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# Edge degrees and dominating cycles

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**Abstract.** The edge degree $d(e)$ of the edge $e = uv$ is defined as the number of neighbours of $e$, i.e., $|N(u) \cup N(v)| - 2$. Two edges are called remote if they are disjoint and there is no edge joining them. In this article, we prove that in a 2-connected graph $G$, if $d(e_1) + d(e_2) > |V(G)| - 4$ for any remote edges $e_1, e_2$, then all longest cycles $C$ in $G$ are dominating, i.e., $G - V(C)$ is edgeless. This lower bound is best possible.

As a corollary, it holds that if $G$ is a 2-connected triangle-free graph with $\sigma_2(G) > |V(G)|/2$, then all longest cycles are dominating.

## 1 Introduction

The order of a simple graph $G$ is denoted by $n$ throughout this article, and a cycle $C$ is called *dominating* if $G - V(C)$ is a stable set. Nash-Williams [13] showed that if $G$ is 2-connected and $\delta(G) \geq (n+2)/3$, then all longest cycles of $G$ are dominating. Bondy generalized this fact as follows. Let

$$\sigma_k(G) = \min\{\sum_{i \leq k} d(x_i) \mid x_1, x_2, \ldots, x_k \text{ are independent vertices in } G\}.$$

**Theorem 1** (Bondy [6]). *Let $G$ be a 2-connected graph. If $\sigma_3(G) \geq n + 2$, then all longest cycles in $G$ are dominating.*

For studying dominating cycles in triangle-free graphs, an invariant called an *edge degree* is useful, and it seems essential. The edge degree $d(e)$ of an edge $e = uv$ is defined as the number of neighbours of $e$, i.e., $|N(u) \cup N(v)| - 2$. Two edges are called *remote* if they are disjoint and there is no edge joining them. Veldman [15] proved a $k$-connected graph has a dominating cycle if $\sum_{l \leq k} d(e_l) > k(n-k)/2$ for any $k+1$ mutually remote edges $e_0, e_1, \ldots, e_k$. Yamashita [17] improved this result by replacing the sufficient condition with the existence of three edges $e_0, e_1, e_2$ such that $\sum_{l \leq 2} d(e_l) > n - 2$ in any $k+1$ mutually remote edges.

For the existence of a longest cycle which is dominating, the following fact holds.

**Theorem 2** (Broersma, Yoshimoto and Zhang [7]). *Let $G$ be a 2-connected graph. If $d(e_0) + d(e_1) + d(e_2) > n - 2$ for any mutually remote edges $e_0, e_1, e_2$, then $G$ contains a longest cycle which is dominating.*

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The lower bound in Theorem 2 is best possible. Consider the vertex disjoint graphs $K_{m_1,k_1}, K_{m_2,k_2}, K_{m_3,k_3}$ and $\overline{K_2} = \{x, y\}$, and let $X_i$ and $Y_i$ be the partite sets of $K_{m_i,k_i}$. Then the graph

$$H_1 = K_{m_1,k_1} \cup K_{m_2,k_2} \cup K_{m_3,k_3} \cup \overline{K_2} \cup \{xx' \mid x' \in \bigcup_{i\le 3} X_i\} \cup \{yy' \mid y' \in \bigcup_{i\le 3} Y_i\}$$

has no dominating cycle, and the degree sum of any three mutually remote edges is $n-2$. See Figure 1.

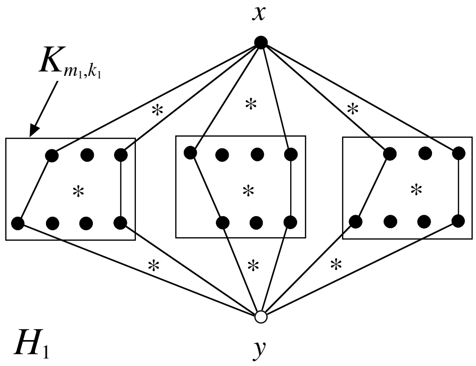


Figure 1:

The purpose of this article is to establish the following.

