjacent to $u_{\tau(j)+2}$. If $u_{\tau(j)+2} = u_{\tau(j+1)-1}$, then the cycle:

$$C' = u_{\tau(l)+2} u_{\tau(j+1)-1} w u_{\tau(l)+3} \cup [u_{\tau(l)+3}, u_{\tau(j)}] \cup u_{\tau(j)} x_2 w' u_{\tau(j+1)} \cup [u_{\tau(j+1)}, u_{\tau(l)+2}]$$

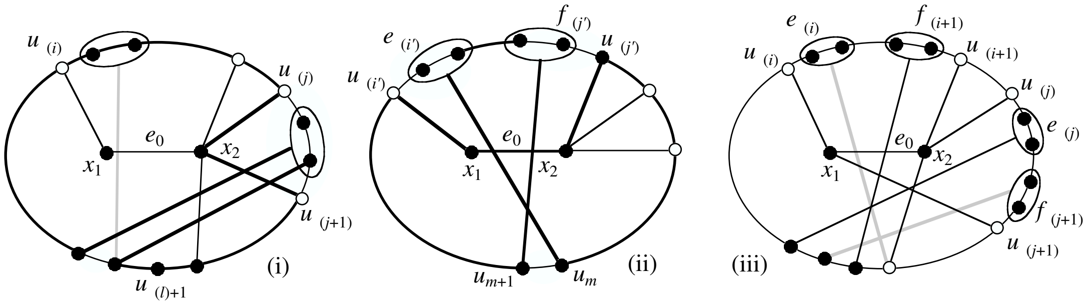

Figure 2:

is longest and $|E(G-C')| < |E(G-C)|$ where $w \in V(e_{\tau(j)}) \cap N(u_{\tau(l)+3})$ and $w' \in V(e_0) \cap N(u_{\tau(j+1)})$. See Figure 2i. This is a contradiction. Hence we have:

$$|(u_{\tau(j)}, u_{\tau(j+1)}]| \geq 4 \text{ for all } u_{\tau(j)} \in N(e_0). \tag{2}$$

Using this fact, we can prove Claim 5 as follows. Let $u_{\tau(i')} \in N(x_1)$ and $u_{\tau(j')} \in N(x_2)$ and $K = [u_{\tau(j')}, u_{\tau(i')}]$. Suppose that there is a vertex $u_m \in N_K(e_{\tau(i')}) \cap \sigma^{-1}(N_K(f_{\tau(j')}))$, and let $w \in V(e_{\tau(i')}) \cap N(u_m)$ and $w' \in V(f_{\tau(j')}) \cap N(u_{m+1})$. Let:

$$C' = u_m w \cup [w, w'] \cup w' u_{m+1} \cup [u_{m+1}, u_{\tau(i')}] \cup u_{\tau(i')} x_1 x_2 u_{\tau(j')} \cup [u_{\tau(j')}, u_m].$$

See Figure 2ii. If $\{w, w'\} \cap \{u_{\tau(i')+1}, u_{\tau(j')-1}\} \neq \emptyset$, then C' is longer than C. Hence $w = u_{\tau(i')+2}$ and $w' = u_{\tau(j')-2}$, and $|C'| = |C|$. If $u_{\tau(i')+1} u_{\tau(j')-1} \notin E(G)$, then $|E(G-C')| < |E(G-C)|$. This contradicts the assumption on C. Therefore $u_{\tau(i')+1} u_{\tau(j')-1} \in E(G)$, and then $|C'| = |C|$ and $|E(G-C')| = |E(G-C)|$, and this cycle contradicts the fact (2). Hence we have $N_K(e_{\tau(i')}) \cap \sigma^{-1}(N_K(f_{\tau(j')})) = \emptyset$. By symmetry, we obtain Claim 5.