**Theorem 3.** *Let $G$ be a 2-connected graph. If $d(e_1) + d(e_2) > n - 4$ for any remote edges $e_1, e_2$, then all longest cycles in $G$ are dominating.*

The lower bound in Theorem 3 is also best possible. Consider the graphs $K_{m_4,k_4}$ and $K_{m_5,k_5}$, where $|X_i| \ge 2$ and $|Y_i| = |X_i| + 2$, and let $\{y_1^i, y_2^i, y_3^i\} \subset Y_i$ for $i = 4, 5$. Then the graph

$$H_2 = K_{m_4,k_4} \cup K_{m_5,k_5} \cup \{y_1^4 y_1^5, y_2^4 y_2^5, y_3^4 y_3^5\}$$

has a longest cycle which is not dominating and the minimum edge degree is $(n-4)/2$. See Figure 2. The graphs $H_1$ and $H_2$ generalize the examples due to Ash and Jackson in [1].

Ore [14] showed that the circumference of a 2-connected graph is at least $\sigma_2$ or the graph is hamiltonian. In the same way, can we measure the circumference using edge degrees? For this question, we have the following conjecture.

**Conjecture 4.** *If $G$ is a 1-tough graph, then the circumference of $G$ is at least*

$$2 + \min\{d(e_1) + d(e_2) \mid e_1, e_2 \text{ are remote edges}\}$$

*or all longest cycles in $G$ are dominating.*

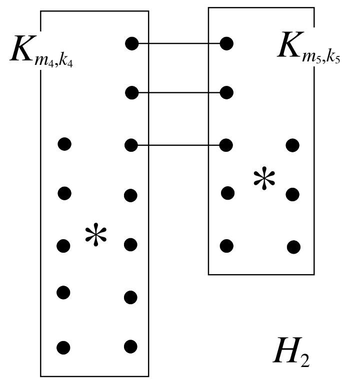


Figure 2:

In this conjecture, we cannot replace 1-toughness with 2-connectedness by $H_1$.

If a graph is triangle-free, then an edge degree is obtained immediately from the degree sum of it's ends, and so $d(e_0) + d(e_1) + d(e_2) \geq 2(\sigma_3(G) - 3)$ for mutually remote edges $e_0, e_1, e_2$. Hence, using Theorem 2 we can improve Aung's theorem [3], which states that a 2-connected triangle-free graph with $\delta(G) > (n + 5)/6$ contains a longest cycle which is dominating.

**Corollary 5.** *Let $G$ be a 2-connected triangle-free graph. If $\sigma_3(G) > (n+4)/2$, then $G$ contains a longest cycle which is dominating.*

Theorem 1.1 in [3] implies that in a 2-connected triangle-free graph with $\delta > n/4$, all longest cycles are dominating. Theorem 3 improves this fact.

**Corollary 6.** *Let $G$ be a 2-connected triangle-free graph. If $\sigma_2(G) > n/2$, then all longest cycles in $G$ are dominating.*

Let $G$ be a bipartite graph with partite sets $X$ and $Y$, and $\sigma_{1,1}(G) = \min\{d(x)+d(y) \mid xy \notin E(G), x \in X, y \in Y\}$. Moon and Moser showed that a 2-connected balanced bipartite graph with $\sigma_{1,1} > n/2$ is hamiltonian. Kaneko and Yoshimoto [10] generalized this by showing that if $G$ is a 2-connected balanced bipartite graph and is not hamiltonian, then $G$ has a cycle of length at least $2\sigma_{1,1} - 2$. For dominating cycles, Theorem 3 implies the following.

**Corollary 7.** *Let $G$ be a 2-connected bipartite graph. If $\sigma_{1,1}(G) > n/2$, then all longest cycles in $G$ are dominating.*

These results lead to a question.

*Is a 1-tough triangle-free graph with $\sigma_2 > (n + 2)/2$ hamiltonian?*

The unbalance complete bipartite graphs show that 1-toughness cannot be replaced with 2-connectedness. But the minimum degree of the Petersen graph is $(n+2)/4$, perhaps the graph is a special case. However, it is not possible to replace $(n+2)/2$ by $n/2$ because Bauer et al. [5] constructed a class of non-hamiltonian 1-tough triangle-free graphs with $\delta = (n+1)/4$.

Finally, we give some additional definitions and notations. The set of all the neighbours of a vertex $x \in V(G)$ is denoted by $N_G(x)$ or simply $N(x)$, and its cardinality by $d_G(x)$ or $d(x)$. For a subgraph $H \subset G$, we denote $N_G(x) \cap V(H)$ by $N_H(x)$ and its cardinality by $d_H(x)$. The set of neighbours $\bigcup_{v \in H} N_G(v) \setminus V(H)$ is written by $N_G(H)$ or $N(H)$. For a subgraph $F \subset G$, $N_G(H) \cap V(F)$ is denoted by $N_F(H)$. If the meaning is clear, we denote the vertex subset $V(H)$ by simply $H$.

All notation and terminology not explained here is given in [8].

# 2 The Proof of Theorem 3

We assume that $G$ has a longest cycle $C = u_1u_2 \dots u_{|C|}u_1$ such that $E(G-C) \neq \emptyset$, and reach a contradiction.

The successor $u_{i+1}$ of $u_i$ is denoted by $u_i^+$ and the predecessor by $u_i^-$. For $A \subset V(C)$, we write $\{u_i^+ \mid u_i \in A\}$ and $\{u_i^- \mid u_i \in A\}$ by $A^+$ and $A^-$, respectively. The segment $u_iu_{i+1} \dots u_j$ is denoted by $u_i \overrightarrow{C} u_j$ where the subscripts are to be taken modulo $|C|$. The reverse segment $u_ju_{j-1} \dots u_i$ is given by $u_j \overleftarrow{C} u_i$. For each $u_i \in C$, we denote the edge $u_iu_{i+1}$ by $e_i$.

Let $H$ be a component in $G-C$ containing at least two vertices and $P$ a longest path in $H$ such that it's ends $x, y$ are adjacent to distinct vertices on $C$. If $|V(P)| = 1$, then $G$ has a cut vertex, and so $|V(P)| \geq 2$. Let $N_C(x) \cup N_C(y) = \{u_{\tau(1)}, u_{\tau(2)}, \dots\}$ which occur on $C$ in the order of their indices.

**Case 1.** $|V(P)| \geq 3$**.**

Let $u_{\tau(i)} \in N(x)$ and $u_{\tau(j)} \in N(y)$ such that $i \neq j$ and $u_{\tau(i+1)} \in N(y)$ and $u_{\tau(j+1)} \in N(x)$. If $e_{\tau(i)+1} = u_{\tau(i)+1}u_{\tau(i)+2}$ is adjacent to $H$, then $N_H(e_{\tau(i)+1})$ contains a vertex $z \neq x$ because $u_{\tau(i+1)} \in N(y)$. Hence the cycle

$$xu_{\tau(i)} \overleftarrow{C} wzQx,$$

where $w \in N_{e_{\tau(i)+1}}(z)$ and $Q$ is a path joining $z$ and $x$ in $H$, is longer than $C$. Thus neither $e_{\tau(i)+1}$ nor (by symmetry) $e_{\tau(j)+1}$ is adjacent to $H$.

Let $I = u_{\tau(i)+1}\overrightarrow{C}u_{\tau(j)}$ and $J = u_{\tau(j)+1}\overrightarrow{C}u_{\tau(i)}$. If there exists a vertex $u_l \in N_I(e_{\tau(i)+1})^- \cap N_I(e_{\tau(j)+1})$, then the cycle:

$$xPyu_{\tau(j)}\overleftarrow{C}u_l^+ w\overrightarrow{C}u_l w'\overrightarrow{C}u_{\tau(i)}x$$

is longer than $C$, where $w \in N_{e_{\tau(i)+1}}(u_l^+)$ and $w' \in N_{e_{\tau(j)+1}}(u_l)$. See Figure 3. Hence

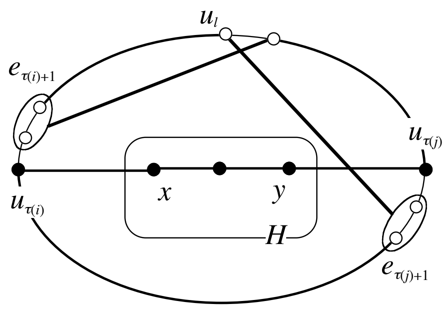


Figure 3:

by symmetry, we have:

$$N_I(e_{\tau(i)+1})^- \cap N_I(e_{\tau(j)+1}) = \emptyset \text{ and } N_J(e_{\tau(i)+1}) \cap N_J(e_{\tau(j)+1})^- = \emptyset.$$

Similarly, if $u_{\tau(i)+1} \in N(e_{\tau(j)+1})$, then the cycle:

$$xPyu_{\tau(j)}\overleftarrow{C}u_{\tau(i)+1}w\overrightarrow{C}u_{\tau(i)}x$$

is longer than $C$, where $w \in N_{e_{\tau(j)+1}}(u_{\tau(i)+1})$. Thus, $u_{\tau(i)+1} \notin N(e_{\tau(j)+1})$ and $u_{\tau(j)+1} \notin N(e_{\tau(i)+1})$ by symmetry.