From (2) and the assumption $|(u_{\tau(j)}, u_{\tau(j+1)}]| \leq 4$, we have $u_{\tau(j)+2} = u_{\tau(j+1)-2}$. Since $|M| \geq |T|+1$, there is a vertex $u_{\tau(l)+2} \in M$ such that $\sigma((u_{\tau(l)}, u_{\tau(l+1)}]) \cap T = \emptyset$. Notice that if $u_m \in N_J(u_{\tau(j)+2})$, then $u_{m+1} \notin N_J(u_{\tau(j)+1})$; otherwise there is a cycle which contradicts the assumption on C as $u_{\tau(j)+2} = u_{\tau(j+1)-2}$. Especially, $u_{\tau(l)+3} \in N(u_{\tau(j)+2})$. By Fact 1(2), $u_{\tau(l+1)+1} \notin N(u_{\tau(j)+2})$, and so $\sigma((u_{\tau(l)}, u_{\tau(l+1)}]) \setminus N(u_{\tau(j)+2}) \neq \emptyset$. Since $\{u_{\tau(l)+2}, u_{\tau(l)+3}\} \subset N(u_{\tau(j)+2})$, there exists a vertex $u_{\tau(l)+k} \in \sigma((u_{\tau(l)}, u_{\tau(l+1)}]) \setminus N(u_{\tau(j)+2})$ such that:

$$\{u_{\tau(l)+k-1}, u_{\tau(l)+k-2}\} \subset N(u_{\tau(j)+2}).$$

Then $u_{\tau(l)+k}, u_{\tau(l)+k-1} \notin N(u_{\tau(j)+1}) \cup N(e_{\tau(i)})$ by the above notice and Claim 5. Hence $u_{\tau(l)+k-1} \notin \sigma^{-1}(N(e_{\tau(j)})) \cup N(e_{\tau(i)})$. Because $u_{\tau(l)+k-1} \notin T$, the vertex $u_{\tau(l)+k-2}$ is adjacent to e_0. This is a contradiction. □

In the case that $d(e_0) = 2$, clearly both of $\sigma(N_I(e_0)) \cap \sigma^{-1}(N_I(e_{\tau(1)}))$ and $\sigma(N_J(e_0)) \cap \sigma^{-1}(N_J(e_{\tau(2)}))$ are empty where $I = (u_{\tau(1)}, u_{\tau(2)}]$ and $J = (u_{\tau(2)}, u_{\tau(1)}]$. This contradicts Claim 3. Therefore $d(e_0) \geq 3$. By symmetry, we may assume $d(x_1) \geq 2$. Let $u_{\tau(i)}, u_{\tau(j+1)} \in N_C(x_1)$ $(j > i)$ such that $u_{\tau(i+1)} \in N_C(x_2)$ and $(u_{\tau(i)}, u_{\tau(j)}] \cap N_C(e_0) \subset N_C(x_2)$. See Figure 2iii. Let $I = (u_{\tau(i)}, u_{\tau(j)}]$ and $J = (u_{\tau(j)}, u_{\tau(i)}]$. By Fact 1(3), we have $\sigma(N_I(e_0)) \cap \sigma^{-1}(N_I(e_{\tau(i)})) = \emptyset$ and $\sigma(N_J(x_1)) \cap \sigma^{-1}(N_J(e_{\tau(j)})) = \emptyset$.

Let $L = J \cap N(e_0) \setminus N(x_1)$, and

$$
\begin{aligned}
S &= \sigma^{-2}(L) \setminus \{N(e_{\tau(i)}) \cup \sigma^{-1}(N(e_{\tau(j)}))\} \\
T &= L \setminus N(e_{\tau(i)}) \\
U &= \sigma^{2}(L) \setminus \sigma^{-1}(N(e_j)).
\end{aligned}
$$

Reversing the orientation on C, similar arguments as above hold for $e_0, f_{\tau(j+1)}, f_{\tau(i+1)}$. Hence we have $\sigma^{-1}(N_{I'}(e_0)) \cap \sigma(N_{I'}(f_{\tau(j+1)})) = \emptyset$ where $I' = [u_{\tau(i+1)}, u_{\tau(j+1)})$, and:

$$[u_{\tau(j+1)}, u_{\tau(i+1)}) \cap N(e_0) \setminus N(x_1) = L$$

from the definition of $u_{\tau(i)}, u_{\tau(j+1)}$, and $S' = \sigma^2(L) \setminus \{N(f_{\tau(j+1)}) \cup \sigma(N(f_{\tau(i+1)}))\}$ corresponding to S. By symmetry, we may assume that $|S| \geq |S'|$.