Since $N_I(e_{\tau(i)+1})^- \cup N_I(e_{\tau(j)+1}) \subset I - u_{\tau(i)+1}$ and $N_J(e_{\tau(i)+1}) \cup N_J(e_{\tau(j)+1})^- \subset J - u_{\tau(j)+1}$,

$$\begin{aligned} |C| &\geq |N_I(e_{\tau(i)+1})^-| + |N_I(e_{\tau(j)+1})| + |N_J(e_{\tau(i)+1})| + |N_J(e_{\tau(j)+1})^-| \\ &\quad + |\{u_{\tau(i)+1}, u_{\tau(j)+1}\}| = |N_C(e_{\tau(i)+1})| + |N_C(e_{\tau(j)+1})| + 2. \end{aligned}$$

Similarly we can show that $e_{\tau(i)+1}$ and $e_{\tau(j)+1}$ have no common neighbours in $G - (C \cup H)$. Since neither $e_{\tau(i)+1}$ nor $e_{\tau(j)+1}$ is adjacent to $H$,

$$\begin{aligned} n &\geq |N_{G-C}(e_{\tau(i)+1})| + |N_{G-C}(e_{\tau(j)+1})| + |N_C(e_{\tau(i)+1})| + |N_C(e_{\tau(j)+1})| \\ &\quad + 2 + |H| \geq d(e_{\tau(i)+1}) + d(e_{\tau(j)+1}) + 5 > (n-4) + 5 > n, \end{aligned}$$

a contradiction.

**Case 2.** $|V(P)| = 2$**.**

Let $P = e_0 = xy$ and $\widetilde{N}_C(e_0) = N_C(e_0)^+ \cup N_C(e_0)^-$. For an edge $e_i = u_iu_{i+1}$ on $C$, we denote $N_C(e_i) \cup \{u_i, u_{i+1}\}$ by $N_C[e_i]$.

**Fact 1.** *If an edge $e_i$ on $C$ is remote to $e_0$, then $|\widetilde{N}_C(e_0) \setminus N_C[e_i]| < d_C(e_0)$.*

*Proof.* Suppose $e_i$ is remote to $e_0$. If $e_i$ is adjacent to a vertex $z \in H - \{x, y\}$, then there exists a path joining $e_0$ and $z$ in $H$, which contradicts our assumption of $P$. Hence, $N_H(e_i) = \emptyset$ and

$$N(e_i) \subset G - H - \{u_i, u_{i+1}\} - \widetilde{N}_C(e_0) \setminus N_C[e_i].$$

If $|\widetilde{N}_C(e_0) \setminus N_C[e_i]| \geq d_C(e_0)$, then:

$$\begin{aligned} d(e_i) &\leq n - |H| - 2 - |\widetilde{N}_C(e_0) \setminus N_C[e_i]| \\ &\leq n - (d_H(e_0) + 2) - 2 - d_C(e_0) = n - d(e_0) - 4 \end{aligned}$$

since $|H| \geq d_H(e_0) + 2$ and $d(e_0) = d_H(e_0) + d_C(e_0)$. Hence $d(e_0) + d(e_i) \leq n - 4$, a contradiction. $\square$

Let $u_{\tau(i)} \in N_C(y)$ such that $u_{\tau(i+1)} \in N_C(x)$, and let

$$X = (N_C(x) \setminus N_C(y)) \cup u_{\tau(i+1)} \text{ and } Y = N_C(e_0) \setminus X.$$

If there exists a vertex $u^-_{\tau(l)} \in Y^- \cap N(e_{\tau(i+1)-2})$, then the cycle:

$$xyu_{\tau(l)}\overrightarrow{C}wu^-_{\tau(l)}\overleftarrow{C}u_{\tau(i+1)}x$$

is longer than $C$, where $w \in N_{e_{\tau(i+1)-2}}(u^-_{\tau(l)})$. Hence $Y^- \cap N(e_{\tau(i+1)-2}) = \emptyset$. If $X^+ \cap N(e_{\tau(i+1)-2}) = \emptyset$, then

$$|\widetilde{N}_C(e_0) \setminus N_C[e_{\tau(i+1)-2}]| \geq |Y^-| + |X^+| \geq d_C(e_0)$$

because $Y^-, X^+$ and $N_C[e_{\tau(i+1)-2}]$ are pairwise disjoint. Since this contradicts Fact 1, $X^+ \cap N(e_{\tau(i+1)-2}) \neq \emptyset$.

Let $k = \min\{l \mid u_l \in u^+_{\tau(i)}\overrightarrow{C}u^-_{\tau(i+1)} \text{ and } X^+ \cap N(u_l) \neq \emptyset\}$. Clearly $u_k \notin e_{\tau(i)+1}$; otherwise there exists a cycle longer than $C$. Thus $e_{k-2} \in u^+_{\tau(i)}\overrightarrow{C}u^-_{\tau(i+1)}$ and $X^+ \cap N(e_{k-2}) = \emptyset$.

Let $u^+_{\tau(l)} \in X^+ \cap N(u_k)$ and $Y_1 = Y \cap u_{\tau(l)}\overrightarrow{C}u_{\tau(i)}$ and $Y_2 = Y \cap u_{\tau(i+1)}\overrightarrow{C}u_{\tau(l)}$. Notice that $u_{\tau(l)} \notin Y = Y_1 \cup Y_2$ since $u_{\tau(l)} \in X$. If there exists $u^-_{\tau(m)} \in Y_1^- \cap N(e_{k-2})$, then the cycle

$$xyu_{\tau(m)}\overrightarrow{C}wu^-_{\tau(m)}\overleftarrow{C}u^+_{\tau(l)}u_k\overrightarrow{C}u_{\tau(l)}x$$

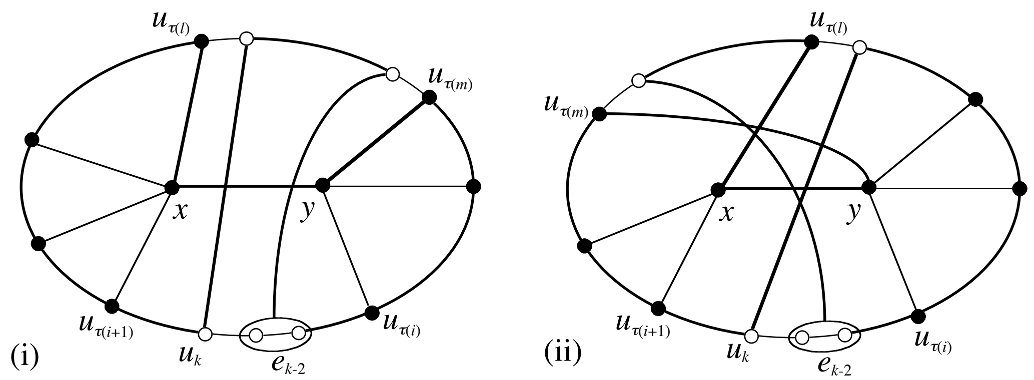


Figure 4:

is longer than $C$, where $w \in N_{e_{k-2}}(u^-_{\tau(m)})$. See Figure 4(i).

If there exists $u^+_{\tau(m)} \in Y^+_2 \cap N(e_{k-2})$, then the cycle

$$xyu_{\tau(m)}\overleftarrow{C}u_k u^+_{\tau(l)}\overrightarrow{C}wu^+_{\tau(m)}\overrightarrow{C}u_{\tau(l)}x$$

is longer than $C$, where $w \in N_{e_{k-2}}(u^+_{\tau(m)})$. See Figure 4(ii). Hence $N(e_{k-2}) \cap (Y^-_1 \cup Y^+_2) = \emptyset$. Since $X^+, Y^-_1, Y^+_2$ and $N_C[e_{k-2}]$ are pairwise disjoint,

$$|\widetilde{N}_C(e_0) \setminus N[e_{k-2}]| \geq |X^+| + |Y^-_1| + |Y^+_2| \geq d_C(e_0).$$

This contradicts Fact 1. The proof is completed now.

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