By Claim 6 and Fact 1, S,T and U are mutually disjoint, and it holds that:

$$(S \cup T \cup U) \cap \{\sigma^{-1}(N_J(e_{\tau(j)}) \cup N_J(e_{\tau(i)}) \cup \sigma(N_J(e_0))\} = \emptyset.$$

As $u_{\tau(i)}, u_{\tau(j+1)} \notin L$, we have $S \cup U \subset \sigma(J)$, and thus the following inequalities hold:

$$
\begin{aligned}
|C| &= |\sigma(I) \cup \sigma(J)| = |\sigma(I)| + |\sigma(J)| \\
&\geq |\sigma^{-1}(N_I(e_{\tau(i)}))| + |N_I(e_{\tau(j)})| + |\sigma(N_I(e_0))| \\
&\quad + |\sigma^{-1}(N_J(e_{\tau(j)}))| + |N_J(e_{\tau(i)})| + |\sigma(N_J(e_0))| - |L| + |S| + |T| + |U| \\
&\geq |N_I(e_{\tau(i)})| + |N_I(e_{\tau(j)})| + |N_I(e_0)| + |N_J(e_{\tau(j)})| + |N_J(e_{\tau(i)})| + |N_J(e_0)| \\
&\quad - |L| + |S| + |T| + |U| \\
&= |N_C(e_{\tau(i)})| + |N_C(e_{\tau(j)})| + |N_C(e_0)| - |L| + |S| + |T| + |U|.
\end{aligned}
$$

Because $|C| + (|N_{G-C}(e_0)| + |N_{G-C}(e_{\tau(i)})| + |N_{G-C}(e_{\tau(j)})|) \leq |G| - 2$, we have $|L| - |U| \geq |S| + |T| + 1$ as in (1). Hence:

$$|\sigma^2(L) \cap \sigma^{-1}(N(e_{\tau(j)}))| = |L| - |U| \geq |S| + |T| + 1.$$

See Figure 2iii. On the other hand, by Claim 5, $\sigma^{-1}(N_J(e_{\tau(j)})) \cap N_J(f_{\tau(j+1)}) = \emptyset$. Therefore:

$$|\{\sigma^2(L) \cap \sigma^{-1}(N(e_{\tau(j)}))\} \cap \sigma(N(f_{\tau(i+1)}))| \geq |S| + |T| + 1 - |S'| \geq |T| + 1.$$

Then again by Claim 5, $\{\sigma^2(L) \cap \sigma^{-1}(N(e_{\tau(j)})) \cap \sigma(N(f_{\tau(i+1)}))\} \cap \sigma^2(N(e_{\tau(i)})) = \emptyset$. Therefore $\sigma^2(L) \cap \sigma^{-1}(N(e_{\tau(j)})) \cap \sigma(N(f_{\tau(i+1)})) \subset \sigma^2(T)$, and thus:

$$|T| + 1 \leq |\sigma^2(L) \cap \sigma^{-1}(N(e_{\tau(j)})) \cap \sigma(N(f_{\tau(i+1)}))| \leq |\sigma^2(T)| = |T|.$$

This is a contradiction. The proof is completed now.

Remark.

A cycle C is called a *vertex-dominating* cycle if all vertices of $G - C$ have a neighbour on C. Recently Yamashita [15] showed: let G be a 2-connected bipartite graph with partite sets B and W. If both of $|B|$ and $|W|$ are less than $3\delta(G)$, then there is a vertex-dominating cycle. However, edge degrees do not work for the existence of a vertex-dominating cycle (See the examples of Wang [14]).

Aung [3] showed: if G is a 2-connected triangle-free graph with order at most $6\delta(G) - 6$, then $|E(G - C)| \leq 1$ for any longest cycle C. For bipartite graphs, the following conjecture seems to hold.

Conjecture 10. *Let G be a 2-connected bipartite graph. If $|G| < 3\sigma_{1,1}(G) - 4$, then $|E(G - C)| \leq 1$ for any longest cycle C.*

Perhaps an even stronger conjecture holds.

Conjecture 11. *Let G be a 2-connected graph. If $d(e_0) + d(e_1) + d(e_2) > |G| - 2$ for any mutually remote edges e_0, e_1, e_2, then $|E(G - C)| \leq 1$ for any longest cycle C.*

If $\sigma_3(G) - 2 \geq |G|$, then $d(e_0) + d(e_1) + d(e_2) \geq 2\sigma_3(G) - 3 > |G| - 2$; In this case the conjecture follows from Theorem 1.

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